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NEW INEQUALITIES FOR INTERPOLATIONAL OPERATOR MEANS

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Abstract. The main goal of this article is to present several refinements and reverses of well known operator inequalities. These inequalities include operator means, operator monotone functions, operator log-convex functions and positive linear maps.

Among many other results, we show that for any $0 \leqslant \alpha, \beta \leqslant 1$,

$$f(A\nabla_\alpha B) \leqslant f\left((A\nabla_\alpha B)\nabla_\beta A\right)\sharp_\alpha f\left((A\nabla_\alpha B)\nabla_\beta B\right) \leqslant f(A)\sharp_\alpha f(B)$$

whenever f is a non-negative operator log-convex function, $A, B \in \mathscr{B}(\mathscr{H})$ are positive operators, and $0 \leqslant \alpha, \beta \leqslant 1$. Further, we consider some inequalities of Ando's type, and prove that if Φ is a positive linear map, then

$$\Phi(A\sharp_\alpha B) \leqslant \Phi\left((A\sharp_\alpha B)\sharp_\beta A\right)\sharp_\alpha \Phi\left((A\sharp_\alpha B)\sharp_\beta B\right) \leqslant \Phi(A)\sharp_\alpha \Phi(B).$$

Many other refinements and reverses are shown by invoking ideas related to the so called interpolational operator means.

1. Introduction and preliminaries

We denote the set of all bounded linear operators on a Hilbert space $\mathscr{H}$ by $\mathscr{B}(\mathscr{H})$. An operator $A \in \mathscr{B}(\mathscr{H})$ is said to be positive (denoted by $A > 0$) if $\langle Ax, x\rangle > 0$ for all non zero $x \in \mathscr{H}$.

The axiomatic theory for connections and means for pairs of positive operators has been studied by Kubo and Ando [11]. A binary operation σ defined on the cone of positive operators is called an operator mean if for $A, B > 0$,

(i) $I\sigma I = I$, where I is the identity operator;

(ii) $C^*(A\sigma B)C \leqslant (C^*AC)\,\sigma\,(C^*BC)$, $\forall C \in \mathscr{B}(\mathscr{H})$;

(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$, where $A_n \downarrow A$ means that $A_1 \geqslant A_2 \ldots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;

(iv)
$$A \leqslant B \quad \& \quad C \leqslant D \quad \Rightarrow \quad A\sigma C \leqslant B\sigma D,\ \forall\, C, D > 0. \tag{1.1}$$

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For a symmetric operator mean σ (in the sense that $A\sigma B = B\sigma A$), a parametrized operator mean σ_α ($\alpha \in [0,1]$) is called an interpolational path for σ (or Uhlmann's interpolation for σ) if it satisfies

(c1) $A\sigma_0 B = A$ (here we recall the convention $T^0 = I$ for any positive operator T), $A\sigma_1 B = B$, and $A\sigma_{\frac{1}{2}} B = A\sigma B$;

(c2) $(A\sigma_\alpha B)\,\sigma\,\left(A\sigma_\beta B\right) = A\sigma_{\frac{\alpha+\beta}{2}} B$ for all $\alpha, \beta \in [0,1]$;

(c3) the map $\alpha \in [0,1] \mapsto A\sigma_\alpha B$ is norm continuous for each A and B.

It is straightforward to see that the set of all $\gamma \in [0,1]$ satisfying

$$(A\sigma_\alpha B)\,\sigma_\gamma\,\left(A\sigma_\beta B\right) = A\sigma_{(1-\gamma)\alpha+\gamma\beta} B, \tag{1.2}$$

for all α, β is a convex subset of $[0,1]$ including 0 and 1. Therefore (1.2) is valid for all $\alpha, \beta, \gamma \in [0,1]$ (see [7, Lemma 1]).

