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FRACTIONAL INTEGRAL OPERATORS WITH HOMOGENEOUS KERNELS ON GENERALIZED LORENTZ–MORREY SPACES

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Abstract. We establish the mapping properties of the fractional integral operators with homogeneous kernels on generalized Lorentz-Morrey spaces.

1. Introduction

This paper aims to establish the mapping properties of the fractional integral operators with homogeneous kernels on Morrey spaces built on generalized Lorentz spaces.

The fractional integral operators with homogeneous kernels were introduced in [29]. It is a nature generalization of the fractional integral operators. The weighted norm inequalities for the fractional integral operators with homogeneous kernels were obtained in [12]. By applying extrapolation theory on these inequalities, we have the mapping properties for the fractional integral operators with homogeneous kernels on Lebesgue spaces with variable exponents [22, Theorem 4.4]. Moreover, the mapping properties for the fractional integral operators with homogeneous kernels had been further extended to the Morrey spaces with variable exponent in [22, Theorems 3.1 and 3.2]. In particular, the results in [22] also give the boundedness of the fractional integral operators with homogeneous kernels on the classical Morrey spaces.

The main results of this paper are motivated by the generalized Lorentz spaces Λ_w^p introduced by Lorentz in [25], the Morrey-Lorentz spaces introduced by Ragusa in [33] and the local Morrey-Lorentz spaces introduced by Aykol, Guliyev and Serbetci in [3, 4, 18].

The generalized Lorentz spaces include the classical Lorentz spaces $L_{p,q}$, Lorentz-Zygmund spaces [5] and the Lorentz-Karamata spaces [13, Section 3.4.3]. Since the introduction of the generalized Lorentz space, it becomes the main topic of the study of function spaces [1, 2, 6, 8, 9, 11, 27, 34]. It is impossible to give a detail survey on this huge topic, the reader is referred to [10] for the recent development on the generalized Lorentz spaces.

The classical Morrey spaces were introduced by Morrey in [28] for the study of the solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces on partial differential equation, the reader is referred to [31, 32].

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Recently, the study of Morrey spaces had been extended to the Morrey-Lorentz spaces [19, 33], the Orlicz-Morrey spaces [30] and the Morrey spaces with variable exponents [16, 17, 22]. The study of these Morrey type spaces has applications on partial differential equations, for example, they are related to the viscosity solutions of some fully nonlinear elliptic equations [35].

This paper is organized as follows. The definition of the generalized Lorentz space and some of its fundamental properties are given in Section 2. We also present the mapping properties of the fractional integral operators with homogeneous kernels on Lebesgue spaces in this section. The boundedness of the fractional integral operators with homogeneous kernels on generalized Lorentz spaces is established in 3. The main result of this paper, the mapping properties of the fractional integral operators with homogeneous kernels on generalized Lorentz-Morrey spaces, is given in Section 4.

2. Preliminaries and Definitions

Let $\mathscr{M}$ and $\mathscr{M}(0,\infty)$ be the sets of Lebesgue measurable functions on $\mathbb{R}^n$ and $(0,\infty)$, respectively.

The fractional integral operators with homogeneous kernels are introduced by Muckenhoupt and Wheeden in [29]. We recall the definition of fractional integral operator with homogeneous kernel from [29]. Let $0<\alpha<n$ and Ω be a homogeneous function on $\mathbb{R}^n$ with degree zero. That is, for any $x\in\mathbb{R}^n$ and $\lambda>0$

$$\Omega(\lambda x)=\Omega(x). \tag{2.1}$$

The fractional integral operator with homogeneous kernel is defined by

$$T_{\Omega,\alpha}f(x)=\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^{n-\alpha}}f(y)dy.$$

We present the mapping properties of $T_{\Omega,\alpha}$ on Lebesgue spaces in the following [26, Theorem 3.3.1].

THEOREM 2.1. *Let $0<\alpha<n$ and $\Omega\in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ satisfy (2.1). Suppose that $1<p<\frac{n}{\alpha}$ and*

$$\frac{1}{p}=\frac{1}{q}+\frac{\alpha}{n}.$$

There exists a constant $C>0$ such that for any $f\in L_p$, we have

$$\|T_{\Omega,\alpha}f\|_{L_q}\leqslant C\|f\|_{L_p}.$$

We recall the definition of the generalized Lorentz spaces [2, 8, 9, 25, 34]. In order to present the definition of the generalized Lorentz space, we first recall the definition of the decreasing rearrangement for Lebesgue measurable functions.

For any Lebesgue measurable function $f\in\mathscr{M}$, define

$$d_f(\lambda)=|\{x\in\mathbb{R}^n:|f(x)|>\lambda\}|,\quad \lambda>0.$$

The decreasing rearrangement of f is defined as

$$f^*(t) = \inf\{\lambda : d_f(\lambda) \leqslant t\}, \quad t \geqslant 0.$$

We now give the definition of the generalized Lorentz spaces in the following.

