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HERMITE–HADAMARD INEQUALITIES FOR CO–ORDINATED $\log - h$–CONVEX FUNCTIONS

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Abstract. In this paper, we establish some Hermite-Hadamard type inequalities for co-ordinated $\log - h$-convex functions on rectangles from $\mathbb{R}^n$, which extend some known results. Some mappings connected with these inequalities and related results are also obtained.

1. Introduction

The concept of h-convexity was first introduced by Varošanec [19] in 2007, and then has been studied extensively by many mathematicians, see e.g. [4, 10, 13, 16] and the references therein.

DEFINITION 1.1. Let $h : [0,1] \to [0,\infty)$ be a given function. We say that $f : \mathscr{D} \to \mathbb{R}$, where $\mathscr{D}$ is a convex subset of $\mathbb{R}^n$, is h-convex if for any $\mathbf{x}, \mathbf{y} \in \mathscr{D}$ and $\alpha \in [0,1]$,

$$f(\alpha\mathbf{x} + (\mathbf{1} - \alpha\mathbf{y})) \leqslant h(\alpha)f(\mathbf{x}) + h(1-\alpha)f(\mathbf{y}). \tag{1.1}$$

This notion unifies and generalizes the known classes of the usual convex functions, s-convex functions (in the second sense) [5], P-functions [17] and Godunova-Levin functions (or Q-functions) [9], which are obtained by putting in (1.1)

$$h(\alpha) = \alpha,\ h(\alpha) = \alpha^s\ (0 < s \leqslant 1),\ h(\alpha) = 1, \tag{1.2}$$

and

$$h(\alpha) = \begin{cases} 1/\alpha, & 0 < \alpha \leqslant 1, \\ 0, & \alpha = 0, \end{cases} \tag{1.3}$$

respectively.

Convexity and its generalizations are very important both in pure mathematics and in applications. One of the significant application involved in convex type functions is the following well-known Hermite-Hadamard inequality.

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THEOREM A. *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a convex mapping. Then*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\frac{f(a)+f(b)}{2}.$$

In 1995, Dragomier, Pečarić and Persson [8] established similar results for P-functions and Q-functions. About four years later, Dragomir and Fitzpatrick [7] extended an analogous inequality for s-convex functions (in the second sense). In 2008, Sarikaya, Saglam and Yildririm obtained the following inequality for h-convex functions.

THEOREM B. [18] *Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be an h-convex function on $[a,b]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant[f(a)+f(b)]\int_0^1 h(t)dt.$$

It is notable that Theorem B reduces to the results in [8] and [7] by taking $h(\alpha)=1$, $h(\alpha)=1/\alpha$ and $h(\alpha)=\alpha^s$, respectively.

In 2013, Noor, Qi and Awan [15] introduced the concept of the $\log-h$-convex function, that is

DEFINITION 1.2. Let $h:[0,1]\to[0,\infty)$ be a given function. We say that $f:\mathscr{D}\to(0,+\infty)$, where $\mathscr{D}$ is a convex subset of $\mathbb{R}^n$, is $\log-h$-convex if for any $\mathbf{x},\mathbf{y}\in\mathscr{D}$ and $\alpha\in[0,1]$,

$$f(\alpha\mathbf{x}+(\mathbf{1}-\alpha)\mathbf{y})\leqslant[f(\mathbf{x})]^{h(\alpha)}[f(\mathbf{y})]^{h(1-\alpha)}. \tag{1.4}$$

Particularly, if letting h be as in (1.2) and (1.3), then the $\log-h$-convex function reduces to the $\log$-convex function, the $\log-s$-convex function (in the second sense), the $\log-P$-convex function and the $\log-Q$-convex function, respectively. Readers interested in learning more about these functions are referred to the papers [14, 21, 22].

As an application, the authors [15] proved the following result.

THEOREM C. *Let $f:[a,b]\subset\mathbb{R}\to(0,+\infty)$ be a $\log-h$-convex function. Then*

$$f\left(\frac{a+b}{2}\right)^{1/(2h(1/2))}\leqslant\exp\left[\frac{1}{b-a}\int_a^b\ln f(x)dx\right]\leqslant[f(a)f(b)]^{\int_0^1 h(t)dt}.$$

On the other hand, there is an extensive literature devoted to develop Hermite-Hadamard's type inequalities to higher-dimensions. In 2001, Dragomir [6] extended Theorem A to co-ordinated convex functions on a rectangle from the plane $\mathbb{R}^2$.

THEOREM D. [6] *If $f:\Delta=[a_1,b_1]\times[a_2,b_2]\subset\mathbb{R}^2\to\mathbb{R}$ is a co-ordinated convex on Δ, then*

$$\begin{aligned}f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)&\leqslant\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_2}^{b_2}\int_{a_1}^{b_1}f(x_1,x_2)\,dx_1dx_2\\&\leqslant\frac{f(a_1,a_2)+f(a_1,b_2)+f(b_1,a_2)+f(b_1,b_2)}{4}.\end{aligned}$$

The above inequalities are sharp. In 2008, Alomari and Darus proved similar inequalities for co-ordinated s-convex functions (in the first sense [2], in the second sense [1]) on a rectangle from the plane $\mathbb{R}^2$. In 2009, Latif and Alomari [12] introduced the concept of co-ordinated h-convex functions on rectangles from the plane $\mathbb{R}^2$.

DEFINITION 1.3. Let $h:[0,1]\subset\mathbb{R}\to\mathbb{R}$ be a non-negative function. A function $f:\Delta=[a_1,b_1]\times[a_2,b_2]\subset\mathbb{R}^2\to\mathbb{R}$ is said to be a co-ordinated h-convex function on Δ, if the partial mappings $f_1:[a_1,b_1]\to\mathbb{R}, f_1(u)=f(u,x_2)$ and $f_2:[a_2,b_2]\to\mathbb{R}, f_2(u)=f(x_1,u)$ are h-convex for all $x_j\in[a_j,b_j],\ j=1,2$.

The authors also proved the Hermite-Hadamard inequality for co-ordinated h-convex functions as follows.

THEOREM E. [12] *If $f:\Delta=[a_1,b_1]\times[a_2,b_2]\subset\mathbb{R}^2\to\mathbb{R}$ is a co-ordinated h-convex on Δ, then*

$$\begin{aligned}
&\frac{1}{4h^2(1/2)}f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)\\
&\leqslant \frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x_1,x_2)\,dx_1dx_2\\
&\leqslant [f(a_1,a_2)+f(a_1,b_2)+f(b_1,a_2)+f(b_1,b_2)]\left(\int_0^1 h(t)dt\right)^2.
\end{aligned}$$

In particular, if setting $h(\alpha)=\alpha^s$, Theorem E reduces to the results in [1].