An example of typical interpolational means the so-called power means

$$Am_\upsilon B = A^{\frac{1}{2}}\left(\frac{1}{2}\left(I + \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\upsilon}\right)\right)^{\frac{1}{\upsilon}} A^{\frac{1}{2}}, \qquad -1 \leqslant \upsilon \leqslant 1$$

and their interpolational paths are [8, Theorem 5.24],

$$Am_{\upsilon,\alpha} B = A^{\frac{1}{2}}\left((1-\alpha)I + \alpha\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\upsilon}\right)^{\frac{1}{\upsilon}} A^{\frac{1}{2}}, \quad 0 \leqslant \alpha \leqslant 1.$$

In particular, we have

$$Am_{1,\alpha} B = A\nabla_\alpha B := (1-\alpha)A + \alpha B,$$

$$Am_{0,\alpha} B = A\sharp_\alpha B := A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha} A^{\frac{1}{2}},$$

$$Am_{-1,\alpha} B = A!_\alpha B := \left(A^{-1}\nabla_\alpha B\right)^{-1}.$$

These are called the weighted arithmetic, weighted geometric, and weighted harmonic means respectively. It is well-known that

$$A!_\alpha B \leqslant A\sharp_\alpha B \leqslant A\nabla_\alpha B, \quad 0 \leqslant \alpha \leqslant 1. \tag{1.3}$$

In [5], Aujla et al. introduced the notion of operator log-convex functions in the following way: A continuous real function $f : (0,\infty) \to (0,\infty)$ is called operator log-convex if

$$f(A\nabla_\alpha B) \leqslant f(A)\,\sharp_\alpha f(B), \quad 0 \leqslant \alpha \leqslant 1 \tag{1.4}$$

for all positive operators A and B. After that, Ando and Hiai [2] gave the following characterization of operator monotone decreasing functions.

LEMMA 1.1. *Let f be a continuous non-negative function on $(0,\infty)$. Then the following conditions are equivalent:*

(a) f is operator monotone decreasing;

(b) f is operator log-convex;

(c) $f(A\nabla B) \leqslant f(A)\sigma f(B)$ for all positive operators A, B and for all symmetric operator means σ.

In Theorem 2.1 below, we provide a more precise estimate than (1.4) for operator log-convex functions. As a by-product, we improve both inequalities in (1.3). Additionally, we present a refinement and two reverse inequalities for the triangle inequality.

Our main application of Theorem 2.1 is a subadditive behavior of operator monotone decreasing functions. Recall that a concave function (not necessarily operator concave) $f:(0,\infty)\to[0,\infty)$ enjoys the subadditive inequality

$$f(a+b) \leqslant f(a)+f(b), \qquad a,b>0. \tag{1.5}$$

Operator concave functions do not enjoy the same subadditive behavior. However, in [3] it was shown that an operator concave function $f:(0,\infty)\to(0,\infty)$ satisfies the norm version of (1.5) as follows

$$|||f(A+B)||| \leqslant |||f(A)+f(B)|||, \tag{1.6}$$

for positive matrices A,B and any unitraily invariant norm $|||\ \ |||$. Later, the authors in [6] showed that (1.6) is still valid for concave functions $f:(0,\infty)\to(0,\infty)$ (not necessarily operator concave).

We emphasize that (1.6) does not hold without the norm. We refer the interested reader to [15], where additional assumption implies (1.6) without the norm and to [9] for a reversed version of (1.6). In [4], it is shown that an operator monotone decreasing function $f:(0,\infty)\to(0,\infty)$ satisfies the subadditive inequality

$$f(A+B) \leqslant f(A)\nabla f(B), \tag{1.7}$$

for the positive matrices A,B.

In Corollary 2.1, we present multiple refinements of (1.7).

The celebrated Ando inequality asserts that if Φ is a positive unital linear map on $\mathscr{B}(\mathscr{H})$ and $A,B\in\mathscr{B}(\mathscr{H})$ are positive operators, then

$$\Phi(A\sharp_\alpha B) \leqslant \Phi(A)\sharp_\alpha\Phi(B), \qquad 0\leqslant\alpha\leqslant 1. \tag{1.8}$$

Recall that, a linear map $\Phi:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{H})$ is positive if $\Phi(A)$ is positive whenever A is positive. In Section 3, we improve and extend this result to Uhlmann's interpolation $\sigma_{\alpha\beta}$ $(0\leqslant\alpha,\beta\leqslant 1)$.