DEFINITION 2.1. Let $0 < p < \infty$ and $w : [0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. The generalized Lorentz space Λ_w^p consists of all Lebesgue measurable function f satisfying

$$\|f\|_{\Lambda_w^p} = \left(\int_0^\infty (f^*(t))^p w(t)dt\right)^{1/p} < \infty.$$

For any $0 < p < \infty$ and $w : [0,\infty) \to [0,\infty)$, we have the weighted Lebesgue space

$$L_w^p = \left\{f \in \mathscr{M}(0,\infty) : \|f\|_{L_w^p} < \infty\right\}$$

where

$$\|f\|_{L_w^p} = \left(\int_0^\infty |f(y)|^p w(y)dy\right)^{1/p}.$$

Consequently, $\|f\|_{\Lambda_w^p} = \|f^*\|_{L_w^p}$.

Furthermore, since $(|f|^q)^* = (f^*)^q$, $0 < q < \infty$, [6, Chapter 2, (1.20)] we have

$$\||f|^q\|_{\Lambda_w^p} = \left(\int_0^\infty (f^*(t))^{pq} w(t)dt\right)^{1/p} = \|f\|_{\Lambda_w^{pq}}^q. \tag{2.2}$$

The generalized Lorentz space was introduced by Lorentz [25]. In addition, Lorentz showed that when $p \geqslant 1$, $\|\cdot\|_{\Lambda_w^p}$ is a norm if and only if w is non-increasing.

In view of [34, Theorem 4], [2] and [8, Theorem 2.3], $\|\cdot\|_{\Lambda_w^p}$ is equivalent to a norm if w satisfies

$$t^p \int_t^\infty y^{-p} w(y)dy \leqslant C \int_0^y w(y)dy, \quad 0 < t, \quad \text{when} \quad p > 1,$$
$$\frac{1}{t}\int_0^t w(y)dy \leqslant \frac{C}{s}\int_0^s w(y)dy \quad 0 < s \leqslant t, \quad \text{when} \quad p = 1$$

for some $C > 0$.

According to [9, 10], $\|\cdot\|_{\Lambda_w^p}$ is a quasi-norm if and only if the fundamental function

$$W(t) = \int_0^t w(s)ds$$

satisfies the $\triangle_2$ condition

$$W(2t) \leqslant CW(t), \quad t > 0$$

for some $C > 1$, see also [14]. For any Lebesgue measurable set E with $|E| < \infty$, we have

$$\|\chi_E\|_{\Lambda_w^p}^p = \int_0^\infty (\chi_{[0,|E|]}(t))^p w(t)dt = W(|E|).$$

Therefore, we have the following lemma. The following lemma involves the notion of rearrangement-invariant quasi-Banach function space, for brevity, we refer the reader to [21, Definition 2.1] for the definition.

LEMMA 2.2. *Let* $0 < p < \infty$ *and* $w : [0,\infty) \to [0,\infty)$. *If* $W(t) = \int_0^t w(s)ds$ *is finite a.e.a and satisfies the* $\triangle_2$ *condition, then* Λ_w^p *is a rearrangement-invariant quasi-Banach function space.*

At the end of this section, we recall an important member of the generalized Lorentz space, the Lorentz-Karamata space [13, Section 3.4.3].

DEFINITION 2.2. A Lebesgue measurable function $b : [1,\infty) \to (0,\infty)$ is called as a slowly varying function if for any $\varepsilon > 0$

1. the function $t \to t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function on $[1,\infty)$, and

2. the function $t \to t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function on $[1,\infty)$.

For any slowly varying function, define

$$\gamma_b(t) = b(\max\{t, t^{-1}\}), \quad t > 0.$$

When $0 < r, p < \infty$, b is a slowly varying function and $w(t) = t^{\frac{p}{r}-1}(\gamma_b(t))^p$, the generalized Lorentz space Λ_w^p is the Lorentz-Karamata space $L_{r,p,b}$.

According to [13, Proposition 3.4.33 (i) and (v)], there is a constant $C > 0$ such that for any $t > 0$

$$\int_0^t s^{\frac{p}{r}-1}(\gamma_b(s))^p ds \approx t^{\frac{p}{r}}(\gamma_b(t))^p. \tag{2.3}$$

Therefore, [13, Proposition 3.4.33 (iii)] guarantees that

$$\begin{aligned} \int_0^{2t} s^{\frac{p}{r}-1}(\gamma_b(s))^p ds &\leqslant C(2t)^{\frac{p}{r}}(\gamma_b(2t))^p \\ &\leqslant C\int_0^t s^{\frac{p}{r}-1}(\gamma_b(s))^p ds \leqslant C(2t)^{\frac{p}{r}}(\gamma_b(t))^p \end{aligned} \tag{2.4}$$

for some $C > 0$. The above inequalities show that the fundamental function for $L_{r,p,b}$ is finite a.e. and satisfies the $\triangle_2$ condition. That is, $L_{r,p,b}$ is a rearrangement-invariant quasi-Banach function space.

When $1 < r < \infty$ and $1 \leqslant p < \infty$, the Lorentz-Karamata space is equivalent to a rearrangement-invariant Banach function space [13, Theorem 3.4.41].

In addition, when $0 < r, p < \infty$, we have $p_{L_{r,p,b}} = q_{L_{r,p,b}} = r$ [23, Proposition 6.1]. The statement and the proof of [23, Proposition 6.1] are for the case $1 < r, p < \infty$. It is easy to see that they are also valid for $0 < r, p < \infty$.