In 2001, Dragomir [6] also studied some properties of mappings connected to the Hermite-Hadamard type inequalities of co-ordinated convex functions on rectangles from the plane $\mathbb{R}^2$.

THEOREM F. [6] *Define the mapping $H:[0,1]^2\subset\mathbb{R}^2\to\mathbb{R}$ by*

$$\begin{aligned}
H(t_1,t_2) = &\frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_2}^{b_2}\\
&\times\int_{a_1}^{b_1} f\left(t_1x_1+(1-t_1)\frac{a_1+b_1}{2},t_2x_2+(1-t_2)\frac{a_2+b_2}{2}\right)dx_1dx_2.
\end{aligned}$$

If $f:\Delta=[a_1,b_1]\times[a_2,b_2]\subseteq\mathbb{R}^2\to\mathbb{R}$ is co-ordinated convex on Δ, then:

(i) *The mapping H is co-ordinated convex on $[0,1]^2$.*

(ii)

$$\sup_{(t_1,t_2)\in[0,1]^2} H(t_1,t_2)=H(1,1),\qquad \inf_{(t_1,t_2)\in[0,1]^2} H(t_1,t_2)=f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)=H(0,0).$$

With these motivations, the main purpose in this paper is to establish analogues of Hermite-Hadamard inequalities for co-ordinated $\log -h$-convex functions on n-dimensional rectangles and study some related mappings.

2. Main results

In the sequel, unless otherwise specified, $\mathbb{R}^n$ denotes the Euclidean space of dimension n and $\mathbb{R}^1=\mathbb{R}$. $[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n$ denotes the usual Cartesian product by $[\mathbf{a},\mathbf{b}]=[a_1,b_1]\times[a_2,b_2]\times\cdots\times[a_n,b_n]$ and the Lebesgue measure of it by $|[\mathbf{a},\mathbf{b}]|=\prod_{i=1}^n(b_i-a_i)$. Denote $L(E)$ by the set of Lebesgue integrable functions on the measurable set $E\subset\mathbb{R}^n$. Define the product of vectors by

$$\mathbf{t}\circ\mathbf{x}=(t_1x_1,t_2x_2,\dots,t_nx_n),$$

and the linear combination of vectors by

$$a\mathbf{t}+b\mathbf{x}=(at_1+bx_1,at_2+bx_2,\dots,at_n+bx_n),$$

where $\mathbf{t}=(t_1,t_2,\dots,t_n)$, $\mathbf{x}=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$, $a,b\in\mathbb{R}$.

Similar to Definition 1.2, one can give the notion of co-ordinated $\log-h$-convex functions on rectangles from $\mathbb{R}^n$ $(n\geqslant 2)$.

DEFINITION 2.1. Let $h:[0,1]\subset\mathbb{R}\to\mathbb{R}$ be a non-negative function. A function $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ is said to be a co-ordinated $\log-h$-convex function on $[\mathbf{a},\mathbf{b}]$, if for every $i\in\{1,2,\dots,n\}$ the partial mapping $f_i:[a_i,b_i]\to(0,+\infty), f_i(u)=f(x_1,\dots,x_{i-1},u,x_{i+1},\dots,x_n)$ is $\log-h$-convex for all $x_j\in[a_j,b_j]$, $j\neq i$.

In particular, if f satisfies the conditions in the proceeding definition with h defined by (1.2) and (1.3), then f is said to be *the co-ordinated* $\log$*-convex function, the co-ordinated* $\log-s$*-convex function (in the second sense), the co-ordinated* $\log-P$*-function* and the *co-ordinated* $\log-Q$ *function*, respectively.

It is not difficult to check that every $\log-h$-convex function $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ is co-ordinate $\log-h$-convex on $[\mathbf{a},\mathbf{b}]$, but the converse is not generally true (see the details in Appendix).

Throughout the paper, we assume that the function h in the above definitions is always Lebesgue integrable on the interval $[0,1]$ and satisfies $h(1/2)>0$.

Now we are in a position to state our results.

THEOREM 2.1. *Let $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ and $\ln f\in L([\mathbf{a},\mathbf{b}])$. If the partial mapping f_i is a $\log-h_i$-convex function on $[a_i,b_i]$ for $i=1,2,\dots,n$ respectively, then*

$$\begin{aligned}\left[f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)\right]^{1/\left(2^n\prod_{i=1}^n h_i(1/2)\right)} &\leqslant \exp\left[\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}\ln f(\mathbf{x})d\mathbf{x}\right]\\ &\leqslant\left[\prod_{c_i=a_i \text{ or } b_i} f(c_1,c_2,\dots,c_n)\right]^{\prod_{i=1}^n\int_0^1 h_i(t)dt}.\end{aligned}$$

Proof. It follows from Theorem B that

$$\begin{aligned}&\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}\ln f(\mathbf{x})d\mathbf{x} \qquad (2.1)\\ &=\frac{1}{\prod_{j=1}^n(b_j-a_j)}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{a_1}^{b_1}\ln f(x_1,x_2,\dots,x_n)dx_1dx_2\dots dx_n\end{aligned}$$

$$\geqslant \frac{1}{2h_1(1/2)\prod_{j=2}^{n}(b_j-a_j)}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\ln f\left(\frac{a_1+b_1}{2},x_2,\ldots,x_n\right)dx_2\ldots dx_n$$
$$\geqslant \cdots \geqslant \frac{1}{2^n\prod_{i=1}^{n}h_i(1/2)}\ln f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},\ldots,\frac{a_n+b_n}{2}\right).$$

On the other hand, the right part of Theorem B and Fubini's theorem imply that

$$\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}\ln f(\mathbf{x})d\mathbf{x}$$
$$\leqslant \frac{\int_0^1 h_1(t)dt}{\prod_{j=2}^{n}(b_j-a_j)}\int_{a_n}^{b_n}\cdots\int_{a_3}^{b_3}\int_{a_2}^{b_2}\left[\ln f(a_1,x_2,\ldots,x_n)+\ln f(b_1,x_2,\ldots,x_n)\right]dx_2\ldots dx_n$$
$$\leqslant \frac{\int_0^1 h_1(t)dt\int_0^1 h_2(t)dt}{\prod_{j=3}^{n}(b_j-a_j)}\int_{a_n}^{b_n}\cdots\int_{a_4}^{b_4}\int_{a_3}^{b_3}\left[\ln f(a_1,a_2,\ldots,x_n)+\ln f(a_1,b_2,\ldots,x_n)\right.$$
$$\left.+\ln f(b_1,a_2,\ldots,x_n)+\ln f(b_1,b_2,\ldots,x_n)\right]dx_3dx_4\ldots dx_n.$$

Then, by induction, we have

$$\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}\ln f(\mathbf{x})d\mathbf{x}\leqslant\left(\prod_{i=1}^{n}\int_0^1 h_i(t)dt\right)\left(\sum_{c_i=a_i\text{ or }b_i}\ln f(c_1,c_2,\ldots,c_n)\right)$$
$$=\left(\prod_{i=1}^{n}\int_0^1 h_i(t)dt\right)\ln\left(\prod_{c_i=a_i\text{ or }b_i}f(c_1,c_2,\ldots,c_n)\right). \quad (2.2)$$

Therefore we finish the proof of Theorem 2.1 by (2.1) and (2.2). □

If taking $h_1=h_2=\cdots=h_n$ in Theorem 2.1, we have the following result.