A reverse of (1.8) has been shown in [13, Theorem 4] as follows

$$\Phi(A)\sharp\Phi(B) \leqslant \left(\frac{(M/m)^{\frac{1}{4}}+(m/M)^{\frac{1}{4}}}{2}\right)\Phi(A\sharp B), \tag{1.9}$$

whenever $A,B \in \mathscr{B}(\mathscr{H})$ are two positive operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m < M$, and Φ is a unital positive linear map. In compliance with the theme of this article, we present a refinement of (1.9) as follows

$$\begin{aligned}&\Phi(A)\sharp\Phi(B)\\&\leqslant \frac{1}{2}\int_0^1 \left[(Mm)^{-\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(B)\right)+(Mm)^{\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(A)\right)\right]dt\\&\leqslant \left(\frac{(M/m)^{\frac{1}{4}}+(m/M)^{\frac{1}{4}}}{2}\right)\Phi(A\sharp B).\end{aligned}$$

2. On the operator log-convexity

We begin our main results with a refinement of (1.4), as follows.

THEOREM 2.1. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators and $0 \leqslant \alpha \leqslant 1$. If $f:(0,\infty)\to(0,\infty)$ is an operator monotone decreasing function, then*

$$f(A\nabla_\alpha B)\leqslant f\left((A\nabla_\alpha B)\nabla_\beta A\right)\sharp_\alpha f\left((A\nabla_\alpha B)\nabla_\beta B\right)\leqslant f(A)\sharp_\alpha f(B), \tag{2.1}$$

for any $0 \leqslant \beta \leqslant 1$.

Proof. Assume f is operator monotone decreasing (equivalently, operator-log convex by Lemma 1.1). The following useful identity is easily verified:

$$A\nabla_\alpha B=\left((A\nabla_\alpha B)\nabla_\beta A\right)\nabla_\alpha\left((A\nabla_\alpha B)\nabla_\beta B\right), \tag{2.2}$$

which follows from (1.2) with $A = A\nabla_0 B$ and $B = A\nabla_1 B$. Then we have

$$\begin{aligned}f(A\nabla_\alpha B)&=f\left(\left((A\nabla_\alpha B)\nabla_\beta A\right)\nabla_\alpha\left((A\nabla_\alpha B)\nabla_\beta B\right)\right)\\&\leqslant f\left((A\nabla_\alpha B)\nabla_\beta A\right)\sharp_\alpha f\left((A\nabla_\alpha B)\nabla_\beta B\right) &(2.3)\\&\leqslant \left(f(A\nabla_\alpha B)\sharp_\beta f(A)\right)\sharp_\alpha\left(f(A\nabla_\alpha B)\sharp_\beta f(B)\right) &(2.4)\\&\leqslant \left((f(A)\sharp_\alpha f(B))\sharp_\beta f(A)\right)\sharp_\alpha\left((f(A)\sharp_\alpha f(B))\sharp_\beta f(B)\right) &(2.5)\\&=\left((f(A)\sharp_\alpha f(B))\sharp_\beta(f(A)\sharp_0 f(B))\right)\sharp_\alpha\left((f(A)\sharp_\alpha f(B))\sharp_\beta(f(A)\sharp_1 f(B))\right) &(2.6)\\&=f(A)\sharp_{(1-\beta)\alpha+\beta\alpha}f(B) &(2.7)\\&=f(A)\sharp_\alpha f(B),\end{aligned}$$

where the inequalities (2.3), (2.4) and (2.5) follow directly from the log-convexity assumption on f together with (1.1), the equalities (2.6) and (2.7) are obtained from the property (c1) and (1.2), respectively. This completes the proof. □

As promised in the introduction, we present the following refinement of Aujla inequality (1.7), as a main application of Theorem 2.1.

COROLLARY 2.1. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators. If $f:(0,\infty)\to(0,\infty)$ is an operator monotone decreasing function, then*

$$\begin{aligned} f(A+B) &\leqslant f(3A\nabla B)\sharp f(A\nabla 3B)\\ &\leqslant f(2A)\sharp f(2B)\\ &\leqslant f(2A)\nabla f(2B)\\ &\leqslant f(A)\nabla f(B). \end{aligned}$$

Proof. In Theorem 2.1, let $\alpha=\beta=\frac{1}{2}$ and replace (A,B) by $(2A,2B)$. This implies the first and second inequalities immediately. The third inequality follows from the second inequality in (1.3), while the last inequality follows from the properties of operator means and the fact that f is operator monotone decreasing. □

REMARK 2.1. Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators and $0\leqslant\alpha\leqslant 1$. If f is a function satisfying

$$f(A\nabla_\alpha B)\leqslant f\left((A\nabla_\alpha B)\nabla_\beta A\right)\sharp_\alpha f\left((A\nabla_\alpha B)\nabla_\beta B\right), \tag{2.8}$$

for $0\leqslant\beta\leqslant 1$, then f is operator monotone decreasing. This follows by taking $\beta=1$ in (2.8) and equivalence of (a) and (b) in Lemma 1.1 above.