3. Fractional integrals with homogeneous kernels on generalized Lorentz spaces

In this section, we establish the mapping properties of $T_{\Omega,\alpha}$ on generalized Lorentz spaces. We accomplish this result by interpolating the mapping properties $T_{\Omega,\alpha}$ on Lebesgue spaces with the interpolation functor used in [21]. To apply this interpolation functor, we need to the notion of Boyd's indices. Therefore, we first recall the definition of Boyd's indices for Λ^p_w.

For any $s > 0$ and $f \in \mathscr{M}(0,\infty)$, define $(D_s f)(t) = f(st)$.

DEFINITION 3.1. Let $0 < p < \infty$ and $w : [0,\infty) \to [0,\infty)$. Define the lower Boyd index of Λ^p_w, $p_{\Lambda^p_w}$, and the upper Boyd index of Λ^p_w, $q_{\Lambda^p_w}$, by

$$p_{\Lambda^p_w} = \sup\{p > 0 : \exists C > 0 \textit{ such that } \forall 0 \leqslant s < 1,\ \|D_s\|_{L^p_w \to L^p_w} \leqslant Cs^{-1/p}\},$$
$$q_{\Lambda^p_w} = \inf\{q > 0 : \exists C > 0 \textit{ such that } \forall 1 \leqslant s,\ \|D_s\|_{L^p_w \to L^p_w} \leqslant Cs^{-1/q}\},$$

respectively.

The above definition of Boyd's indices follows from the definition of Boyd's indices for rearrangement-invariant quasi-Banach function spaces [21, 27]. In view of the formula for the Boyd's indices of Λ^p_w [1], we have

$$\frac{1}{p_{\Lambda^p_w}} = \lim_{s\to\infty} \frac{\log \bar{W}^{1/p}(s)}{\log s} \quad \text{and} \quad \frac{1}{q_{\Lambda^p_w}} = \lim_{s\to 0} \frac{\log \bar{W}^{1/p}(s)}{\log s} \tag{3.1}$$

where

$$\bar{W}(s) = \sup_{t>0} \frac{W(st)}{W(t)}, \quad s > 0.$$

In particular, the above formulas assure that $p_0 < p_{\Lambda^p_w} \leqslant q_{\Lambda^p_w} < p_1$ whenever there exist $s_0, s_\infty > 0$ and $C_0, C_\infty > 0$ such that

$$W^{1/p}(st) \leqslant C_\infty W^{1/p}(t) s^{1/p_0}, \quad \forall t > 0 \quad \text{and} \quad s > s_\infty, \tag{3.2}$$
$$W^{1/p}(st) \leqslant C_0 W^{1/p}(t) s^{1/p_1}, \quad \forall t > 0 \quad \text{and} \quad 0 < s < s_0. \tag{3.3}$$

Furthermore, for any $0 < r < \infty$, we find that

$$\frac{1}{p_{\Lambda^{pr}_w}} = \lim_{s\to\infty} \frac{\log \bar{W}^{1/pr}(s)}{\log s} = \lim_{s\to\infty} \frac{\log \bar{W}^{1/p}(s)}{r \log s} = \frac{1}{r p_{\Lambda^p_w}} \tag{3.4}$$
$$\frac{1}{q_{\Lambda^{pr}_w}} = \lim_{s\to\infty} \frac{\log \bar{W}^{1/pr}(s)}{\log s} = \lim_{s\to\infty} \frac{\log \bar{W}^{1/p}(s)}{r \log s} = \frac{1}{r q_{\Lambda^p_w}}. \tag{3.5}$$

Since Lemma 2.2 guarantees that Λ^p_w is a rearrangement-invariant quasi-Banach function space, we are allowed to use the interpolation functor introduced in [21, Definition 4,2].

DEFINITION 3.2. Let $0 < r, \theta < \infty$, $0 < p < \infty$ and $w : (0,\infty) \to (0,\infty)$. Let (X_0, X_1) be a compatible couple of quasi-normed spaces. The space $(X_0, X_1)_{\theta,r,\Lambda_w^p}$ consists of all f in $X_0 + X_1$ such that

$$\|f\|_{(X_0,X_1)_{\theta,r,\Lambda_w^p}} = \left(\int_0^\infty (t^{-\frac{1}{r}} K(f, t^{\frac{1}{\theta}}, X_0, X_1))^p w(t) dt \right)^{1/p} < \infty. \tag{3.6}$$

It was shown in [21, Theorem 4.1] that $(\cdot,\cdot)_{\theta,r,\Lambda_w^p}$ is an interpolation functor. It also present the corresponding result on the interpolation of linear operators.

We introduce a new function space used to study the mapping properties of $T_{\Omega,\alpha}$.

DEFINITION 3.3. Let $0 \leqslant \alpha < \infty$, $0 < p < \infty$ and $w : (0,\infty) \to (0,\infty)$. The set $\Lambda_w^{p,\alpha}$ consists of all $f \in \mathscr{M}$ satisfying

$$\|f\|_{\Lambda_w^{p,\alpha}} = \left(\int_0^\infty (t^{-\frac{\alpha}{n}} f^*(t))^p w(t) dt \right)^{1/p} < \infty.$$

We see that $\Lambda_w^{p,\alpha}$ is also a generalized Lorentz space with the weight $t \to t^{-\frac{\alpha p}{n}} w(t)$. In particular, when $\alpha = 0$, it reduces to Λ_w^p.