COROLLARY 2.2. *Let $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ be a co-ordinated $\log -h$-convex function and $\ln f\in L([\mathbf{a},\mathbf{b}])$. Then*

$$\left[f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)\right]^{1/(2h(1/2))^n}\leqslant\exp\left[\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}\ln f(\mathbf{x})d\mathbf{x}\right]$$
$$\leqslant\left[\prod_{c_i=a_i\text{ or }b_i}f(c_1,c_2,\ldots,c_n)\right]^{\left(\int_0^1 h(t)dt\right)^n}.$$

Especially, if $h(t)=t$ and $n=2$, then Corollary 2.2 reduces to Corollary 3.1 in [3].

Next we introduce a key lemma as follows.

LEMMA 2.3. *Let $f:[a,b]\subset\mathbb{R}\to(0,+\infty)$ be a $\log -h$-convex function on the interval $[a,b]$ and $\ln f\in L([a,b])$. Define the mapping $\mathfrak{L}:[0,1]\subset\mathbb{R}\to\mathbb{R}$ by*

$$\mathfrak{L}(t)=\exp\left[\frac{1}{b-a}\int_a^b\ln f\left(tx+(1-t)\frac{a+b}{2}\right)dx\right].$$

Then:

(i) $\mathfrak{L}$ *is a* $\log -h$*-convex function on* $[0,1]$.

(ii) *For* $t \in [0,1]$, *we have*

$$\left[f\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h(1/2)}} \leqslant \mathfrak{L}(t) \leqslant \mathfrak{L}(1)^{h(t)+2h(1/2)h(1-t)}.$$

Especially, if f *is a* $\log$*-convex function on* $[a,b]$, *we have*

$$\sup_{t\in[0,1]} \mathfrak{L}(t) = \mathfrak{L}(1), \quad \inf_{t\in[0,1]} \mathfrak{L}(t) = \mathfrak{L}(0).$$

Proof. (i) Let t_1, t_2, $\alpha,\beta \in [0,1]$ and $\alpha+\beta = 1$. For any $x \in [a,b]$, by the definition of $\log -h$-convexity, we have

$$\begin{aligned}
&\ln f\left((\alpha t_1+\beta t_2)x+[1-(\alpha t_1+\beta t_2)]\,\frac{a+b}{2}\right)\\
&= \ln f\left[\alpha\left(t_1x+(1-t_1)\frac{a+b}{2}\right)+\beta\left(t_2x+(1-t_2)\,\frac{a+b}{2}\right)\right]\\
&\leqslant h(\alpha)\ln f\left(t_1x+(1-t_1)\,\frac{a+b}{2}\right)+h(\beta)\ln f\left(t_2x+(1-t_2)\,\frac{a+b}{2}\right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\ln \mathfrak{L}(\alpha t_1+\beta t_2)\\
&\leqslant \frac{h(\alpha)}{b-a}\int_a^b \ln f\left(t_1x+(1-t_1)\,\frac{a+b}{2}\right)dx+\frac{h(\beta)}{b-a}\int_a^b \ln f\left(t_2x+(1-t_2)\,\frac{a+b}{2}\right)dx\\
&= h(\alpha)\ln \mathfrak{L}(t_1)+h(\beta)\ln \mathfrak{L}(t_2).
\end{aligned}$$

That is

$$\mathfrak{L}(\alpha t_1+\beta t_2) \leqslant \mathfrak{L}(t_1)^{h(\alpha)}\,\mathfrak{L}(t_2)^{h(\beta)},$$

which completes the proof of (i).

(ii) A changing of variable shows that

$$\mathfrak{L}(t) = \exp\left[\frac{1}{t(b-a)}\int_{\frac{(1+t)a+(1-t)b}{2}}^{\frac{(1-t)a+(1+t)b}{2}} \ln f(\xi)d\xi\right].$$

Since

$$\begin{aligned}
\frac{(1+t)a+(1-t)b}{2}-\frac{(1-t)a+(1+t)b}{2} &= t(b-a),\\
\frac{(1+t)a+(1-t)b}{2}+\frac{(1-t)a+(1+t)b}{2} &= a+b,
\end{aligned}$$

by Theorem B,

$$\mathfrak{L}(t) \geqslant \left[f\left(\frac{a+b}{2}\right)\right]^{\frac{1}{2h(1/2)}} \tag{2.3}$$

holds for all $t \in [0,1]$.

On the other hand,

$$\begin{aligned}
\mathfrak{L}(t) &\leqslant \exp\left[\frac{h(t)}{b-a}\int_a^b \ln f(x)dx + h(1-t)\ln f\left(\frac{a+b}{2}\right)\right] \\
&= \exp\left[h(t)\ln \mathfrak{L}(1) + h(1-t)\ln f\left(\frac{a+b}{2}\right)\right] \\
&= (\mathfrak{L}(1))^{h(t)} \cdot \left(f\left(\frac{a+b}{2}\right)\right)^{h(1-t)} \\
&\leqslant \mathfrak{L}(1)^{h(t)} \cdot \mathfrak{L}(1)^{2h(1/2)h(1-t)} \\
&= \mathfrak{L}(1)^{h(t)+2h(1/2)h(1-t)},
\end{aligned}$$

the last inequality is obtained by (2.3). Thus we finish the proof of Lemma 2.3. □

THEOREM 2.4. *Let* $f : [\mathbf{a},\mathbf{b}] \subset \mathbb{R}^n \to (0,+\infty)$ *and* $\ln f \in L([\mathbf{a},\mathbf{b}])$. *Define the mapping* $\mathfrak{L} : [\mathbf{0},\mathbf{1}] \subset \mathbb{R}^n \to \mathbb{R}$ *by*

$$\mathfrak{L}(\mathbf{t}) = \exp\left[\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]} \ln f\left(\mathbf{t}\circ\mathbf{x} + (\mathbf{1}-\mathbf{t})\circ\frac{\mathbf{a}+\mathbf{b}}{2}\right)d\mathbf{x}\right]. \tag{2.4}$$

If the partial mapping f_i *is* $\log -h_i$*-convex on* $[a_i,b_i]$ *for* $i = 1,2,\dots,n$ *respectively, then:*

(i) *The partial mapping* $\mathfrak{L}_i(u) = \mathfrak{L}(t_1,\dots,t_{i-1},u,t_{i+1},\dots,t_n)$ *is a* $\log -h_i$*-convex function on* $[0,1]$ *for every* $i \in \{1,2,\dots,n\}$.

