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WEIGHTED INTEGRAL INEQUALITY AND APPLICATION IN UNIFORM STABILITY FOR A NONLINEAR SYSTEM WITH MEMORY

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Abstract. In this paper, we consider the viscoelastic system

$$\boldsymbol{u}_{tt}-\mu\Delta\boldsymbol{u}-(\lambda+\mu)\nabla(\mathrm{div}\boldsymbol{u})+\int_0^t g(t-s)\mathrm{div}[a(x)\nabla\boldsymbol{u}(s)]ds+b(x)\boldsymbol{h}(\boldsymbol{u}_t)=\boldsymbol{f}(\boldsymbol{u})$$

with initial conditions and boundary conditions. Under some assumptions on the relaxation function g, and other functions $\boldsymbol{h}$ and $\boldsymbol{f}$, without constructing any auxiliary functional, by establishing weighted integral inequality on the energy functional, we obtain a general energy decay formula for the solution, such that the usual exponential decay results and the polynomial decay results are only special cases, respectively.

1. Introduction

In this paper, we investigate the asymptotic behavior of the solution to the problem

$$\begin{cases}\boldsymbol{u}_{tt}-\mu\Delta\boldsymbol{u}-(\lambda+\mu)\nabla(\mathrm{div}\boldsymbol{u})+\displaystyle\int_0^t g(t-s)\mathrm{div}\left[a(x)\nabla\boldsymbol{u}(s)\right]ds+b(x)\boldsymbol{h}(\boldsymbol{u}_t)\\ \qquad =\boldsymbol{f}(\boldsymbol{u})\,\mathrm{in}\,\Omega\times\mathbb{R}^+,\\ \boldsymbol{u}(x,t)=0 \text{ on } \partial\Omega\times(0,\infty),\\ \boldsymbol{u}(x,0)=\boldsymbol{u}_0(x),\ \ \boldsymbol{u}_t(x,0)=\boldsymbol{u}_1(x)\,\mathrm{in}\,\Omega,\end{cases} \tag{1.1}$$

where μ,λ are Lamé constants, Ω is a bounded domain in $\mathbb{R}^n$ $(n\geqslant 1)$ with smooth boundary $\partial\Omega$, g is a positive function that represents the kernel of the memory term which satisfies some conditions to be specified below, a,b are real functions and control functions $\boldsymbol{h}$, $\boldsymbol{f}$ are real vector valued functions which satisfy appropriate conditions. Let $\boldsymbol{u}=(u^1,u^2\cdots,u^n)$ be a vector function, $\mathrm{div}\boldsymbol{u}=u^1_{x_1}+u^2_{x_2}+\cdots+u^n_{x_n}$ is the divergence of $\boldsymbol{u}$, Δ denotes $\sum\limits_{i=1}^n\frac{\partial^2}{\partial x_i^2}$, $\nabla\boldsymbol{u}$ is the gradient of $\boldsymbol{u}$, $|\nabla\boldsymbol{u}|$ is the length of $\nabla\boldsymbol{u}$. We write

$$\Delta\boldsymbol{u}=\left(\sum_{i=1}^n u^1_{x_ix_i},\sum_{i=1}^n u^2_{x_ix_i},\cdots,\sum_{i=1}^n u^n_{x_ix_i}\right)^T,$$

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$$\nabla \boldsymbol{u} = \begin{pmatrix} u^1_{x_1} & \cdots & u^1_{x_n} \\ \vdots & \ddots & \vdots \\ u^n_{x_1} & \cdots & u^n_{x_n} \end{pmatrix}, \quad |\nabla \boldsymbol{u}| = \left(\sum_{i,j=1}^{n} |u^i_{x_j}|^2 \right)^{\frac{1}{2}}.$$

This problem has its origin in the mathematical description of memory-type elastic materials. It is well known that memory-type elastic materials exhibit nature damping, which is due to the special property of these materials to keep memory of their past history. From the mathematical point of view, these damping effects are modeled by integro-differential operators. Therefore, dynamics of memory-type elastic materials are very important and interesting as they have wide applications in natural sciences. From the physical point of view, the problem (1.1) describes the position $u(x,t)$ of the material particle $x = (x_1, x_2, \cdots, x_n)$ at time t, which is controlled by memory function g, dissipative function $\boldsymbol{h}$, and force function $\boldsymbol{f}$.