According to [21, Proposition 3.1], whenever $W(t) = \int_0^t w(s) ds$ is finite a.e. and satisfies the $\triangle_2$ condition, then $\Lambda_w^{p,\alpha}$ is a rearrangement-invariant quasi-Banach function spaces. Furthermore, if w is non-increasing, $\Lambda_w^{p,\alpha}$ is a Banach space.

The following result is a special case of [21, Theorem 3.2].

LEMMA 3.1. *Let* $0 \leqslant \alpha < \infty$, $0 < p < \infty$ *and* $w : (0,\infty) \to (0,\infty)$. *If* W *is finite a.e., satisfies the* $\triangle_2$ *condition and* $0 < p_{\Lambda_w^p} \leqslant q_{\Lambda_w^p} < \frac{n}{\alpha}$, *then*

$$W_\alpha(t) = \int_0^t y^{-\frac{\alpha p}{n}} w(y) dy$$

is finite a.e. and satisfies the $\triangle_2$ *condition.*

Proof. Since W is finite a.e., satisfies the $\triangle_2$ condition, Λ_w^p is a rearrangement-invariant quasi-Banach function space. Therefore, [21, Theorem 3.2] guarantees that $\Lambda_w^{p,\alpha}$ is also a rearrangement-invariant quasi-Banach function space.

As for any Lebesgue measurable set E with finite measure, $\|\chi_E\|_{\Lambda_w^{p,\alpha}}^p = W_\alpha(|E|)$, W_α is finite a.e. In addition, $\|\cdot\|_{\Lambda_w^{p,\alpha}}$ is a quasi-norm guarantees that W_α fulfills the $\triangle_2$ condition [9]. $\square$

In view of [21, Theorem 4.2] and [24, Theorem 3.3], we have the following result which shows that generalized Lorentz spaces can be generated by the action of the functor $(\cdot,\cdot)_{\theta,r,\Lambda_w^p}$ on Lebesgue spaces.

PROPOSITION 3.2. *Let* $0 \leqslant \alpha < \infty$, $0 < p_0 < p_1 < \infty$, $0 < p < \infty$ *and* $w : (0,\infty) \to (0,\infty)$. *Let*

$$\frac{1}{\theta} = \frac{1}{p_0} - \frac{1}{p_1} \quad \text{and} \quad \frac{1}{r} = \frac{1}{p_0} + \frac{\alpha}{n}.$$

Suppose that w *satisfies* (3.2)–(3.3) *and*

$$\frac{1}{p_1}+\frac{\alpha}{n}<\frac{1}{q_{\Lambda_w^p}}\leqslant\frac{1}{p_{\Lambda_w^p}}<\frac{1}{p_0}+\frac{\alpha}{n}.$$

We have

$$(L^{p_0},L^{p_1})_{\theta,r,\Lambda_w^p}=\Lambda_w^{p,\alpha}.$$

We are now ready to present the mapping properties of the fractional integrals with homogeneous kernels on generalized Lorentz spaces.

THEOREM 3.3. *Let* $0<p<\infty$, $0<\alpha<n$ *and* $w:[0,\infty)\to[0,\infty)$. *Suppose that* $\Omega\in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ *satisfies* (2.1). *If* $1<p_{\Lambda_w^p}\leqslant q_{\Lambda_w^p}<\frac{n}{\alpha}$, *then there exists a constant* $C>0$ *such that for any* $f\in\Lambda_w^p$

$$\|T_{\Omega,\alpha}f\|_{\Lambda_w^{p,\alpha}}\leqslant C\|f\|_{\Lambda_w^p}.$$

Proof. In view of (3.1) and the assumption $1<p_{\Lambda_w^p}\leqslant q_{\Lambda_w^p}<\frac{n}{\alpha}$, there exists $1<u_0<p_{\Lambda_w^p}\leqslant q_{\Lambda_w^p}<u_1<\frac{n}{\alpha}$ such that

$$W^{1/p}(st)\leqslant C_\infty W^{1/p}(t)s^{1/u_0},\quad \forall t>0 \quad\text{and}\quad s>s_\infty, \tag{3.7}$$
$$W^{1/p}(st)\leqslant C_0 W^{1/p}(t)s^{1/u_1},\quad \forall t>0 \quad\text{and}\quad 0<s<s_0. \tag{3.8}$$

for some $C_0,C_\infty,s_0,s_\infty>0$.