(ii) *For all* $\mathbf{t} \in [\mathbf{0},\mathbf{1}] \subset \mathbb{R}^n$, *we have*

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^{1/\left(2^n\prod_{i=1}^n h_i(1/2)\right)} \leqslant \mathfrak{L}(\mathbf{t}) \leqslant [\mathfrak{L}(\mathbf{1})]^{\prod_{i=1}^n[h_i(t_i)+2h_i(1/2)h_i(1-t_i)]}.$$

Proof. (i) Without loss of generality, we just prove that $\mathfrak{L}_1(\cdot)$ is a $\log -h$-convex function on $[0,1] \subset \mathbb{R}$, the others follow the same procedure. For any $\xi,\eta,\alpha,\beta \in [0,1] \subset \mathbb{R}$ and $\alpha+\beta = 1$, Fubini's theorem and the similar argument as in the proof of Lemma 2.3 (i) tell us that

$$\begin{aligned}
&\ln \mathfrak{L}_1(\alpha\xi+\beta\eta) \\
&= \frac{1}{\prod_{j=1}^n(b_j-a_j)}\int_{a_n}^{b_n}\dots\int_{a_1}^{b_1}\ln f\left((\alpha\xi+\beta\eta)x_1+\left[1-(\alpha\xi+\beta\eta)\right]\frac{a_1+b_1}{2},\dots,\right. \\
&\quad \left. t_nx_n+(1-t_n)\frac{a_n+b_n}{2}\right)dx_1\dots dx_n \\
&\leqslant \frac{h_1(\alpha)}{\prod_{j=1}^n(b_j-a_j)}\int_{a_n}^{b_n}\dots\int_{a_1}^{b_1}\ln f\left(\xi x_1+(1-\xi)\frac{a+b}{2},\dots,\right. \\
&\quad \left. t_nx_n+(1-t_n)\frac{a_n+b_n}{2}\right)dx_1\dots dx_n \\
&\quad + \frac{h_1(\beta)}{\prod_{j=1}^n(b_j-a_j)}\int_{a_n}^{b_n}\dots\int_{a_1}^{b_1}\ln f\left(\eta x_1+(1-\eta)\frac{a+b}{2},\dots,t_nx_n+(1-t_n)\frac{a_n+b_n}{2}\right)dx_1\dots dx_n
\end{aligned}$$

$$= h_1(\alpha)\ln\mathfrak{L}(\xi,t_2,\dots,t_n)+h_1(\beta)\ln\mathfrak{L}(\eta,t_2,\dots,t_n)$$
$$= h_1(\alpha)\ln\mathfrak{L}_1(\xi)+h_1(\beta)\ln\mathfrak{L}_1(\eta),$$

which yields that

$$\mathfrak{L}_1(\alpha\xi+\beta\eta)\leqslant\mathfrak{L}_1(\xi)^{h_1(\alpha)}\mathfrak{L}_1(\eta)^{h_1(\beta)}.$$

This proves Theorem 2.4 (i).

(ii) Changes of variables tell us that

$$\mathfrak{L}(\mathbf{t})=\exp\left[\frac{1}{\prod_{i=1}^n t_i(b_i-a_i)}\int_{\frac{(1-t_n)a_n+(1+t_n)b_n}{2}}^{\frac{(1+t_n)a_n+(1-t_n)b_n}{2}}\cdots\int_{\frac{(1-t_1)a_1+(1+t_1)b_1}{2}}^{\frac{(1+t_1)a_1+(1-t_1)b_1}{2}}\ln f(\xi_1,\dots,\xi_n)d\xi_1\dots d\xi_n\right].$$

Since, for $i=1,2,\dots,n$,

$$\frac{(1-t_i)a_i+(1+t_i)b_i}{2}-\frac{(1+t_i)a_i+(1-t_i)b_i}{2}=t_i(b_i-a_i),$$
$$\frac{(1-t_i)a_i+(1+t_i)b_i}{2}+\frac{(1+t_i)a_i+(1-t_i)b_i}{2}=a_i+b_i,$$

by induction, Fubini's theorem and Lemma 2.3 show that

$$\ln\mathfrak{L}(\mathbf{t})\geqslant\frac{1}{2^n\prod_{i=1}^n h_i(1/2)}\ln f\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},\dots,\frac{a_n+b_n}{2}\right)$$
$$=\frac{1}{2^n\prod_{i=1}^n h_i(1/2)}\ln f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right).$$

Therefore

$$\mathfrak{L}(\mathbf{t})\geqslant\left[f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)\right]^{1/\left(2^n\prod_{i=1}^n h_i(1/2)\right)}.$$

This completes the proof of the left part of Theorem 2.4 (ii).

On the other hand, using Lemma 2.3 and Fubini's theorem again, we derive that

$$\ln\mathfrak{L}(\mathbf{t})=\ln\mathfrak{L}(t_1,t_2,\dots,t_n)$$
$$=\frac{1}{\prod_{j=1}^n(b_j-a_j)}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{a_1}^{b_1}\ln f\left(t_1x_1+(1-t_1)\frac{a_1+b_1}{2},\right.$$
$$\left.t_2x_2+(1-t_2)\frac{a_2+b_2}{2},\dots,t_nx_n+(1-t_n)\frac{a_n+b_n}{2}\right)dx_1dx_2\dots dx_n$$
$$\leqslant\frac{h_1(t_1)+2h_1(1/2)h_1(1-t_1)}{\prod_{j=1}^n(b_j-a_j)}$$
$$\times\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{a_1}^{b_1}\ln f\left(x_1,t_2x_2+(1-t_2)\frac{a_2+b_2}{2},\dots,\right.$$
$$\left.t_nx_n+(1-t_n)\frac{a_n+b_n}{2}\right)dx_1dx_2\dots dx_n$$

$$\leqslant \frac{[h_1(t_1)+2h_1(1/2)h_1(1-t_1)][h_2(t_2)+2h_2(1/2)h_2(1-t_2)]}{\prod_{j=1}^{n}(b_j-a_j)}$$
$$\times \int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{a_1}^{b_1}\ln f\Big(x_1,x_2,t_3x_3+(1-t_3)\frac{a_3+b_3}{2},\ldots,$$
$$t_nx_n+(1-t_n)\frac{a_n+b_n}{2}\Big)dx_1dx_2\ldots dx_n$$
$$\leqslant \cdots$$
$$\leqslant \frac{\prod_{i=1}^{n}\left[h_i(t_i)+2h_i\left(\frac{1}{2}\right)h_i(1-t_i)\right]}{\prod_{j=1}^{n}(b_j-a_j)}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{a_1}^{b_1}\ln f(x_1,x_2,\ldots,x_n)\,dx_1dx_1\ldots dx_n$$
$$=\prod_{i=1}^{n}\left[h_i(t_i)+2h_i\left(\frac{1}{2}\right)h_i(1-t_i)\right]\ln\mathfrak{L}(\mathbf{1}),$$

which means that

$$\mathfrak{L}(\mathbf{t})\leqslant\mathfrak{L}(\mathbf{1})^{\prod_{i=1}^{n}\left[h_i(t_i)+2h_i\left(\frac{1}{2}\right)h_i(1-t_i)\right]}$$

holds for all $\mathbf{t}\in[\mathbf{0},\mathbf{1}]$. Thus the proof is completed. □