Cavalcanti and Oquendo [2] considered the viscoelastic wave equation

$$u_{tt} - \kappa_0 \Delta u + \int_0^t g(t-s) \operatorname{div}[a(x)\nabla u(s)]\, ds + b(x)h(u_t) + f(u) = 0 \text{ in } \Omega \times (0, +\infty),$$

under some assumption conditions on the memory function g and $a(x) + b(x) \geqslant \delta > 0$, they obtained the exponential stability when g decays exponentially and h is linear, the polynomial stability when g decays polynomially and h is nonlinear. Li and Bao [10] studied the viscoelastic problem

$$\begin{cases} u_{tt} - \mu \Delta \boldsymbol{u} - (\mu + \lambda)\nabla(\operatorname{div} \boldsymbol{u}) + \displaystyle\int_0^t g(t-s)\Delta \boldsymbol{u}(s)ds = 0 & \text{in } \Omega \times (0, \infty), \\ \boldsymbol{u} = 0 & \text{on } \Gamma_0 \times [0, \infty), \\ \mu \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}} - \displaystyle\int_0^t g(t-s) \dfrac{\partial \boldsymbol{u}}{\partial \boldsymbol{\nu}}(s)ds + (\mu + \lambda)(\operatorname{div} \boldsymbol{u})\boldsymbol{\nu} + \boldsymbol{h}(\boldsymbol{u}_t) = 0 & \text{on } \Gamma_1 \times [0, \infty), \\ \boldsymbol{u}(x,0) = \boldsymbol{u}_0, \boldsymbol{u}_t(x,0) = \boldsymbol{u}_1 & \text{in } \Omega, \end{cases}$$

under suitable assumptions on boundary memory function g and boundary control function $\boldsymbol{h}$, showed a uniform stability result of the solution by constructing appropriate auxiliary functionals and establishing differential inequalities. For more uniform stability results of evolution equations, we refer the readers to see [12, 5, 4, 15, 9, 8, 6].

In [1], applying integral inequalities and multiplier technique, Alabau-Boussouira et al. studied the abstract integro-differential evolution equations under the assumption $g'(t) \leqslant -kg^{1+\frac{1}{p}}(t)$ with $p \in (2, \infty)$, $k > 0$ and proved that the energy of the mild solution decays exponentially or polynomially as $t \to \infty$. In this work, we develop a weighted integral inequality on the original energy to derive general decay result. Another advantage of integral approach is that instead of using Lyapunov technique for some perturbed energy, we rather concentrate on the original energy. The key contribution of our work is to show a general energy decay formula for the solution to the initial boundary value problem (1.1). By the formula, we obtain several explicit energy decay rates (e.g., exponential decay rate and polynomial decay rate) for the solutions to several evolution equations. From the application's point of view, our result may provide some qualitative analysis and intuition for the researchers in other fields when they study concrete models. The methods in this paper can be applied to various partial differential equations to obtain some more general results.

Our aim in this work is to establish the general decay result of the problem (1.1) without constructing any auxiliary functional. The method used in this paper is different from the methods in some literatures. The usual exponential decay results and the polynomial decay results in many literatures are only special cases of our result given in this paper.

The outline of this paper is as follows. In section 2, we present some notations, assumptions and lemmas needed throughout our proofs. Section 3 is devoted to proving the general decay result.

2. Notations and preliminaries

In this section, we present some materials needed in the proofs of our results. We use the standard Lebesgue space $L^p(\Omega)$ and Sobolev space $H_0^1(\Omega)$ with their usual scalar products and norms. In this paper, we denote

$$\|u\|_s = \|u\|_{L^s(\Omega)}.$$

Now, we present some hypotheses as follows.

(H_1) $a, b : \Omega \to \mathbb{R}$ are nonnegative functions and $a \in C^1(\overline{\Omega})$, $b \in C^\infty(\Omega)$ with

$$a(x) \geqslant \alpha > 0,\ b(x) \geqslant \beta > 0.$$

(H_2) $g : [0,\infty) \to [0,\infty)$ is a non-increasing C^1 function with

$$g(0) > 0,\ \ \mu - \|a\|_\infty \int_0^\infty g(s)ds = l > 0.$$

Furthermore, there exists a non-increasing positive differentiable function ξ with $\int_0^{+\infty} \xi(\tau)d\tau = +\infty$ such that

$$g'(t) \leqslant -\xi(t)g(t),\ \ \forall t \geqslant 0.$$

(H_3) $\boldsymbol{f} = (f_1, \cdots, f_n) : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\boldsymbol{f}(\boldsymbol{s}) \cdot \boldsymbol{s} \leqslant 2\sum_{i=1}^n F_i(\boldsymbol{s}) \leqslant 0, \quad \forall \boldsymbol{s} = (s_1, \cdots, s_n) \in \mathbb{R}^n,$$

where

$$F_i(\boldsymbol{s}) = \int_0^{s_i} f_i(s_1, \cdots, s_{i-1}, z, s_{i+1}, \cdots, s_n)dz.$$

(H_4) $\boldsymbol{h} : \mathbb{R}^n \to \mathbb{R}^n$ satisfies $\boldsymbol{h}(\boldsymbol{v}) \cdot \boldsymbol{v} \geqslant 0$, $\ \forall \boldsymbol{v} \in \mathbb{R}^n$, and there exist two positive constants c_1 and c_2 such that

$$c_1|\boldsymbol{v}| \leqslant |\boldsymbol{h}(\boldsymbol{v})| \leqslant c_2|\boldsymbol{v}|,\ \ \forall \boldsymbol{v} \in \mathbb{R}^n.$$

REMARK 2.1. Inspired by the model in [2], we generalize the model in [10] by assumption (H_1). The assumption (H_2) guarantees that the model is hyperbolic and shows the decay rate of the memory function g. The assumptions (H_3) and (H_4) give the strength of the force function $\boldsymbol{f}$ and dissipative function $\boldsymbol{h}$, respectively.