Define q_i, $i=0,1$ by

$$\frac{1}{q_i}=\frac{1}{u_i}-\frac{\alpha}{n}.$$

Theorem 2.1 assures that $T_{\Omega,\alpha}:L^{u_i}\to L^{q_i}$, $i=0,1$, are bounded. Let $\frac{1}{\theta}=\frac{1}{u_0}-\frac{1}{u_1}=\frac{1}{q_0}-\frac{1}{q_1}$. We have

$$\frac{1}{q_1}+\frac{\alpha}{n}=\frac{1}{u_1}<\frac{1}{q_{\Lambda_w^p}}\leqslant\frac{1}{p_{\Lambda_w^p}}<\frac{1}{u_0}=\frac{1}{q_0}+\frac{\alpha}{n}.$$

By applying the functor $(\cdot,\cdot)_{\theta,u_0,\Lambda_w^p}$ on $T_{\Omega,\alpha}$, we obtain

$$\|T_{\Omega,\alpha}f\|_{\Lambda_w^{p,\alpha}}=\|T_{\Omega,\alpha}\|_{(L^{q_0},L^{q_1})_{\theta,u_0,\Lambda_w^p}}\leqslant C\|f\|_{(L^{p_0},L^{p_1})_{\theta,u_0,\Lambda_w^p}}=C\|f\|_{\Lambda_w^p}. \qquad \square$$

Let $0<\alpha<n$ and f be a locally integrable function. The fractional integral operator is defined by

$$I_\alpha f(x)=\int_{\mathbb{R}^n}\frac{f(y)}{|x-y|^{n-\alpha}}dy.$$

The fractional maximal operator M_α is defined as

$$M_\alpha f(x)=\sup_{x\ni B}\frac{1}{|B|^{1-\frac{\alpha}{n}}}\int_B|f(y)|dy$$

where the supremum is taken over all $B \in \mathbb{B}$ containing x.

It is well known that there exists a constant $C > 0$ such that for any locally integrable function f, we have

$$M_\alpha(f)(x) \leqslant CI_\alpha(|f|)(x), \quad \forall x \in \mathbb{R}^n. \tag{3.9}$$

By using Theorem 3.3 with $\Omega \equiv 1$, we establish the subsequent corollary.

COROLLARY 3.4. *Let* $0 < p < \infty$, $0 < \alpha < n$ *and* $w : [0,\infty) \to [0,\infty)$. *If* $1 < p_{\Lambda_w^p} \leqslant q_{\Lambda_w^p} < \frac{n}{\alpha}$, *then there exists a constant* $C > 0$ *such that for any* $f \in \Lambda_w^p$

$$\|M_\alpha f\|_{\Lambda_w^{p,\alpha}} \leqslant C\|f\|_{\Lambda_w^p}.$$

4. Main result

The main result of this paper is established in this section. We also apply our main result to study the mapping properties of $T_{\Omega,\alpha}$ on the Lorentz-Karamata-Morrey spaces.

We start with the definition of the generalized Lorentz-Morrey spaces.

For any $r > 0$ and $x \in \mathbb{R}^n$, define $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. Write $\mathbb{B} = \{B(x,r) : x \in \mathbb{R}^n, r > 0\}$.

DEFINITION 4.1. Let $0 < p < \infty$, $w : [0,\infty) \to [0,\infty)$ and $u : \mathbb{R}^n \times [0,\infty) \to (0,\infty)$. Suppose that $W(t) = \int_0^t w(s)ds$ is finite a.e. and satisfies the $\triangle_2$ condition. The generalized Lorentz-Morrey space $\mathfrak{M}_{w,u}^p$ consists of all Lebesgue measurable function f satisfying

$$\|f\|_{\mathfrak{M}_{w,u}^p} = \sup_{B(x,r)\in\mathbb{B}} \frac{1}{u(x,r)} \|\chi_B f\|_{\Lambda_w^p} < \infty.$$

Let $0 < \alpha < n$. Suppose that $W_\alpha(t) = \int_0^t s^{-\frac{\alpha p}{n}} w(s)ds$ is finite a.e. and satisfies the $\triangle_2$ condition. The generalized Lorentz-Morrey space $\mathfrak{M}_{w,u}^{p,\alpha}$ consists of all Lebesgue measurable function f satisfying

$$\|f\|_{\mathfrak{M}_{w,u}^{p,\alpha}} = \sup_{B(x,r)\in\mathbb{B}} \frac{1}{u(x,r)} \|\chi_B f\|_{\Lambda_w^{p,\alpha}} < \infty.$$

The conditions, $W(t) = \int_0^t w(s)ds$ is finite a.e. and satisfies the $\triangle_2$ condition, guarantee that Λ_w^p is a quasi-Banach space, therefore, $\mathfrak{M}_{w,u}^p$ is also a quasi-Banach space.

Whenever $w \equiv 1$, the generalized Lorentz-Morrey space becomes the classical Morrey space. When $w(t) = t^{\frac{p}{q}-1}$ and $1 < p,q < \infty$, $\mathfrak{M}_{w,u}^p$ reduces to the Lorentz-Morrey space [33].

Moreover, when Λ_w^p is the Lorentz-Karamata space $L_{r,p,b}$, we denote he Morrey space built on $L_{r,p,b}$ by $\mathfrak{M}_{u,b}^{r,p}$. We call this function space as the Lorentz-Karamata-Morrey space.

Notice that (2.3)–(2.4) assure that $\mathfrak{M}_{u,b}^{r,p}$ is a well defined quasi-Banach space.