COROLLARY 2.2. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators. If $g:(0,\infty)\to(0,\infty)$ is operator monotone increasing, then*

$$g(A\nabla_\alpha B)\geqslant g\left((A\nabla_\alpha B)\nabla_\beta A\right)\sharp_\alpha g\left((A\nabla_\alpha B)\nabla_\beta B\right)\geqslant g(A)\sharp_\alpha g(B),$$

for any $0\leqslant\alpha,\beta\leqslant 1$.

Proof. It was shown in [2] that operator monotonicity of g is equivalent to operator log-concavity ($g(A\nabla_\alpha B)\geqslant g(A)\sharp_\alpha g(B)$). The proof goes in a similar way to the proof of Theorem 2.1. □

REMARK 2.2. In [2, Remark 2.6], we have for non-negative operator monotone decreasing function f, and any operator mean σ and $A,B>0$,

$$f(A\nabla_\alpha B)\leqslant f(A)!_\alpha f(B)\leqslant f(A)\sigma f(B),\quad 0\leqslant\alpha\leqslant 1. \tag{2.9}$$

Better estimates than (2.9) may be obtained as follows, where $0\leqslant\alpha,\beta\leqslant 1$,

$$\begin{aligned} f(A\nabla_\alpha B) &= f\left(\left((A\nabla_\alpha B)\nabla_\beta A\right)\nabla_\alpha\left((A\nabla_\alpha B)\nabla_\beta B\right)\right)\\ &\leqslant f\left((A\nabla_\alpha B)\nabla_\beta A\right)!_\alpha f\left((A\nabla_\alpha B)\nabla_\beta B\right)\\ &\leqslant \left(f(A\nabla_\alpha B)!_\beta f(A)\right)!_\alpha\left(f(A\nabla_\alpha B)!_\beta f(B)\right)\\ &\leqslant \left((f(A)!_\alpha f(B))!_\beta f(A)\right)!_\alpha\left((f(A)!_\alpha f(B))!_\beta f(B)\right)\\ &= \left((f(A)!_\alpha f(B))!_\beta(f(A)!_0 f(B))\right)!_\alpha\left((f(A)!_\alpha f(B))!_\beta(f(A)!_1 f(B))\right)\\ &= f(A)!_{(1-\beta)\alpha+\beta\alpha}f(B)\\ &= f(A)!_\alpha f(B)\\ &\leqslant f(A)\sigma f(B). \end{aligned}$$

In the following we improve the well-known weighted operator arithmetic-geometric-harmonic mean inequalities (1.3).

THEOREM 2.2. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators. Then*

$$\begin{aligned} A!_\alpha B &\leqslant \left((A\sharp_\alpha B)\sharp_\beta A\right) !_\alpha \left((A\sharp_\alpha B)\sharp_\beta B\right) \\ &\leqslant A\sharp_\alpha B \\ &\leqslant \left((A\sharp_\alpha B)\sharp_\beta A\right) \nabla_\alpha \left((A\sharp_\alpha B)\sharp_\beta B\right) \\ &\leqslant A\nabla_\alpha B, \end{aligned}$$

for $0 \leqslant \alpha,\beta \leqslant 1$.

Proof. It follows from the proof of Theorem 2.1 that

$$A\sharp_\alpha B = \left((A\sharp_\alpha B)\sharp_\beta A\right)\sharp_\alpha \left((A\sharp_\alpha B)\sharp_\beta B\right), \quad 0 \leqslant \alpha,\beta \leqslant 1. \tag{2.10}$$