Our results are based on the following existence and uniqueness theorem of solution to problem (1.1).

THEOREM 2.1. *Let the assumptions $(H_1)-(H_4)$ hold. If $(\boldsymbol{u}_0,\boldsymbol{u}_1)\in\mathscr{H}:=[H^2(\Omega)\cap H_0^1(\Omega)]^n\times[H_0^1(\Omega)]^n$, then for all $T>0$, there exists a unique solution u of (1.1) satisfying*

$$\boldsymbol{u}\in\left(L^\infty\left(0,T;H^2(\Omega)\cap H_0^1(\Omega)\right)\right)^n,\boldsymbol{u}_t\in\left(L^\infty\left(0,T;H_0^1(\Omega)\right)\right)^n,\boldsymbol{u}_{tt}\in\left(L^\infty\left(0,T;L^2(\Omega)\right)\right)^n.$$

Proof. The proof can be obtained by the Faedo-Galerkin method and calculus theorem in an abstract space (c.f. [7, 13, 11, 16, 3]). □

To prove our main results, we give some important lemmas.

LEMMA 2.2. *If $\boldsymbol{u}$ is the solution to (1.1) and $g\in C^1[0,\infty)$, then*

$$\begin{aligned}\int_\Omega\nabla\boldsymbol{u}_t\cdot\int_0^t g(t-s)a(x)\nabla\boldsymbol{u}(s)dsdx=&\frac{1}{2}(g'\circ\nabla\boldsymbol{u})(t)-\frac{1}{2}g(t)\int_\Omega a(x)|\nabla\boldsymbol{u}(t)|^2dx\\&+\frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds\int_\Omega a(x)|\nabla\boldsymbol{u}(t)|^2dx-(g\circ\nabla\boldsymbol{u})(t)\right],\end{aligned}$$

where

$$(g\circ\nabla\boldsymbol{u})(t)=\int_\Omega\int_0^t g(t-s)a(x)|\nabla\boldsymbol{u}(t)-\nabla\boldsymbol{u}(s)|^2dsdx.$$

Proof. Differentiating $(g\circ\nabla\boldsymbol{u})(t)$ with respect to t and noting

$$\int_0^t g(t-s)ds=\int_0^t g(s)ds,$$

we get

$$\begin{aligned}\frac{d}{dt}(g\circ\nabla\boldsymbol{u})(t)&=\frac{d}{dt}\int_\Omega\int_0^t g(t-s)a(x)|\nabla\boldsymbol{u}(t)-\nabla\boldsymbol{u}(s)|^2dsdx\\&=\int_\Omega\int_0^t g'(t-s)a(x)|\nabla\boldsymbol{u}(t)-\nabla\boldsymbol{u}(s)|^2dsdx\\&\quad+2\int_\Omega\int_0^t g(t-s)a(x)\left[\nabla\boldsymbol{u}(t)-\nabla\boldsymbol{u}(s)\right]\cdot\nabla\boldsymbol{u}_t dsdx\\&=(g'\circ\nabla\boldsymbol{u})(t)+2\int_\Omega\int_0^t g(t-s)a(x)\nabla\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}_t(t)dsdx\\&\quad-2\int_\Omega\int_0^t g(t-s)a(x)\nabla\boldsymbol{u}(s)\cdot\nabla\boldsymbol{u}_t(t)dsdx\\&=(g'\circ\nabla\boldsymbol{u})(t)+\frac{d}{dt}\left[\int_0^t g(s)ds\int_\Omega a(x)|\nabla\boldsymbol{u}(t)|^2dx\right]\\&\quad-2\int_\Omega\nabla\boldsymbol{u}_t(t)\cdot\int_0^t g(t-s)a(x)\nabla\boldsymbol{u}(s)dsdx\\&\quad-g(t)\int_\Omega a(x)|\nabla\boldsymbol{u}(t)|^2dx,\end{aligned}$$

which implies

$$\int_\Omega \nabla \boldsymbol{u}_t \cdot \int_0^t g(t-s)a(x)\nabla \boldsymbol{u}(s)dsdx = \frac{1}{2}(g' \circ \nabla \boldsymbol{u})(t) - \frac{1}{2}g(t)\int_\Omega a(x)|\nabla \boldsymbol{u}(t)|^2 dx + \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds \int_\Omega a(x)|\nabla \boldsymbol{u}(t)|^2 dx - (g \circ \nabla \boldsymbol{u})(t)\right].$$

The proof is completed. □

LEMMA 2.3. *Let $\boldsymbol{w}, \boldsymbol{v} \in [H^1(\Omega)]^n$. We have the following formulas*

$$\int_\Omega \boldsymbol{v} \cdot \nabla(div\boldsymbol{w})dx = \int_{\partial\Omega} \boldsymbol{v} \cdot (div\boldsymbol{w})\boldsymbol{\nu} d\Gamma - \int_\Omega (div\boldsymbol{w})(div\boldsymbol{v})dx,$$

$$\int_\Omega \boldsymbol{v} \cdot \Delta \boldsymbol{w} dx = \int_{\partial\Omega} \boldsymbol{v} \cdot \frac{\partial \boldsymbol{w}}{\partial \nu} d\Gamma - \int_\Omega \nabla \boldsymbol{v} \cdot \nabla \boldsymbol{w} dx.$$