In addition, if $1 < r < \frac{n}{\alpha}$, we have

$$\int_0^t s^{\frac{p}{r}-\frac{\alpha p}{n}-1}(\gamma_b(s))^p ds \approx t^{\frac{p}{r}-\frac{\alpha p}{n}}(\gamma_b(t))^p$$

and

$$\int_0^{2t} s^{\frac{p}{r}-\frac{\alpha p}{n}-1}(\gamma_b(s))^p ds \leqslant C(2t)^{\frac{p}{r}-\frac{\alpha p}{n}}(\gamma_b(2t))^p \leqslant C\int_0^t s^{\frac{p}{r}-\frac{\alpha p}{n}-1}(\gamma_b(s))^p ds \leqslant C(2t)^{\frac{p}{r}-\frac{\alpha p}{n}}(\gamma_b(t))^p$$

for some $C > 0$.

Hence, $\mathfrak{M}_{u,b}^{r,p,\alpha}$ is a quasi-Banach function space.

LEMMA 4.1. *Let $0 < p < \infty$, $0 < \alpha < n$ and $\Omega \in L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$ satisfy (2.1). Suppose that $1 < p_{\Lambda_w^p} \leqslant q_{\Lambda_w^p} < \frac{n}{\alpha}$. For any $1 < v < p_{\Lambda_w^p}$, there is a constant $C > 0$ such that for any $g \in \Lambda_w^p$ and ball $B \in \mathbb{B}$, we have*

$$\left(\int_B |g(y)|^v dy\right)^{\frac{1}{v}} \leqslant C\frac{1}{\|\chi_B\|_{\Lambda_w^{p,\alpha}}}\|g\|_{\Lambda_w^p}|B|^{\frac{1}{v}-\frac{\alpha}{n}}. \tag{4.1}$$

Proof. In view of the assumptions imposed on $v, p_{\Lambda_w^p}, q_{\Lambda_w^p}$, we have

$$1 < \frac{p_{\Lambda_w^p}}{v} \leqslant \frac{q_{\Lambda_w^p}}{v} < \frac{n}{\alpha v}.$$

Therefore, (3.4) and (3.5) yields

$$1 < p_{\Lambda_w^{p/v}} \leqslant q_{\Lambda_w^{p/v}} < \frac{n}{\alpha v}.$$

For any $g \in \Lambda_w^p$ and $B \in \mathbb{B}$, we find that

$$\frac{1}{|B|^{1-\frac{\alpha v}{n}}}\left(\int_B |g(y)|^v dy\right)\chi_B(x) \leqslant M_{\alpha v}(|g|^v)(x), \quad \forall x \in B.$$

We are allowed to apply Corollary 3.4 on $M_{\alpha v}$ and $\Lambda_w^{p/v}$. According to (2.2), we have

$$\begin{aligned}\frac{1}{|B|^{1-\frac{\alpha v}{n}}}\left(\int_B |g(y)|^v dy\right)\|\chi_B\|_{\Lambda_w^{p/q,\alpha v}} &\leqslant \|\chi_B M_{\alpha v}(|g|^v)\|_{\Lambda_w^{p/q,\alpha v}} \\ &\leqslant \|M_{\alpha v}(|g|^v)\|_{\Lambda_w^{p/q,\alpha v}} \leqslant C\||g|^v\|_{\Lambda_w^{p/v}} \\ &\leqslant C\|g\|_{\Lambda_w^p}^v\end{aligned}$$

Moreover,

$$\begin{aligned}\|\chi_B\|_{\Lambda_w^{p/q,\alpha v}} &= \left(\int_0^{|B|}(t^{-\frac{\alpha v}{n}})^{p/v}w(t)dt\right)^{v/p}\\ &= \left(\int_0^{|B|}(t^{-\frac{\alpha}{n}})^{p}w(t)dt\right)^{v/p} = \|\chi_B\|^{v}_{\Lambda_w^{p,\alpha}}.\end{aligned}$$

Therefore, (4.1) follow from the above inequalities. □

We are now ready to present our main result, the mapping properties of the fractional integral operators with homogeneous kernels on generalized Lorentz-Morrey spaces.

THEOREM 4.2. *Let* $0<p<\infty$, $0<\alpha<n$, $w:[0,\infty)\to[0,\infty)$ *and* $u:\mathbb{R}^n\times[0,\infty)\to(0,\infty)$. *Suppose that* $1<p_{\Lambda_w^p}\leqslant q_{\Lambda_w^p}<\frac{n}{\alpha}$, $W(t)$ *is finite a.e. and satisfies the* $\triangle_2$ *condition.*

If there is a $C>0$ *such that for any* $x\in\mathbb{R}^n$ *and* $r>0$

$$\sum_{j=0}^{\infty}\left(\frac{W_\alpha(B(x,r))}{W_\alpha(B(x,2^{j+1}r))}\right)^{1/p}u((x,2^{j+1}r)<Cu(x,r) \tag{4.2}$$

and $\Omega\in L^\theta(\mathbb{S}^{n-1})$ *for some* $\theta'<p_{\Lambda_w^p}$, *then there exists a constant* $C>0$ *such that for any* $f\in\mathfrak{M}^p_{w,u}$

$$\|T_{\Omega,\alpha}f\|_{\mathfrak{M}^{p,\alpha}_{w,u}}\leqslant C\|f\|_{\mathfrak{M}^p_{w,u}}.$$

Proof. Lemma (3.1) guarantees that W_α is finite a.e. and satisfies the $\triangle_2$ condition, therefore, $\mathfrak{M}^{p,\alpha}_{w,u}$ is well defined. For any $B=B(z,r)\in\mathbb{B}$ and $f\in\mathfrak{M}^p_{w,u}$, write $f_0=\chi_{B(z,2r)}f$ and $f_j=\chi_{B(z,2^{j+1}r)\backslash B(z,2^jr)}f$, $j\in\mathbb{N}\backslash\{0\}$.