As a consequence, if $h_1=h_2=\cdots=h_n$ in Theorem 2.4, we derive that

COROLLARY 2.5. *Let the mapping* $\mathfrak{L}:[\mathbf{0},\mathbf{1}]\subset\mathbb{R}^n\to\mathbb{R}$ *be as in Theorem 2.4. If* $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ *is a co-ordinated* $\log -h$*-convex function and* $\ln f\in L([\mathbf{a},\mathbf{b}])$*, then:*

(i) *The mapping* $\mathfrak{L}$ *is a co-ordinated* $\log -h$*-convex function on* $[\mathbf{0},\mathbf{1}]$.

(ii) *For all* $\mathbf{t}\in[\mathbf{0},\mathbf{1}]$*, we have*

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^{1/[2h(1/2)]^n}\leqslant\mathfrak{L}(\mathbf{t})\leqslant[\mathfrak{L}(\mathbf{1})]^{\prod_{i=1}^{n}[h(t_i)+2h(1/2)h(1-t_i)]}.$$

Especially, if f *is a co-ordinated* $\log$*-convex function on* $[\mathbf{a},\mathbf{b}]$*, then*

$$\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]}\mathfrak{L}(\mathbf{t})=\mathfrak{L}(\mathbf{1}),\quad\inf_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]}\mathfrak{L}(\mathbf{t})=\mathfrak{L}(\mathbf{0}).$$

If $t_1=t_2=\ldots=t_n=t$ in the mapping (2.4), we have

COROLLARY 2.6. *Let* $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ *and* $\ln f\in L([\mathbf{a},\mathbf{b}])$*. Define the mapping* $\widetilde{\mathfrak{L}}:[0,1]\subset\mathbb{R}\to\mathbb{R}$ *by*

$$\widetilde{\mathfrak{L}}(t)=\exp\left[\frac{1}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}\ln f\left(t\mathbf{x}+(1-t)\frac{\mathbf{a}+\mathbf{b}}{2}\right)d\mathbf{x}\right].$$

(i) *If the partial mapping* f_{x_i} *is a* $\log -h_i$*-convex function on* $[a_i,b_i]$ *for* $i=1,2,\ldots,n$ *respectively, then*

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^{1/\left(2^n\prod_{i=1}^{n}h_i(1/2)\right)}\leqslant\widetilde{\mathfrak{L}}(t)\leqslant\left[\widetilde{\mathfrak{L}}(1)\right]^{\prod_{i=1}^{n}[h_i(t)+2h_i(1/2)h_i(1-t)]}$$

holds for all $t\in[0,1]\subset\mathbb{R}$.

In particular, if $h_1 = h_2 = \ldots = h_n = h$ *and* f *is a co-ordinated* $\log -h$*-convex function on* $[\mathbf{a},\mathbf{b}]$*, then for all* $t \in [0,1]$*,*

$$f\left(\frac{\mathbf{a}+\mathbf{b}}{2}\right)^{1/[2h(1/2)]^n} \leqslant \widetilde{\mathfrak{L}}(t) \leqslant \widetilde{\mathfrak{L}}(1)^{[h(t)+2h(1/2)h(1-t)]^n}.$$

Furthermore, if f *is a co-ordinated* $\log$*-convex function on* $[\mathbf{a},\mathbf{b}]$*, then*

$$\sup_{t\in[0,1]} \widetilde{\mathfrak{L}}(t) = \widetilde{\mathfrak{L}}(1), \ \inf_{t\in[0,1]} \widetilde{\mathfrak{L}}(t) = \widetilde{\mathfrak{L}}(0).$$

(ii) *If* $f : [\mathbf{a},\mathbf{b}] \subset \mathbb{R}^n \to (0,+\infty)$ *is a* $\log -h$*-convex function on* $[\mathbf{a},\mathbf{b}]$*, then* $\widetilde{\mathfrak{L}}$ *is* $\log -h$*-convex on* $[0,1] \subset \mathbb{R}$.

Corollary 2.5 and Corollary 2.6 are easily obtained by Theorem 2.4, we leave the details to readers.

LEMMA 2.7. *Let* $f : [a,b] \subset \mathbb{R} \to (0,+\infty)$ *be a log* $-\,h$*-convex function on the interval* $[a,b]$ *and* $\ln f \in L([a,b])$*. Define the mapping* $\mathfrak{K} : [0,1] \subset \mathbb{R} \to \mathbb{R}$ *by*

$$\mathfrak{K}(t) = \exp\left[\frac{1}{(b-a)^2}\int_a^b\int_a^b \ln f\,(tx+(1-t)y)\,dxdy\right].$$

Then:

(i) $\mathfrak{K}$ *is symmetric about* $1/2$*, i.e.*

$$\mathfrak{K}\left(\frac{1}{2}+t\right) = \mathfrak{K}\left(\frac{1}{2}-t\right), \ t \in [0,1/2],$$

and

$$\mathfrak{K}(1-t) = \mathfrak{K}(t), \ t \in [0,1].$$

(ii) $\mathfrak{K}$ *is a* $\log -h$*-convex function on* $[0,1]$.