Proof. The proof can be found in [10]. □

3. Main result

In order to define the energy function $E(t)$ of the problem (1.1), we give the following computation. Multiplying $\boldsymbol{u}_t$ on both sides of the equation of the problem (1.1), integrating the resulting system over Ω, using the Green formula and Lemmas 2.2-2.3, we have

$$\begin{aligned}
0 =& \int_\Omega \boldsymbol{u}_t \cdot \boldsymbol{u}_{tt} dx - \mu \int_\Omega \boldsymbol{u}_t \cdot \Delta \boldsymbol{u} dx - (\mu+\lambda)\int_\Omega \boldsymbol{u}_t \cdot \nabla(\mathrm{div}\boldsymbol{u})dx \\
&+ \int_\Omega \boldsymbol{u}_t \cdot \int_0^t g(t-s)\mathrm{div}\left[a(x)\nabla \boldsymbol{u}(s)\right]ds \\
&+ \int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot \boldsymbol{u}_t dx - \int_\Omega \boldsymbol{f}(\boldsymbol{u})\cdot \boldsymbol{u}_t dx \\
=& \int_\Omega \boldsymbol{u}_t \cdot \boldsymbol{u}_{tt} dx + \mu \int_\Omega \nabla \boldsymbol{u}_t \cdot \nabla \boldsymbol{u} dx + (\mu+\lambda)\int_\Omega (\mathrm{div}\boldsymbol{u}_t)(\mathrm{div}\boldsymbol{u})dx \\
&- \int_\Omega \nabla \boldsymbol{u}_t \cdot \int_0^t g(t-s)a(x)\nabla \boldsymbol{u}(s)dsdx \\
&+ \int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot \boldsymbol{u}_t dx - \int_\Omega \boldsymbol{f}(\boldsymbol{u})\cdot \boldsymbol{u}_t dx \\
=& \frac{d}{dt}\left[\frac{1}{2}\|\boldsymbol{u}_t\|_2^2 + \frac{\mu}{2}\|\nabla \boldsymbol{u}\|_2^2 + \frac{\mu+\lambda}{2}\|\mathrm{div}\boldsymbol{u}\|_2^2\right] \\
&- \frac{1}{2}\frac{d}{dt}\left[\int_0^t g(s)ds\int_\Omega a(x)|\nabla \boldsymbol{u}(t)|^2 - (g\circ \nabla \boldsymbol{u})(t)\right]dx \\
&+ \frac{1}{2}g(t)\int_\Omega a(x)|\nabla \boldsymbol{u}|^2 dx - \frac{1}{2}\left(g'\circ \nabla \boldsymbol{u}\right)(t) + \int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot \boldsymbol{u}_t dx - \int_\Omega \boldsymbol{f}(\boldsymbol{u})\cdot \boldsymbol{u}_t dx
\end{aligned}$$

$$=\frac{d}{dt}\Big[\frac{1}{2}\|\boldsymbol{u}_t\|_2^2+\frac{1}{2}\int_\Omega\left(\mu-a(x)\int_0^t g(s)ds\right)|\nabla\boldsymbol{u}(t)|^2dx+\frac{\mu+\lambda}{2}\|\mathrm{div}\boldsymbol{u}\|_2^2$$
$$+\frac{1}{2}\left(g\circ\nabla\boldsymbol{u}\right)(t)-\int_\Omega\sum_{i=1}^n F_i(\boldsymbol{u})dx\Big]$$
$$-\frac{1}{2}\left(g'\circ\nabla\boldsymbol{u}\right)(t)+\frac{1}{2}g(t)\int_\Omega a(x)|\nabla\boldsymbol{u}|^2dx+\int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot\boldsymbol{u}_t dx.$$

The above computation inspires us to define the energy functional as the following

$$E(t)=\frac{1}{2}\|\boldsymbol{u}_t\|_2^2+\frac{1}{2}\int_\Omega\left(\mu-a(x)\int_0^t g(s)ds\right)|\nabla\boldsymbol{u}|^2dx+\frac{\mu+\lambda}{2}\|\mathrm{div}\boldsymbol{u}\|_2^2$$
$$+\frac{1}{2}\left(g\circ\nabla\boldsymbol{u}\right)(t)-\int_\Omega\sum_{i=1}^n F_i(\boldsymbol{u})dx.$$

LEMMA 3.1. *The energy function $E(t)$ satisfies $E(t)\geqslant 0$ and*

$$\frac{d}{dt}E(t)=\frac{1}{2}\left(g'\circ\nabla\boldsymbol{u}\right)(t)-\frac{1}{2}g(t)\int_\Omega a(x)|\nabla\boldsymbol{u}|^2dx-\int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot\boldsymbol{u}_t dx\leqslant 0.$$