Since $\theta'<p_{\Lambda_w^p}<\frac{n}{\alpha}$, $\frac{n}{n-\alpha}=(n/\alpha)'<\theta$, we have $\Omega\in L^\theta(\mathbb{S}^{n-1})\subset L^{\frac{n}{n-\alpha}}(\mathbb{S}^{n-1})$. Theorem 3.3 yields

$$\|\chi_BT_{\Omega,\alpha}f_0\|_{\Lambda_w^{p,\alpha}}\leqslant C\|T_{\Omega,\alpha}f_0\|_{\Lambda_w^{p,\alpha}}\leqslant C\|f_0\|_{\Lambda_w^p}=C\|\chi_{B(z,2r)}f\|_{\Lambda_w^p} \tag{4.3}$$

for some $C>0$.

As W_α satisfies the $\triangle_2$ condition, (4.2) assures that for any $z\in\mathbb{R}^n$ and $r>0$

$$u(z,2r)\leqslant Cu(z,r) \tag{4.4}$$

for some $C>0$.

Consequently, (4.3) and (4.4) yield

$$\frac{1}{u(z,r)}\|\chi_BT_{\Omega,\alpha}f_0\|_{\Lambda_w^{p,\alpha}}\leqslant C\frac{1}{u(z,2r)}\|\chi_{B(z,2r)}f\|_{\Lambda_w^p}\leqslant C\|f\|_{\mathfrak{M}^p_{w,u}} \tag{4.5}$$

for some $c>0$.

Next, there is a constant $C>0$ such that for any $j\geqslant 1$, we have

$$\chi_B(x)|T_{\Omega,\alpha}f_j(x)|\leqslant C\chi_B(x)\int_{B(z,2^{j+1}r)\backslash B(z,2^jr)}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy$$

By using the Hölder inequality, we find that

$$\begin{aligned}&\int_{B(z,2^{j+1}r)\backslash B(z,2^jr)}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy\\&\leqslant C\left(\int_{B(z,2^{j+1}r)\backslash B(z,2^jr)}\frac{|\Omega(x-y)|^\theta}{|x-y|^{\theta(n-\alpha)}}dy\right)^{1/\theta}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{\theta'}}\\&=C\left(\int_{B(x-z,2^{j+1}r)\backslash B(x-z,2^jr)}\frac{|\Omega(y)|^\theta}{|y|^{\theta(n-\alpha)}}dy\right)^{1/\theta}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{\theta'}}\end{aligned}$$

for some $C>0$.

Since $x\in B(z,r)$ and $y\in B(x-z,2^{j+1}r)\backslash B(x-z,2^jr)$, we obtain

$$\begin{aligned}|y|&\leqslant|y-(x-z)|+|x-z|\leqslant 2^{j+1}r+r\leqslant 2^{j+2}r\\|y|&\geqslant|y-(x-z)|-|x-z|\geqslant 2^jr-r\geqslant 2^{j-1}r.\end{aligned}$$

We find that $B(x-z,2^{j+1}r)\backslash B(x-z,2^jr)\subset B(0,2^{j+2}r)\backslash B(0,2^{j-1})$.

The above result and the assumption $\Omega\in L^\theta(\mathbb{S}^{n-1})$ indicate that

$$\begin{aligned}&\int_{B(z,2^{j+1}r)\backslash B(z,2^jr)}\frac{|\Omega(x-y)|}{|x-y|^{n-\alpha}}|f(y)|dy\\&=C\left(\int_{B(0,2^{j+2}r)\backslash B(0,2^{j-1}r)}\frac{|\Omega(y)|^\theta}{|y|^{\theta(n-\alpha)}}dy\right)^{1/\theta}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{\theta'}}\\&=C\left(\int_{2^{j-1}r}^{2^{j+2}r}\int_{\mathbb{S}^{n-1}}|\Omega(s)|^\theta t^{-\theta(n-\alpha)+n-1}dsdt\right)^{1/\theta}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{\theta'}}\\&\leqslant C2^{-(n-\alpha)(j-1)+n(j-1)/\theta}r^{-(n-\alpha)+n/\theta}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{\theta'}}\\&\leqslant C\frac{1}{|B(z,2^{j+1}r)|^{\frac{1}{\theta'}-\frac{\alpha}{n}}}\|\chi_{B(z,2^{j+1}r)}f\|_{L^{\theta'}}\end{aligned}$$

for some $C>0$.