Proof. (i) For any $t \in [0,1/2]$, changes of variables and Fubini's theorem yield that

$$\begin{aligned}\ln \mathfrak{K}\left(\frac{1}{2}+t\right) &= \frac{1}{(b-a)^2}\int_a^b\int_a^b \ln f\left(\left(\frac{1}{2}+t\right)x+\left(\frac{1}{2}-t\right)y\right)dxdy \\ &= \frac{1}{(b-a)^2}\int_a^b\int_a^b \ln f\left(\left(\frac{1}{2}+t\right)y+\left(\frac{1}{2}-t\right)x\right)dydx \\ &= \ln \mathfrak{K}\left(\frac{1}{2}-t\right),\end{aligned}$$

which means that

$$\mathfrak{K}\left(\frac{1}{2}+t\right) = \mathfrak{K}\left(\frac{1}{2}-t\right)$$

holds for all $t \in [0,1/2]$. And, it follows from the same discussion that, for any $t \in [0,1]$,

$$\mathfrak{K}(1-t) = \mathfrak{K}(t).$$

(ii) Let $t_1,\ t_2,\ \alpha,\beta \in [0,1]$ and $\alpha+\beta=1$. For any $x,y\in[a,b]$, the definition of $\log -h$-convexity shows that

$$\begin{aligned}&\ln f\left((\alpha t_1+\beta t_2)x+[1-(\alpha t_1+\beta t_2)]\,y\right)\\&=\ln f\left[\alpha\left(t_1x+(1-t_1)y\right)+\beta\left(t_2x+(1-t_2)\,y\right)\right]\\&\leqslant h(\alpha)\ln f\left(t_1x+(1-t_1)\,y\right)+h(\beta)\ln f\left(t_2x+(1-t_2)\,y\right).\end{aligned}$$

Therefore,

$$\begin{aligned}&\ln\mathfrak{K}\left(\alpha t_1+\beta t_2\right)\\&\leqslant\frac{h(\alpha)}{(b-a)^2}\int_a^b\int_a^b\ln f\left(t_1x+(1-t_1)\,y\right)dxdy+\frac{h(\beta)}{(b-a)^2}\int_a^b\int_a^b\ln f\left(t_2x+(1-t_2)\,y\right)dxdy\\&=h(\alpha)\ln\mathfrak{K}\left(t_1\right)+h(\beta)\ln\mathfrak{K}\left(t_2\right),\end{aligned}$$

which tells us that

$$\mathfrak{K}\left(\alpha t_1+\beta t_2\right)\leqslant\mathfrak{K}\left(t_1\right)^{h(\alpha)}\mathfrak{K}\left(t_2\right)^{h(\beta)}.$$

This completes the proof of (ii). $\square$

THEOREM 2.8. *Let* $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ *and* $\ln f\in L([\mathbf{a},\mathbf{b}])$. *Define the mapping* $\mathfrak{K}:[\mathbf{0},\mathbf{1}]\subset\mathbb{R}^n\to\mathbb{R}$ *by*

$$\mathfrak{K}(\mathbf{t})=\exp\left[\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]}\ln f\left(\mathbf{t}\circ\mathbf{x}+(\mathbf{1}-\mathbf{t})\circ\mathbf{y}\right)d\mathbf{y}d\mathbf{x}\right]. \tag{2.5}$$

If the partial mapping f_i *is* $\log -h_i$*-convex on* $[a_i,b_i]$ *for* $i=1,2,\ldots,n$ *respectively, then:*

(i) *For every* $i\in\{1,2,\ldots,n\}$, *the partial mapping* $\mathfrak{K}_i(u)=\mathfrak{K}(t_1,\ldots,t_{i-1},u,t_{i+1},\ldots,t_n)$ *is symmetric about* $1/2$, *i.e.*

$$\mathfrak{K}\left(t_1,\ldots,t_{i-1},\frac{1}{2}+t_i,t_{i+1},\ldots,t_n\right)=\mathfrak{K}\left(t_1,\ldots,t_{i-1},\frac{1}{2}-t_i,t_{i+1},\ldots,t_n\right),\ t_i\in[0,1/2],$$
$$\mathfrak{K}\left(t_1,\ldots,t_{i-1},1-t_i,t_{i+1},\ldots,t_n\right)=\mathfrak{K}\left(t_1,\ldots,t_{i-1},t_i,t_{i+1},\ldots,t_n\right),\ t_i\in[0,1].$$

(ii) *The partial mapping* $\mathfrak{K}_i(u)$ *is a* $\log -h_i$*-convex function on* $[0,1]$ *for every* $i\in\{1,2,\ldots,n\}$.

(iii) *For all* $\mathbf{t}\in[\mathbf{0},\mathbf{1}]\subset\mathbb{R}^n$,

$$\begin{aligned}\mathfrak{K}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{1/\left(2^n\prod_{i=1}^n h_i(1/2)\right)}&\leqslant\mathfrak{K}(\mathbf{t})\leqslant\mathfrak{K}(\mathbf{1})^{\sum_{c_i=t_i\text{ or }1-t_i;i=1,2,\ldots,n}h_1(c_1)\cdots h_i(c_i)\cdots h_n(c_n)}\\&=\mathfrak{K}(\mathbf{0})^{\sum_{c_i=t_i\text{ or }1-t_i;i=1,2,\ldots,n}h_1(c_1)\cdots h_i(c_i)\cdots h_n(c_n)}\\&=\mathfrak{L}(\mathbf{1})^{\sum_{c_i=t_i\text{ or }1-t_i;i=1,2,\ldots,n}h_1(c_1)\cdots h_i(c_i)\cdots h_n(c_n)}.\end{aligned}$$

Proof. Since the statements of (i) and (ii) are easy to achieved by a similar argument in Lemma 2.7, we will pay more attention to proving (iii). It follows from

Definition 2.1 that

$$\begin{aligned}
&\frac{1}{h_1(1/2)}\ln f\left(\frac{x_1+y_1}{2},t_2x_2+(1-t_2)y_2,\dots,t_nx_n+(1-t_n)y_n\right)\\
&\leqslant \ln f\left(t_1x_1+(1-t_1)y_1,t_2x_2+(1-t_2)y_2,\dots,t_nx_n+(1-t_n)y_n\right)\\
&\quad+\ln f\left((1-t_1)x_1+t_1y_1,t_2x_2+(1-t_2)y_2,\dots,t_nx_n+(1-t_n)y_n\right),
\end{aligned}$$

which tells us that

$$\begin{aligned}
&\frac{1}{(b_1-a_1)^2}\int_{a_1}^{b_1}\int_{a_1}^{b_1}\ln f\big(t_1x_1+(1-t_1)y_1,t_2x_2+(1-t_2)y_2,\dots,t_nx_n\\
&+(1-t_n)y_n\big)dx_1dy_1\\
\geqslant\ &\frac{1}{2h_1(1/2)(b_1-a_1)^2}\int_{a_1}^{b_1}\int_{a_1}^{b_1}\ln f\bigg(\frac{x_1+y_1}{2},t_2x_2+(1-t_2)y_2,\dots,t_nx_n\\
&+(1-t_n)y_n\bigg)dx_1dy_1.
\end{aligned}$$

By a similar argument, we infer from Fubini's theorem and induction that

$$\begin{aligned}
\ln\mathfrak{K}(\mathbf{t}) &\geqslant \frac{1}{2^n\prod_i^n h_i(1/2)}\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]}\ln f\left(\frac{\mathbf{x}+\mathbf{y}}{2}\right)d\mathbf{x}d\mathbf{y}\\
&=\frac{1}{2^n\prod_i^n h_i(1/2)}\ln\mathfrak{K}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)
\end{aligned}$$

holds for all $\mathbf{t}\in[\mathbf{0},\mathbf{1}]$. Thus we complete the proof of the left part of (iii).