Proof. From the assumption (H_3), we have $\int_\Omega\sum_{i=1}^n F_i(\boldsymbol{u})dx\leqslant 0$. Hence, $E(t)\geqslant 0$. Then by the above computations and the assumptions, it is easy to see that

$$\frac{d}{dt}E(t)=\frac{1}{2}\left(g'\circ\nabla\boldsymbol{u}\right)(t)-\frac{1}{2}g(t)\int_\Omega a(x)|\nabla\boldsymbol{u}|^2dx-\int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot\boldsymbol{u}_t dx\leqslant 0.$$

The proof is completed. □

LEMMA 3.2. *Let $E:\mathbb{R}^+\to\mathbb{R}^+$ be a non-increasing function and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ be a strictly increasing C^2 function such that $\psi(0)=0$ and $\lim_{t\to+\infty}\psi(t)=+\infty$. Assume that there exists $c>0$ such that*

$$\int_t^{+\infty}\psi'(s)E(s)ds\leqslant cE(t),\ \ \forall t\geqslant 0,$$

then

$$E(t)\leqslant\lambda E(0)e^{-\omega\psi(t)},$$

for some constants $\omega,\lambda>0$.

Proof. The proof can be found in [14]. □

On the base of the above lemmas, we give our main result as follows.

THEOREM 3.3. *Assume that $(H_1)-(H_4)$ hold. Let $(\boldsymbol{u}_0,\boldsymbol{u}_1)\in\mathscr{H}:=[H^2(\Omega)\cap H_0^1(\Omega)]^n\times[H_0^1(\Omega)]^n$ be given. Then, the unique solution of problem (1.1) satisfies*

$$E(t)\leqslant KE(0)e^{-k\int_0^t\xi(s)ds},\ \ \forall t\geqslant 0,$$

for some positive constants K and k.

Proof. Multiplying $\xi(t)\boldsymbol{u}(t)$ on both sides of the equation of problem (1.1), integrating resulting system over $\Omega\times[t_1,t_2]$ $(0\leqslant t_1\leqslant t_2)$, using of the boundary condition and Lemma 2.3, we have

$$
\begin{aligned}
0 = &\int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{u}_{tt} dxdt \\
&- \mu \int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \Delta \boldsymbol{u} dxdt - (\mu+\lambda) \int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \nabla(\mathrm{div}\boldsymbol{u}) dxdt \\
&+ \int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \int_0^t g(t-s) \mathrm{div}\left[a(x) \nabla \boldsymbol{u}(s)\right] dsdxdt + \int_{t_1}^{t_2} \xi(t) \int_\Omega b(x) \boldsymbol{u} \cdot \boldsymbol{h}(\boldsymbol{u}_t) dxdt \\
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{f}(\boldsymbol{u}) dxdt \\
= &\int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{u}_{tt} dxdt + \mu \int_{t_1}^{t_2} \xi(t) \int_\Omega |\nabla \boldsymbol{u}|^2 dxdt + (\mu+\lambda) \int_{t_1}^{t_2} \xi(t) \int_\Omega |\mathrm{div}\boldsymbol{u}|^2 dxdt \\
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega \nabla \boldsymbol{u} \cdot \int_0^t g(t-s) a(x) \nabla \boldsymbol{u}(s) dsdxdt + \int_{t_1}^{t_2} \xi(t) \int_\Omega b(x) \boldsymbol{u} \cdot \boldsymbol{h}(\boldsymbol{u}_t) dxdt \\
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{f}(\boldsymbol{u}) dxdt \\
= &\int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{u}_{tt} dxdt + \int_{t_1}^{t_2} \xi(t) \int_\Omega \left(\mu - a(x) \int_0^t g(s) ds\right) |\nabla \boldsymbol{u}|^2 dxdt \\
&+ (\mu+\lambda) \int_{t_1}^{t_2} \xi(t) \int_\Omega |\mathrm{div}\boldsymbol{u}|^2 dxdt + \int_{t_1}^{t_2} \xi(t) \int_\Omega b(x) \boldsymbol{u} \cdot \boldsymbol{h}(\boldsymbol{u}_t) dxdt \\
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{f}(\boldsymbol{u}) dxdt - \int_{t_1}^{t_2} \xi(t) \int_\Omega \nabla \boldsymbol{u} \cdot \int_0^t g(t-s) a(x) (\nabla \boldsymbol{u}(s) \\
&- \nabla \boldsymbol{u}(t)) dsdxdt. \qquad (3.1)
\end{aligned}
$$

According to the definition of energy function $E(t)$, we get

$$
\begin{aligned}
&\int_\Omega \left(\mu - a(x) \int_0^t g(s) ds\right) |\nabla \boldsymbol{u}|^2 dx \\
&\quad = 2E(t) - \|\boldsymbol{u}_t\|_2^2 - (\lambda+\mu) \|\mathrm{div}\boldsymbol{u}\|_2^2 - (g \circ \nabla \boldsymbol{u})(t) + 2 \int_\Omega \sum_{i=1}^n F_i(\boldsymbol{u}) dx. \qquad (3.2)
\end{aligned}
$$