Furthermore, since $\theta'<p_{\Lambda_w^p}$, by applying Lemma (4.1) with $v=\theta'$ and $g=\chi_{B(z,2^{j+1}r)}f$, we get

$$\chi_B(x)|T_{\Omega,\alpha}f_j(x)|\leqslant C\chi_B(x)\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{\Lambda_w^p}}{\|\chi_{B(z,2^{j+1}r)}\|_{\Lambda_w^{p,\alpha}}}.$$

Therefore,

$$\chi_B(x)\sum_{j=1}^\infty|T_{\Omega,\alpha}f_j(x)|\leqslant C\chi_B(x)\sum_{j=1}^\infty\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{\Lambda_w^p}}{\|\chi_{B(z,2^{j+1}r)}\|_{\Lambda_w^{p,\alpha}}}$$

Applying the quasi-norm $\|\cdot\|_{\Lambda_w^{p,\alpha}}$ and, then, multiplying $\frac{1}{u(z,r)}$ on both sides of the above inequality, we have

$$\begin{aligned}
&\frac{1}{u(z,r)}\left\|\chi_B\sum_{j=1}^{\infty}|T_{\Omega,\alpha}f|\right\|_{\Lambda_w^{p,\alpha}}\\
&\leqslant C\frac{1}{u(z,r)}\|\chi_B\|_{\Lambda_w^{p,\alpha}}\sum_{j=1}^{\infty}\frac{\|\chi_{B(z,2^{j+1}r)}f\|_{\Lambda_w^p}}{\|\chi_{B(z,2^{j+1}r)}\|_{\Lambda_w^{p,\alpha}}}\\
&\leqslant C\sum_{j=1}^{\infty}\frac{u(z,2^{j+1}r)}{u(z,r)}\frac{\|\chi_B\|_{\Lambda_w^{p,\alpha}}}{\|\chi_{B(z,2^{j+1}r)}\|_{\Lambda_w^{p,\alpha}}}\frac{1}{u(z,2^{j+1}r)}\|\chi_{B(z,2^{j+1}r)}f\|_{\Lambda_w^p}\\
&\leqslant C\sum_{j=1}^{\infty}\frac{u(z,2^{j+1}r)}{u(z,r)}\frac{\|\chi_B\|_{\Lambda_w^{p,\alpha}}}{\|\chi_{B(z,2^{j+1}r)}\|_{\Lambda_w^{p,\alpha}}}\|f\|_{\mathfrak{M}_{w,u}^p}.
\end{aligned}$$

Notice that for any $B\in\mathbb{B}$, $\|\chi_B\|_{\Lambda_w^{p,\alpha}}^p=W_\alpha(|B|)$. Therefore, (4.2) ensures that

$$\frac{1}{u(z,r)}\left\|\chi_{B(z,r)}\sum_{j=1}^{\infty}|T_{\Omega,\alpha}f|\right\|_{\Lambda_w^{p,\alpha}}\leqslant C\|f\|_{\mathfrak{M}_{w,u}^p}\tag{4.6}$$

for some $C>0$.

Therefore, (4.5) and (4.6) yield

$$\frac{1}{u(z,r)}\|\chi_{B(z,r)}T_{\Omega,\alpha}f\|_{\Lambda_w^{p,\alpha}}\leqslant C\|f\|_{\mathfrak{M}_{w,u}^p}.$$

By taking supremum over $B\in\mathbb{B}$ on both sides of the above inequality, we obtain our desired result. □

We give an application of Theorem 4.2 on Lorentz-Karamata-Morrey spaces.

COROLLARY 4.3. *Let $0<p<\infty$, $0<\alpha<n$ and $u:\mathbb{R}^n\times[0,\infty)\to(0,\infty)$. Suppose that $1<r<\frac{n}{\alpha}$ and*

$$\frac{1}{r}=\frac{1}{q}+\frac{\alpha}{n}.\tag{4.7}$$

If there is a $C>0$ such that for any $x\in\mathbb{R}^n$ and $r>0$

$$\sum_{j=0}^{\infty}\frac{\|\chi_{B(x,r)}\|_{L_{q,p,b}}}{\|\chi_{B(x,2^{j+1}r)}\|_{L_{q,p,b}}}u((x,2^{j+1}r)<Cu(x,r)\tag{4.8}$$

and $\Omega\in L^\theta(\mathbb{S}^{n-1})$ for some $\theta'<r$, then there exists a constant $C>0$ such that for any $f\in\mathfrak{M}_{w,u}^p$

$$\|T_{\Omega,\alpha}f\|_{\mathfrak{M}_{u,b}^{q,p}}\leqslant C\|f\|_{\mathfrak{M}_{u,b}^{r,p}}.$$

Notice that (4.7) shows that $\mathfrak{M}_{u,b}^{r,p,\alpha}$ is equal to $\mathfrak{M}_{u,b}^{q,p}$. Moreover, $\mathfrak{M}_{u,b}^{r,p}$ and $\mathfrak{M}_{u,b}^{r,p,\alpha}$ are quasi-Banach function spaces, therefore their corresponding fundamental functions are finite a.e. and satisfy the $\triangle_2$ condition. Therefore, we are allowed to apply Theorem 4.2 to obtain the above corollary.

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