On the other hand, according to Definition 2.1 and Fubini's theorem,

$$\begin{aligned}
&\ln\mathfrak{K}(\mathbf{t})\\
\leqslant\ &\frac{h_1(t_1)}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]}\ln f\left(x_1,t_2x_2+(1-t_2)y_2,\dots,t_nx_n+(1-t_n)y_n\right)d\mathbf{x}d\mathbf{y}\\
&+\frac{h_1(1-t_1)}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]}\ln f\left(y_1,t_2x_2+(1-t_2)y_2,\dots,t_nx_n+(1-t_n)y_n\right)d\mathbf{x}d\mathbf{y}\\
=\ &\frac{h_1(t_1)}{\left[\prod_{i=2}^n(b_i-a_i)\right]^2(b_1-a_1)}\\
&\times\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\int_{[\mathbf{a},\mathbf{b}]}\ln f\left(x_1,t_2x_2+(1-t_2)y_2,\dots,t_nx_n+(1-t_n)y_n\right)d\mathbf{x}dy_2\cdots dy_n\\
&+\frac{h_1(1-t_1)}{\left[\prod_{i=2}^n(b_i-a_i)\right]^2(b_1-a_1)}\\
&\times\int_{[\mathbf{a},\mathbf{b}]}\int_{a_n}^{b_n}\cdots\int_{a_2}^{b_2}\ln f\left(y_1,t_2x_2+(1-t_2)y_2,\dots,t_nx_n+(1-t_n)y_n\right)dx_2\cdots dx_nd\mathbf{y}.
\end{aligned}$$

Using Fubini's theorem and induction again, we have

$$\ln\mathfrak{K}(\mathbf{t})\leqslant\frac{\sum_{c_i=t_i\text{or}1-t_i;i=1,2,\dots,n}h_1(c_1)\cdots h_i(c_i)\cdots h_n(c_n)}{|[\mathbf{a},\mathbf{b}]|}\int_{[\mathbf{a},\mathbf{b}]}f(\mathbf{x})d\mathbf{x}$$

$$= \sum_{c_i=t_i \text{or} 1-t_i; i=1,2,\dots,n} h_1(c_1)\cdots h_i(c_i)\cdots h_n(c_n)\ln \mathfrak{K}(\mathbf{0})$$
$$= \sum_{c_i=t_i \text{or} 1-t_i; i=1,2,\dots,n} h_1(c_1)\cdots h_i(c_i)\cdots h_n(c_n)\ln \mathfrak{K}(\mathbf{1}).$$

Thus we can finish the proof of Theorem 2.8. □

It is clearly that the statement (i) in the proceeding theorem is always true only if the function $\ln f$ is Lebesgue integrable on $[\mathbf{a},\mathbf{b}]$.

If $h_1 = h_2 = \dots = h_n$, we have the following results.

COROLLARY 2.9. *Let $\mathfrak{K}$ be as in Theorem 2.8. If $f : [\mathbf{a},\mathbf{b}] \subset \mathbb{R}^n \to (0,+\infty)$ is a co-ordinated $\log -h$-convex function and $\ln f \in L([\mathbf{a},\mathbf{b}])$, then:*

(i) *$\mathfrak{K}$ is a co-ordinated $\log -h$-convex function on $[\mathbf{0},\mathbf{1}] \subset \mathbb{R}^n$.*

(ii) *For all $\mathbf{t} \in [\mathbf{0},\mathbf{1}] \subset \mathbb{R}^n$,*

$$\mathfrak{K}\left(\frac{\mathbf{1}}{\mathbf{2}}\right)^{1/(2h(1/2))^n} \leqslant \mathfrak{K}(\mathbf{t}) \leqslant \mathfrak{L}(\mathbf{1})^{\sum_{c_i=t_i \text{ or } 1-t_i; i=1,2,\dots,n} h(c_1)\cdots h(c_i)\cdots h(c_n)}.$$

(iii) *Especially, if f is a co-ordinated $\log$-convex function on $[\mathbf{a},\mathbf{b}]$, then*

$$\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \mathfrak{K}(\mathbf{t}) = \mathfrak{L}(\mathbf{1}), \quad \inf_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \mathfrak{K}(\mathbf{t}) = \mathfrak{K}\left(\frac{\mathbf{1}}{\mathbf{2}}\right).$$

Proof. Since (i) and (ii) are easily proved by Theorem 2.8, and $\inf_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]} \mathfrak{K}(\mathbf{t}) = \mathfrak{K}(\mathbf{1}/\mathbf{2})$ is directly calculated by (ii), we omit the details and turn to consider the first case of (iii). Here we will provide two methods to prove the assertion.

Method 1. Let $h(t) = t, 0 \leqslant t \leqslant 1$. It is not difficult to see that the value of $\sum_{c_i=t_i \text{ or } 1-t_i; i=1,2,\dots,n} h(c_1)\cdots h(c_i)\cdots h(c_n)$ is the volume of unit cubes, that is

$$\sum_{c_i=t_i \text{ or } 1-t_i; i=1,2,\dots,n} h(c_1)\cdots h(c_i)\cdots h(c_n) \equiv 1.$$

This proves the corollary.

Compare with the above geometric method, the second one may be more general.

Method 2. Firstly, we will show that the partial mapping $\ln \mathfrak{K}_i$ is increasing on the interval $[1/2,1]$ for any given $i \in \{1,2,\dots,n\}$. Without loss of generality, we just consider the mapping $\ln \mathfrak{K}_1$. In fact, by a similar discussion as in the proof of Theorem 2.8, we have

$$\ln \mathfrak{K}_1(t_1)$$
$$= \frac{1}{\prod_{i=1}^n (b_i - a_i)^2} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1}$$
$$\ln f(t_1x_1 + (1-t_1)y_1, t_2x_2 + (1-t_2)y_2, \dots, t_nx_n + (1-t_n)y_n)\, dx_1 \dots dx_n dy_1 \dots dy_n$$

$$\geqslant \frac{1}{\prod_{i=1}^{n}(b_i-a_i)^2}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1}$$
$$\ln f\left(\frac{x_1+y_1}{2},t_2x_2+(1-t_2)y_2,\ldots,t_nx_n+(1-t_n)y_n\right)dx_1\ldots dx_ndy_1\ldots dy_n$$
$$=\ln\mathfrak{K}_1(1/2).$$

Therefore, for any $t_{11},t_{12}\in(1/2,1]$ and $t_{11}<t_{12}$, it follows from the convexity of $\ln\mathfrak{K}_1$ (see Corollary 2.9 (ii)) that

$$\frac{\ln\mathfrak{K}_1(t_{12})-\ln\mathfrak{K}_1(t_{11})}{t_{12}-t_{11}}\geqslant\frac{\ln\mathfrak{K}_1(t_{11})-\ln\mathfrak{K}_1(1/2)}{t_{11}-1/2}\geqslant 0,$$

which implies that $\mathfrak{K}_1$ is an increasing function on $[1/2,1]$.