Combining (3.1) and (3.2), we deduce

$$
\begin{aligned}
0 = &\int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{u}_{tt} dxdt + (\mu+\lambda) \int_{t_1}^{t_2} \xi(t) \int_\Omega |\mathrm{div}\boldsymbol{u}|^2 dxdt \\
&+ \int_{t_1}^{t_2} \xi(t) \int_\Omega b(x) \boldsymbol{u} \cdot \boldsymbol{h}(\boldsymbol{u}_t) dxdt \\
&- \int_{t_1}^{t_2} \xi(t) \int_\Omega \boldsymbol{u} \cdot \boldsymbol{f}(\boldsymbol{u}) dxdt - \int_{t_1}^{t_2} \xi(t) \int_\Omega \nabla \boldsymbol{u} \cdot \int_0^t g(t-s) a(x) (\nabla \boldsymbol{u}(s) - \nabla \boldsymbol{u}(t)) dsdxdt \\
&+ 2 \int_{t_1}^{t_2} \xi(t) E(t) dt - \int_{t_1}^{t_2} \xi(t) \|\boldsymbol{u}_t\|_2^2 dt - (\lambda+\mu) \int_{t_1}^{t_2} \xi(t) \|\mathrm{div}\boldsymbol{u}\|_2^2 dt \\
&- \int_{t_1}^{t_2} \xi(t) (g \circ \nabla \boldsymbol{u})(t) dt + 2 \int_{t_1}^{t_2} \xi(t) \int_\Omega \sum_{i=1}^n F_i(\boldsymbol{u}) dx,
\end{aligned}
$$

that is

$$2\int_{t_1}^{t_2}\xi(t)E(t)dt=\int_{t_1}^{t_2}\xi(t)\|\boldsymbol{u}_t\|_2^2dt+\int_{t_1}^{t_2}\xi(t)(g\circ\nabla\boldsymbol{u})(t)dt-\int_{t_1}^{t_2}\xi(t)\int_\Omega\boldsymbol{u}\cdot\boldsymbol{u}_{tt}dxdt$$
$$+\int_{t_1}^{t_2}\xi(t)\int_\Omega\nabla\boldsymbol{u}\cdot\int_0^t g(t-s)a(x)\left(\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}(t)\right)dsdxdt$$
$$-\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)\boldsymbol{u}\cdot\boldsymbol{h}(\boldsymbol{u}_t)dxdt$$
$$+\int_{t_1}^{t_2}\xi(t)\int_\Omega\left[-2\sum_{i=1}^n F_i(\boldsymbol{u})+\boldsymbol{f}(\boldsymbol{u})\cdot\boldsymbol{u}\right]dxdt. \tag{3.3}$$

Now, we estimate the terms on the right side of (3.3). In fact, using Lemma 3.1 and the assumptions (H_1) and (H_4), we have

$$\frac{d}{dt}E(t)=\frac{1}{2}\left(g'\circ\nabla\boldsymbol{u}\right)(t)-\frac{1}{2}g(t)\int_\Omega a(x)|\nabla\boldsymbol{u}|^2dx-\int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot\boldsymbol{u}_tdx$$
$$\leqslant-\int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot\boldsymbol{u}_tdx\leqslant-c_2\beta\int_\Omega|\boldsymbol{u}_t|^2dx,$$

which implies

$$\int_{t_1}^{t_2}\xi(t)\|\boldsymbol{u}_t\|_2^2dt\leqslant-\frac{1}{c_2\beta}\int_{t_1}^{t_2}\xi(t)E'(t)dt. \tag{3.4}$$

Using Lemma 3.1, we get

$$\frac{d}{dt}E(t)=\frac{1}{2}\left(g'\circ\nabla\boldsymbol{u}\right)(t)-\frac{1}{2}g(t)\int_\Omega a(x)|\nabla\boldsymbol{u}|^2dx-\int_\Omega b(x)\boldsymbol{h}(\boldsymbol{u}_t)\cdot\boldsymbol{u}_tdx$$
$$\leqslant\frac{1}{2}\left(g'\circ\nabla\boldsymbol{u}\right)(t),$$

that is

$$-\left(g'\circ\nabla\boldsymbol{u}\right)(t)\leqslant-2E'(t),$$

which with (H_2) implies

$$\int_{t_1}^{t_2}\xi(t)\left(g\circ\nabla\boldsymbol{u}\right)(t)dt\leqslant-2\int_{t_1}^{t_2}E'(t)dt. \tag{3.5}$$

