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SHARP POWER MEAN BOUNDS FOR THE TANGENT AND HYPERBOLIC SINE MEANS

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Abstract. In the article, we prove that the double inequalities

$$\boldsymbol{M}_{\alpha_1}(a,b) < \boldsymbol{M}_{\tan}(a,b) < \boldsymbol{M}_{\beta_1}(a,b),$$
$$\boldsymbol{M}_{\alpha_2}(a,b) < \boldsymbol{M}_{\sinh}(a,b) < \boldsymbol{M}_{\beta_2}(a,b)$$

hold for all $a,b>0$ with $a \neq b$ if and only if $\alpha_1 \leqslant 1/3$, $\beta_1 \geqslant \log 2/\log(2\tan 1) \approx 0.61007$, $\alpha_2 \leqslant 2/3$ and $\beta_2 \geqslant \log 2/\log(2\sinh 1) \approx 0.81109$, where $\boldsymbol{M}_p$, $\boldsymbol{M}_{\tan}$ and $\boldsymbol{M}_{\sinh}$ are the p^{th} power mean, tangent mean and hyperbolic sine mean, respectively.

1. Introduction

Let $p \in \mathbb{R}$ and $a,b>0$ with $a \neq b$. Then the harmonic mean $\boldsymbol{H}(a,b)$, geometric mean $\boldsymbol{G}(a,b)$, arithmetic mean $\boldsymbol{A}(a,b)$, first Seiffert mean $\boldsymbol{P}(a,b)$, second Seriffert mean $\boldsymbol{T}(a,b)$, logarithmic mean $\boldsymbol{L}(a,b)$, Neuman-Sándor mean $\boldsymbol{NS}(a,b)$ [7, 8] and p^{th} power mean $\boldsymbol{M}_p(a,b)$ are respectively defined by

$$\boldsymbol{H}(a,b) = \frac{2ab}{a+b}, \quad \boldsymbol{G}(a,b) = \sqrt{ab}, \quad \boldsymbol{A}(a,b) = \frac{a+b}{2},$$
$$\boldsymbol{P}(a,b) = \frac{a-b}{2\arcsin\left(\frac{a-b}{a+b}\right)}, \quad \boldsymbol{T}(a,b) = \frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)},$$
$$\boldsymbol{L}(a,b) = \frac{a-b}{2\operatorname{artanh}\left(\frac{a-b}{a+b}\right)}, \quad \boldsymbol{NS}(a,b) = \frac{a-b}{2\operatorname{arcsinh}\left(\frac{a-b}{a+b}\right)},$$
$$\boldsymbol{M}_p(a,b) = \begin{cases} \left(\frac{a^p+b^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0, \end{cases} \tag{1.1}$$

where $\operatorname{artanh} x = \frac{1}{2}\log[(1+x)/(1-x)]$ and $\operatorname{arcsinh} x = \log(x+\sqrt{x^2+1})$ are the inverse hyperbolic tangent and sine function, respectively.

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Inspired by the form of two Seiffert means, Witkowski [19] introduced the so-called *Seiffert-like means*, which are the means of the form

$$\boldsymbol{M}_f(a,b)=\begin{cases}\frac{|a-b|}{2f(\frac{|a-b|}{a+b})}, & a\neq b,\\ a, & a=b,\end{cases}$$

where the function $f:(0,1)\mapsto\mathbb{R}$ (called *Seiffert function*) satisfies

$$\frac{x}{1+x}\leqslant f(x)\leqslant\frac{x}{1-x}.$$

It is worth mentioning that artanh$x$ and arcsinh$x$ are Seiffert functions and so $\boldsymbol{L}(a,b)$ and $\boldsymbol{NS}(a,b)$ are also Seriffert-like means.

In this paper, we mainly study two Seriffert-like means corresponding to tangent and hyperbolic sine functions, which have been introduced in [19],

$$\boldsymbol{M}_{\tan}(a,b)=\begin{cases}\frac{a-b}{2\tan(\frac{a-b}{a+b})}, & a\neq b,\\ a, & a=b,\end{cases}\qquad\text{(tangent mean)}\tag{1.2}$$

$$\boldsymbol{M}_{\sinh}(a,b)=\begin{cases}\frac{a-b}{2\sinh(\frac{a-b}{a+b})}, & a\neq b,\\ a, & a=b\end{cases}\qquad\text{(hyperbolic sine mean)}\tag{1.3}$$

Recently, the bivariate means and their applications to special functions have attracted the attention of several researchers [3, 6, 11, 14, 24, 25]. In particular, many remarkable inequalities involving the power mean can be found in the literature [13, 16, 17, 18, 20, 21, 27]. For instance, sharp power mean bounds for several Seiffert-like means were established by Lin [4], Hästö [2], Li et al. [5], Yang [22], more precisely, the double inequalities

$$\boldsymbol{M}_{p_1}(a,b)<\boldsymbol{L}(a,b)<\boldsymbol{M}_{q_1}(a,b),\quad \boldsymbol{M}_{p_2}(a,b)<\boldsymbol{NS}(a,b)<\boldsymbol{M}_{q_2}(a,b),$$
$$\boldsymbol{M}_{p_3}(a,b)<\boldsymbol{P}(a,b)<\boldsymbol{M}_{q_3}(a,b),\quad \boldsymbol{M}_{p_4}(a,b)<\boldsymbol{T}(a,b)<\boldsymbol{M}_{q_4}(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $p_1\leqslant 0$, $q_1\geqslant 1/3$, $p_2\leqslant \log 2/[2\log(1+\sqrt{2})]$, $q_2\geqslant 4/3$, $p_3\leqslant\log 2/\log\pi$, $q_3\geqslant 2/3$, $p_4\leqslant\log 2/[\log(\pi/2)]$ and $q_4\geqslant 5/3$.