Secondly, we infer from the monotonicity of $\mathfrak{K}_i$ and Theorem 2.8 (i) that

$$\ln\mathfrak{K}(\mathbf{t})\leqslant\ln\mathfrak{K}(\mathbf{1})=\ln\mathfrak{K}(\mathbf{0})=\ln\mathfrak{L}(\mathbf{1}),$$

which implies that

$$\sup_{\mathbf{t}\in[\mathbf{0},\mathbf{1}]}\mathfrak{K}(\mathbf{t})=\mathfrak{L}(\mathbf{1}).$$

Thus we finish the proof of Corollary 2.9. □

Taking $t_1=t_2=\ldots=t_n$ in the mapping (2.5), we conclude that

COROLLARY 2.10. *Let $f:[\mathbf{a},\mathbf{b}]\subset\mathbb{R}^n\to(0,+\infty)$ and $\ln f\in L([\mathbf{a},\mathbf{b}])$. Define the mapping $\widetilde{\mathfrak{K}}:[0,1]\subset\mathbb{R}\to\mathbb{R}$ by*

$$\widetilde{\mathfrak{K}}(t)=\exp\left[\frac{1}{|[\mathbf{a},\mathbf{b}]|^2}\int_{[\mathbf{a},\mathbf{b}]}\int_{[\mathbf{a},\mathbf{b}]}\ln f(t\mathbf{x}+(1-t)\mathbf{y})d\mathbf{y}d\mathbf{x}\right].$$

(i) *If the partial mapping f_{x_i} is a $\log-h_i$-convex function on $[a_i,b_i]$ for $i=1,2,\ldots,n$ respectively, then*

$$\widetilde{\mathfrak{K}}\left(\frac{1}{2}\right)^{1/\left(2^n\prod_{i=1}^{n}h_i(1/2)\right)}\leqslant\widetilde{\mathfrak{K}}(t)\leqslant\widetilde{\mathfrak{L}}(1)^{\sum_{c_i=t\text{ or }1-t;\ i=1,2,\ldots,n}h_1(c_1)\cdots h_i(c_i)\cdots h_n(c_n)}$$

holds for all $t\in[0,1]$.

Particularly, if $h_1=h_2=\ldots=h_n=h$, that is if f is a co-ordinated $\log-h$-convex function on $[\mathbf{a},\mathbf{b}]$, then for all $t\in[0,1]\subset\mathbb{R}$,

$$\widetilde{\mathfrak{K}}\left(\frac{1}{2}\right)^{1/(2h(1/2))^n}\leqslant\widetilde{\mathfrak{K}}(t)\leqslant\widetilde{\mathfrak{L}}(1)^{\sum_{c_i=t\text{ or }1-t;\ i=1,2,\ldots,n}h(c_1)\cdots h(c_i)\cdots h(c_n)}.$$

Furthermore, if f is a co-ordinated $\log$-convex function on $[\mathbf{a},\mathbf{b}]$, then for all $t\in[0,1]\subset\mathbb{R}$,

$$\sup_{t\in[0,1]}\widetilde{\mathfrak{K}}(t)=\widetilde{\mathfrak{L}}(1),\quad\inf_{t\in[0,1]}\widetilde{\mathfrak{K}}(t)=\widetilde{\mathfrak{K}}\left(\frac{1}{2}\right).$$

(ii) *If* $f : [\mathbf{a}, \mathbf{b}] \subset \mathbb{R}^n \to (0, +\infty)$ *is a* $\log -h$*-convex function on* $[\mathbf{a}, \mathbf{b}]$*, then* $\widetilde{\mathfrak{K}}$ *is* $\log -h$*-convex on* $[0,1] \subset \mathbb{R}$.

Corollary 2.10 can be proved by a similar fashion to Corollary 2.6 and Corollary 2.9, we omit the details.

3. Appendix

Suppose that $f : [\mathbf{a}, \mathbf{b}] = [a_1, b_1] \times \ldots \times [a_n, b_n] \to (0, \infty)$ is a $\log -h$-convex function. Then every partial mapping $f_{x_i} : [a_i, b_i] \to (0, \infty), f_{x_i}(u) = f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n), i = 1, 2, \ldots, n$ satisfies that, for any $t \in [0,1]$ and $v, w \in [a_i, b_i]$,

$$\begin{aligned} f_{x_i}(tv + (1-t)w) &= f(x_1, \ldots, x_{i-1}, tv + (1-t)w, x_{i+1}, \ldots, x_n) \\ &\leqslant f(x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n)^{h(t)} \cdot f(x_1, \ldots, x_{i-1}, w, x_{i+1}, \ldots, x_n)^{h(1-t)} \\ &= f_{x_i}(v)^{h(t)} \cdot f_{x_i}(w)^{h(1-t)}, \end{aligned}$$

which means that $f : [\mathbf{a}, \mathbf{b}] \to (0, \infty)$ is a co-ordinated $\log -h$-convex function.

Conversely, let $f(x_1, x_2, \ldots, x_n) = e^{x_1 x_2 \cdots x_n}$. We claim that $e^{x_1 x_2 \cdots x_n}$ is a co-ordinated $\log$-convex function on $[\mathbf{0}, \mathbf{1}] = [0,1]^n$, but it is not a $\log$-convex function on $[\mathbf{0}, \mathbf{1}]$. In fact, let $(0, u_2, u_3, \ldots, u_n), (u_1, 0, u_3, \ldots, u_n) \in (\mathbf{0}, \mathbf{1}) = (0,1)^n$ and $t \in (0,1)$. We have

$$f(t(0, u_2, u_3, \ldots, u_n) + (1-t)(u_1, 0, u_3, \ldots, u_n)) = e^{t(1-t)\prod_{i=1}^n u_i} > 1.$$

On the other hand,

$$f(0, u_2, u_3, \ldots, u_n)^{h(t)} \cdot f(u_1, 0, u_3, \ldots, u_n)^{h(1-t)} = (e^0)^t \cdot (e^0)^{1-t} = 1.$$

Therefore,

$$\begin{aligned} &f(t(0, u_2, u_3, \ldots, u_n) + (1-t)(u_1, 0, u_3, \ldots, u_n)) \\ &> f(0, u_2, u_3, \ldots, u_n)^{h(t)} \cdot f(u_1, 0, u_3, \ldots, u_n)^{h(1-t)}, \end{aligned}$$

which yields that $f(x_1, x_2, \ldots, x_n) = e^{x_1 x_2 \cdots x_n}$ is not a $\log$-convex function on $[\mathbf{0}, \mathbf{1}]$.

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