For the third term on the right side of (3.3), integrating by parts, we have

$$-\int_{t_1}^{t_2}\xi(t)\int_\Omega\boldsymbol{u}\cdot\boldsymbol{u}_{tt}dxdt=-\int_\Omega\xi(t)\boldsymbol{u}\cdot\boldsymbol{u}_tdx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_\Omega(\xi(t)\boldsymbol{u})_t\cdot\boldsymbol{u}_tdxdt$$
$$=-\int_\Omega\xi(t)\boldsymbol{u}\cdot\boldsymbol{u}_tdx\Big|_{t_1}^{t_2}+\int_{t_1}^{t_2}\int_\Omega\xi'(t)\boldsymbol{u}\cdot\boldsymbol{u}_tdxdt$$
$$+\int_{t_1}^{t_2}\int_\Omega\xi(t)|\boldsymbol{u}_t|^2dxdt. \tag{3.6}$$

By Young inequality, Pincaré inequality and the definition of $E(t)$, we have

$$\begin{aligned}\left|-\int_\Omega \xi(t)\boldsymbol{u}\cdot\boldsymbol{u}_t dx\Big|_{t_1}^{t_2}\right| &\leqslant \sum_{i=1}^{2}\left[\xi(t)\left(\frac{1}{2}\|\boldsymbol{u}\|_2^2+\frac{1}{2}\|\boldsymbol{u}_t\|_2^2\right)\right]_{t=t_i}\\ &\leqslant \sum_{i=1}^{2}\left[\xi(t)\left(\frac{B}{2}\|\nabla\boldsymbol{u}\|_2^2+\frac{1}{2}\|\boldsymbol{u}_t\|_2^2\right)\right]_{t=t_i}\\ &\leqslant \sum_{i=1}^{2}\left[k_1\xi(t)E(t)\right]_{t=t_i}\leqslant 2k_1\xi(0)E(t_1),\end{aligned}\tag{3.7}$$

where B is the Pincaré constant, and $k_1 := \max\{\frac{B}{2},\frac{1}{2}\}$. Similarly,

$$\begin{aligned}\left|\int_{t_1}^{t_2}\xi'(t)\int_\Omega \boldsymbol{u}\cdot\boldsymbol{u}_t dxdt\right| &\leqslant \int_{t_1}^{t_2}|\xi'(t)|\left(\frac{B}{2}\|\nabla\boldsymbol{u}\|_2^2+\frac{1}{2}\|\boldsymbol{u}_t\|_2^2\right)dt\\ &\leqslant -k_1\int_{t_1}^{t_2}\xi'(t)E(t)dt.\end{aligned}\tag{3.8}$$

For the last term on the right side of (3.6), we have

$$\int_{t_1}^{t_2}\xi(t)\|\boldsymbol{u}_t\|_2^2 dt\leqslant -\frac{1}{c_2\beta}\int_{t_1}^{t_2}\xi(t)E'(t)dt.\tag{3.9}$$

For the fourth term on the right hand side of (3.3), we have

$$\begin{aligned}&\int_\Omega \nabla\boldsymbol{u}(t)\cdot\int_0^t g(t-s)a(x)(\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}(t))dsdx\\ &\leqslant \delta\int_\Omega a(x)|\nabla\boldsymbol{u}(t)|^2dx+\frac{1}{4\delta}\int_\Omega a(x)\left(\int_0^t g(t-s)(\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}(t))ds\right)^2dx\\ &\leqslant \delta\|a\|_\infty\|\nabla\boldsymbol{u}\|_2^2+\frac{1}{4\delta}\int_0^t g(s)ds\int_\Omega\int_0^t g(t-s)a(x)|\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}(t)|^2dsdx\\ &\leqslant \delta k_2\|a\|_\infty E(t)+\frac{\mu-l}{4\delta\|a\|_\infty}(g\circ\nabla\boldsymbol{u})(t).\end{aligned}$$

Combining with (3.5), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\xi(t)\int_\Omega\nabla\boldsymbol{u}(t)\cdot\int_0^t g(t-s)a(x)\left(\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}(t)\right)dsdxdt\\ &\qquad\leqslant \delta k_2\|a\|_\infty\int_{t_1}^{t_2}\xi(t)E(t)dt-\frac{\mu-l}{2\delta\|a\|_\infty}\int_{t_1}^{t_2}E'(t)dt.\end{aligned}\tag{3.10}$$

By using Poincaré inequality and (3.4), for the fifth term on the right side of (3.3), we obtain

$$\begin{aligned}&\left|-\int_{t_1}^{t_2}\xi(t)\int_\Omega b(x)\boldsymbol{u}\cdot\boldsymbol{h}(\boldsymbol{u}_t)dxdt\right|\leqslant\|b\|_\infty\int_{t_1}^{t_2}\xi(t)\int_\Omega|\boldsymbol{h}(\boldsymbol{u}_t)||\boldsymbol{u}|dxdt\\ &\leqslant\|b\|_\infty\int_{t_1}^{t_2}\xi(t)\int_\Omega c_2|\boldsymbol{u}_t||\boldsymbol{u}|dxdt\leqslant c_2\|b\|_\infty\int_{t_1}^{t_2}\xi(t)\left(\frac{\delta B}{2}\|\nabla\boldsymbol{u}\|_2^2+\frac{1}{2\delta}\|\boldsymbol{u}_t\|_2^2\right)dt\\ &\leqslant\delta k_3\|b\|_\infty\int_{t_1}^{t_2}\xi(t)E(t)dt-\frac{1}{2\delta\beta}\|b\|_\infty\int_{t_1}^{t_2}\xi(t)E'(t)dt.\end{aligned}\tag{3.11}$$