Thus, we have

$$\begin{aligned} A\sharp_\alpha B &= \left((A\sharp_\alpha B)\sharp_\beta A\right)\sharp_\alpha \left((A\sharp_\alpha B)\sharp_\beta B\right) \\ &\leqslant \left((A\sharp_\alpha B)\sharp_\beta A\right)\nabla_\alpha \left((A\sharp_\alpha B)\sharp_\beta B\right) \qquad (2.11) \\ &\leqslant \left((A\nabla_\alpha B)\nabla_\beta A\right)\nabla_\alpha \left((A\nabla_\alpha B)\nabla_\beta B\right) \qquad (2.12) \\ &= \left((A\nabla_\alpha B)\nabla_\beta (A\nabla_0 B)\right)\nabla_\alpha \left((A\nabla_\alpha B)\nabla_\beta (A\nabla_1 B)\right) \\ &= A\nabla_\alpha B, \qquad (2.13) \end{aligned}$$

where in the inequalities (2.11) and (2.12) we used the weighted operator arithmetic-geometric mean inequality and the equality (2.13) follows from (1.2). This proves the third and fourth inequalities.

As for the first and second inequalities, replace A and B by A^{-1} and B^{-1}, respectively in

$$A\sharp_\alpha B \leqslant \left((A\sharp_\alpha B)\sharp_\beta A\right)\nabla_\alpha \left((A\sharp_\alpha B)\sharp_\beta B\right) \leqslant A\nabla_\alpha B,$$

which we have just shown. Then take the inverse to obtain the required results (thanks to the identity $A^{-1}\sharp_\alpha B^{-1} = (A\sharp_\alpha B)^{-1}$). This completes the proof. □

REMARK 2.3. We notice that similar inequalities maybe obtained for any symmetric mean σ, as follows. First, observe that if σ,τ are two symmetric means such that $\sigma \leqslant \tau$, then the set $T = \{t : 0 \leqslant t \leqslant 1 \text{ and } \sigma_t \leqslant \tau_t\}$ is convex. Indeed, assume $t_1,t_2 \in T$. Then for the positive operators A,B, we have

$$\begin{aligned} A\sigma_{\frac{t_1+t_2}{2}}B &= (A\sigma_{t_1}B)\sigma(A\sigma_{t_2}B) \\ &\leqslant (A\tau_{t_1}B)\tau(A\tau_{t_2}B) \\ &= A\tau_{\frac{t_1+t_2}{2}}B, \end{aligned}$$

where we have used the assumptions $\sigma \leqslant \tau$ and $t_1,t_2 \in T$. This proves that T is convex, and hence $T = [0,1]$ since $0,1 \in T$, trivially. Thus, we have shown that if $\sigma \leqslant \tau$ then $\sigma_\alpha \leqslant \tau_\alpha$, for all $0 \leqslant \alpha \leqslant 1$. Now noting that

$$A\sigma_\alpha B = \left((A\sigma_\alpha B)\sigma_\beta A\right)\sigma_\alpha \left((A\sigma_\alpha B)\sigma_\beta B\right),$$

and proceeding as in Theorem 2.1, we obtain

$$f(A\nabla_\alpha B) \leqslant f\left((A\nabla_\alpha B)\nabla_\beta A\right)\sigma_\alpha f\left((A\nabla_\alpha B)\nabla_\beta B\right) \leqslant f(A)\sigma_\alpha f(B), \tag{2.14}$$

for any $0 \leqslant \beta \leqslant 1$ and the operator log-convex function f. This provides a more precise estimate than (c) in Lemma 1.1 above.

On the other hand, proceeding as in Theorem 2.2, we obtain

$$A\sigma_\alpha B \leqslant \left((A\sigma_\alpha B)\sigma_\beta A\right)\nabla_\alpha\left((A\sigma_\alpha B)\sigma_\beta B\right) \leqslant A\nabla_\alpha B, \tag{2.15}$$

observing that $\sigma_\alpha \leqslant \nabla_\alpha$. This provides a refinement of the latter basic inequality.

In the next result, we aim to provide a more precise estimate than (1.9).

COROLLARY 2.3. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators such that $mA \leqslant B \leqslant MA$ for some scalars $0 < m < M$, and let Φ be a positive linear map. Then*

$$\begin{aligned}
&\Phi(A)\sharp\Phi(B)\\
&\leqslant \frac{1}{2}\int_0^1 \left[(Mm)^{-\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(B)\right) + (Mm)^{\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(A)\right)\right]dt\\
&\leqslant \left(\frac{(M/m)^{\frac{1}{4}} + (m/M)^{\frac{1}{4}}}{2}\right)\Phi(A\sharp B).
\end{aligned}$$

Proof. It follows from the assumption $\sqrt{m} \leqslant \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \leqslant \sqrt{M}$, so

$$0 \leqslant \left(\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} - \sqrt{m}\right)\left(\sqrt{M} - \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}\right).$$

This implies that

$$\left(\sqrt{M} + \sqrt{m}\right)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}} \geqslant \sqrt{Mm} + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}.$$

Multiply both sides by $A^{\frac{1}{2}}$ so that

$$\left(\sqrt{M} + \sqrt{m}\right)A\sharp B \geqslant \sqrt{Mm}A + B.$$

Thus,

$$\begin{aligned}
&\left(\sqrt{M} + \sqrt{m}\right)\Phi(A\sharp B)\\
&\geqslant \sqrt{Mm}\Phi(A) + \Phi(B)\\
&\geqslant \left(\sqrt{Mm}\Phi(A)\sharp\Phi(B)\right)\sharp_t\Phi(B) + \left(\sqrt{Mm}\Phi(A)\sharp\Phi(B)\right)\sharp_t\sqrt{Mm}\Phi(A)\\
&= (Mm)^{\frac{1-t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(B)\right) + (Mm)^{\frac{1+t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(A)\right)\\
&\geqslant 2(Mm)^{\frac{1}{4}}\left((\Phi(A)\sharp\Phi(B))\right)
\end{aligned}$$

thanks to Theorem 2.2. Consequently,

$$\begin{aligned}&\left(\frac{\sqrt{M}+\sqrt{m}}{2(Mm)^{\frac{1}{4}}}\right)\Phi(A\sharp B)\\&\geqslant\frac{1}{2}\left[(Mm)^{-\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(B)\right)+(Mm)^{\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(A)\right)\right]\\&\geqslant\Phi(A)\sharp\Phi(B).\end{aligned}$$

Integrating over t from 0 to 1, yields

$$\begin{aligned}&\left(\frac{\sqrt{M}+\sqrt{m}}{2(Mm)^{\frac{1}{4}}}\right)\Phi(A\sharp B)\\&\geqslant\frac{1}{2}\int_0^1\left[(Mm)^{-\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(B)\right)+(Mm)^{\frac{t}{4}}\left((\Phi(A)\sharp\Phi(B))\sharp_t\Phi(A)\right)\right]dt\\&\geqslant\Phi(A)\sharp\Phi(B).\end{aligned}$$

The proof is then completed by noting that $\frac{\sqrt{M}+\sqrt{m}}{2(Mm)^{\frac{1}{4}}}=\frac{(M/m)^{\frac{1}{4}}+(m/M)^{\frac{1}{4}}}{2}$. □

Taking into account (2.2), it follows that

$$A+B=\alpha A+(1-\alpha)(A\nabla B)+\alpha B+(1-\alpha)(A\nabla B).$$

As a consequence of this inequality, we have the following refinement of the well-known triangle inequality

$$\|A+B\|\leqslant\|A\|+\|B\|.$$

COROLLARY 2.4. *Let $A,B\in\mathscr{B}(\mathscr{H})$. Then, for $\alpha\in\mathbb{R}$,*

$$\|A+B\|\leqslant\|\alpha A+(1-\alpha)(A\nabla B)\|+\|\alpha B+(1-\alpha)(A\nabla B)\|\leqslant\|A\|+\|B\|.$$

REMARK 2.4. Using Corollary 2.4, we obtain the reverse triangle inequalities

$$\|A\|-\|B\|\leqslant\frac{1}{2}\left(\|A\nabla_{-\alpha}(2B)\|+\|A\nabla_{\alpha}(2B)\|-2\|B\|\right)\leqslant\|A-B\|,$$

and

$$\|B\|-\|A\|\leqslant\frac{1}{2}\left(\|B\nabla_{-\alpha}(2A)\|+\|B\nabla_{\alpha}(2A)\|-2\|A\|\right)\leqslant\|A-B\|,$$

where $A\nabla_\alpha B:=(1-\alpha)A+\alpha B$ for positive operators $A,B\in\mathscr{B}(\mathscr{H})$ and $\alpha\in\mathbb{R}$.