For the last term on the right hand side of (3.3), on the base of the assumption (H_3), we obtain

$$\int_{t_1}^{t_2} \xi(t) \int_\Omega \left[-2\sum_{i=1}^{n} F_i(\boldsymbol{u}) + \boldsymbol{f}(\boldsymbol{u}) \cdot \boldsymbol{u} \right] dxdt \leqslant 0. \tag{3.12}$$

Combining (3.3)–(3.12), we arrive at

$$\begin{aligned} 2\int_{t_1}^{t_2} \xi(t)E(t)dt \leqslant{}& (k_2\|a\|_\infty + k_3\|b\|_\infty)\,\delta \int_{t_1}^{t_2} \xi(t)E(t)dt \\ &- \left(\frac{2}{c_2\beta} + \frac{\|b\|_\infty}{2\delta\beta}\right)\int_{t_1}^{t_2} \xi(t)E'(t)dt - k_1\int_{t_1}^{t_2} \xi'(t)E(t)dt \\ &- \left(2 + \frac{\mu - l}{2\delta\|a\|_\infty}\right)\int_{t_1}^{t_2} E'(t)dt + 2k_1\xi(0)E(t_1). \end{aligned} \tag{3.13}$$

Integrating by parts, using Lemma 3.1 and (H_2), we have

$$\begin{aligned} -\int_{t_1}^{t_2} \xi'(t)E(t)dt ={}& -\xi(t)E(t)|_{t_1}^{t_2} + \int_{t_1}^{t_2} \xi(t)E'(t)dt \\ ={}& -\xi(t_2)E(t_2) + \xi(t_1)E(t_1) + \int_{t_1}^{t_2} \xi(t)E'(t)dt \\ \leqslant{}& \xi(0)E(t_1). \end{aligned} \tag{3.14}$$

Similarly,

$$\begin{aligned} -\int_{t_1}^{t_2} \xi(t)E'(t)dt ={}& -\xi(t)E(t)|_{t_1}^{t_2} + \int_{t_1}^{t_2} \xi'(t)E(t)dt \\ ={}& -\xi(t_2)E(t_2) + \xi(t_1)E(t_1) + \int_{t_1}^{t_2} \xi'(t)E(t)dt \\ \leqslant{}& \xi(0)E(t_1), \end{aligned} \tag{3.15}$$

and

$$-\int_{t_1}^{t_2} E'(t)dt = E(t_1) - E(t_2) \leqslant E(t_1). \tag{3.16}$$

Owing to (3.13)–(3.16), we get

$$\begin{aligned} 2\int_{t_1}^{t_2} \xi(t)E(t)dt \leqslant{}& (k_2\|a\|_\infty + k_3\|b\|_\infty)\,\delta \int_{t_1}^{t_2} \xi(t)E(t)dt \\ &+ \left[\left(\frac{2}{c_2\beta} + \frac{\|b\|_\infty}{2\delta\beta} + 3k_1\right)\xi(0) + \left(2 + \frac{\mu - l}{2\delta\|a\|_\infty}\right)\right]E(t_1). \end{aligned} \tag{3.17}$$

Denote

$$\begin{aligned} \theta_1(\delta) &= (k_2\|a\|_\infty + k_3\|b\|_\infty)\,\delta, \\ \theta_2(\delta) &= \left(\frac{2}{c_2\beta} + \frac{\|b\|_\infty}{2\delta\beta} + 3k_1\right)\xi(0) + \left(2 + \frac{\mu - l}{2\delta\|a\|_\infty}\right). \end{aligned}$$

Obviously, for all $\delta > 0$, $\theta_1(\delta) > 0$ and $\theta_2(\delta) > 0$. In order to ensure $\theta_1(\delta) < 2$, we need to take a suitable $\delta := \delta_0$.

In particular, we choose

$$\delta_0 := \frac{1}{k_2\|a\|_\infty + k_3\|b\|_\infty},$$

then

$$2-\theta_1(\delta_0) > 0.$$

Combining with (3.17), we get

$$(2-\theta_1(\delta_0))\int_{t_1}^{t_2}\xi(t)E(t)dt \leqslant CE(t_1),$$

that is

$$\int_{t_1}^{t_2}\xi(t)E(t)dt \leqslant CE(t_1),$$

for some $C>0$. Letting t_2 go to infinity, the assumptions of Lemma 3.2 are satisfied with

$$\psi(t) := \int_0^t \xi(s)ds.$$

Applying Lemma 3.2, the proof of Theorem 3.3 is completed. □

REMARK 3.1. In particular, if $\xi(t)\equiv a$, we get the exponential decay result

$$E(t)\leqslant ce^{-kt},\ \forall t\geqslant 0.$$

If $\xi(t)=\frac{1}{1+t}$, we get the polynomial decay result

$$E(t)\leqslant c(1+t)^{-k},\ \forall t\geqslant 0.$$

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