3. A glimpse at the Ando inequality

In this section, we present some versions and improvements of the Ando inequality (1.8).

THEOREM 3.1. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators and Φ be a unital positive linear map. Then for any $0 \leqslant \alpha,\beta \leqslant 1$,*

$$\Phi(A\sharp_\alpha B) \leqslant \Phi\left((A\sharp_\alpha B)\sharp_\beta A\right)\sharp_\alpha \Phi\left((A\sharp_\alpha B)\sharp_\beta B\right) \leqslant \Phi(A)\sharp_\alpha \Phi(B). \tag{3.1}$$

In particular,

$$\begin{aligned}\sum_{j=1}^{m} A_j\sharp_\alpha B_j &\leqslant \left(\sum_{j=1}^{m}(A_j\sharp_\alpha B_j)\sharp_\beta A_j\right)\sharp_\alpha\left(\sum_{j=1}^{m}(A_j\sharp_\alpha B_j)\sharp_\beta B_j\right)\\ &\leqslant \left(\sum_{j=1}^{m}A_j\right)\sharp_\alpha\left(\sum_{j=1}^{m}B_j\right).\end{aligned} \tag{3.2}$$

Proof. We omit the proof of (3.1) because it is proved in a way similar to that of (2.1) in Theorem 2.1. Now, if in (3.1) we take $\Phi : M_{nk}(\mathbb{C}) \to M_k(\mathbb{C})$ defined by

$$\Phi\left(\begin{pmatrix} X_{1,1} & & \\ & \ddots & \\ & & X_{n,n}\end{pmatrix}\right) = X_{1,1} + \ldots + X_{n,n},$$

and apply Φ to $A = \mathrm{diag}(A_1,\ldots,A_n)$ and $B = \mathrm{diag}(B_1,\ldots,B_n)$, we get (3.2). □

In the following, we present a more general form of (3.1).

THEOREM 3.2. *Let $A,B \in \mathscr{B}(\mathscr{H})$ be positive operators and Φ be a unital positive linear map. Then we have the following inequalities for Uhlmann's interpolation $\sigma_{\alpha\beta}$ and $0 \leqslant \alpha,\beta \leqslant 1$,*

$$\begin{aligned}\Phi\left(A\sigma_{\alpha\beta}B\right) &\leqslant \Phi\left((A\sigma_\alpha B)\sigma_\beta(A\sigma_0 B)\right)\sigma_\alpha\Phi\left((A\sigma_\alpha B)\sigma_\beta(A\sigma_1 B)\right)\\ &\leqslant \Phi(A)\sigma_\alpha\Phi(B).\end{aligned}$$

Proof. From (1.2), we obviously have

$$\begin{aligned}&\left((A\sigma_\alpha B)\sigma_\beta(A\sigma_0 B)\right)\sigma_\alpha\left((A\sigma_\alpha B)\sigma_\beta(A\sigma_1 B)\right)\\ &= \left(A\sigma_{\alpha(1-\beta)}B\right)\sigma_\alpha\left(A\sigma_{\alpha(1-\beta)+\beta}B\right)\\ &= A\sigma_\alpha B.\end{aligned}$$

Now, the desired result follows directly from the above identities. □

REMARK 3.1. From simple calculations, we have the following inequalities for positive operators $A,B \in \mathscr{B}(\mathscr{H})$, a unital positive linear map Φ, and $0 \leqslant \alpha,\beta,\gamma,\delta \leqslant 1$,

$$\begin{aligned}\Phi\left(A\sigma_{\alpha(1-\beta)+\beta((1-\alpha)\gamma+\alpha\delta)}B\right) &\leqslant \Phi\left((A\sigma_\alpha B)\sigma_\beta\left((A\sigma_\gamma B)\right)\right)\sigma_\alpha\Phi\left((A\sigma_\alpha B)\sigma_\beta\left((A\sigma_\delta B)\right)\right)\\ &\leqslant \Phi(A)\,\sigma_{\alpha(1-\beta)+\beta((1-\alpha)\gamma+\alpha\delta)}\Phi(B).\end{aligned} \tag{3.3}$$

Apparently, (3.3) reduces to (3.1) when $\gamma = 0,\ \delta = 1,\ \sigma_\alpha = \sharp_\alpha$ and $\sigma_\beta = \sharp_\beta$.

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