As shown in [19, Lem. 3.2], the inequalities

$$\boldsymbol{H}(a,b)<\boldsymbol{G}(a,b)<\boldsymbol{L}(a,b)<\boldsymbol{M}_{\tan}(a,b)<\boldsymbol{M}_{\sinh}(a,b)<\boldsymbol{A}(a,b)\tag{1.4}$$

hold for all $a,b>0$ with $a\neq b$. By virtue of (1.4), in their two very close papers [9, 10] Nowicka and Witkowski presented the linear, harmonic, quadratic, the weighted -2^{nd} power mean and homotopy-type bounds for $\boldsymbol{M}_{\tan}(a,b)$ and $\boldsymbol{M}_{\sinh}(a,b)$ in terms of $\boldsymbol{A}(a,b),\boldsymbol{G}(a,b)$ (or $\boldsymbol{A}(a,b),\boldsymbol{H}(a,b)$).

It is well-known that several classical means are the special cases of power means. So the chain of inequalities (1.4) can be rewritten as

$$\boldsymbol{M}_{-1}(a,b)<\boldsymbol{M}_0(a,b)<\boldsymbol{L}(a,b)<\boldsymbol{M}_{\tan}(a,b)<\boldsymbol{M}_{\sinh}(a,b)<\boldsymbol{M}_1(a,b)$$

hold for all $a,b>0$ with $a\neq b$. It makes sense to ask what the optimal numbers α_1,α_2 and β_1,β_2 are satisfying

$$\boldsymbol{M}_{\alpha_1}(a,b)<\boldsymbol{M}_{\tan}(a,b)<\boldsymbol{M}_{\beta_1}(a,b) \quad \text{and} \quad \boldsymbol{M}_{\alpha_2}(a,b)<\boldsymbol{M}_{\sinh}(a,b)<\boldsymbol{M}_{\beta_2}(a,b)$$

for all $a,b>0$ with $a\neq b$. The main purpose of this paper is to answer this question.

2. Lemmas

In order to prove our main results, we need some notations and several technical lemmas which we present in this section.

The following two lemmas offer a simple criterion to determine the sign of a class of special polynomial or series .

LEMMA 2.1. ([26, Lem. 2.2]). *Let $n,m\in\mathbb{N}\cup\{0\}$ with $n>m$ and $P_n(t)$ be the polynomial of degree n defined by*

$$P_n(t)=\sum_{i=0}^{m}a_it^i-\sum_{i=m+1}^{n}a_it^i,$$

where $a_m,a_n>0$ and $a_i\geqslant 0$ for $0\leqslant i\leqslant n-1$ with $i\neq m$. Then there exist $t_0\in(0,\infty)$ such that $P_n(t_0)=0$ and $P_n(t)>0$ for $t\in(0,t_0)$ and $P_n(t)<0$ for $t\in(t_0,\infty)$.

LEMMA 2.2. ([23, Lem. 2]). *Let $\{a_k\}_{k=0}^{\infty}$ be a nonnegative real sequence with $a_m>0$ and $\sum_{k=m+1}^{\infty}a_k>0$ and let*

$$S(t)=\sum_{k=0}^{m}a_kt^k-\sum_{k=m+1}^{\infty}a_kt^k$$

be a convergent power series on the interval $(0,R)$ $(R>0)$. Then the following statements are true:

(i) If $S(R^-)\geqslant 0$, then $S(t)>0$ for all $t\in(0,R)$;

(ii) If $S(R^-)<0$, then there is a unique $t_0\in(0,R)$ such that $S(t)>0$ for $t\in(0,t_0)$ and $S(t)<0$ for $t\in(t_0,R)$.

LEMMA 2.3. ([1, 4.3.68, 4.5.67]). *Let $|x|<\pi$. Then we have Taylor expansion*

$$\frac{x}{\sin x}=1+\sum_{n=1}^{\infty}\frac{2^{2n}-2}{(2n)!}|B_{2n}|x^{2n}, \tag{2.1}$$

$$\frac{x}{\tanh x}=1+\sum_{n=1}^{\infty}\frac{2^{2n}}{(2n)!}B_{2n}x^{2n}, \tag{2.2}$$

where B_{2n} is the even-index Bernoulli numbers for $n=1,2,3,\cdots$.

For the readers' convenience, recall from [1, p.804, 23.1.1] that the Bernoulli numbers B_n may be defined by the power series expansion

$$\frac{z}{e^z-1}=\sum_{n=0}^{\infty}B_n\frac{z^n}{n!}=1-\frac{z}{2}+\sum_{k=1}^{\infty}B_{2k}\frac{z^{2k}}{(2k)!},\quad |z|<2\pi.$$

The first few Bernoulli numbers B_{2k} are

$$B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad ,B_8=-\frac{1}{30},\quad B_{10}=\frac{5}{66},\quad B_{12}=-\frac{691}{2730}$$

and have the property $(-1)^{k+1}B_{2k}>0$ for $k\geqslant 1$. More properties of B_{2k} are stated in the following lemma.

LEMMA 2.4. ([12]). *For $k\in\mathbb{N}$, Bernoulli numbers B_{2k} satisfy*

$$\frac{2^{2k-1}-1}{2^{2k+1}-1}\frac{(2k+1)(2k+2)}{\pi^2}<\left|\frac{B_{2k+2}}{B_{2k}}\right|<\frac{2^{2k}-1}{2^{2k+2}-1}\frac{(2k+1)(2k+2)}{\pi^2}.$$

LEMMA 2.5. *Let $0<p<1$ and*

$$f_p(u)=\frac{(1+u)^{p-1}+(1-u)^{p-1}}{(1+u)^p+(1-u)^p}.$$

Then the following statements are true:

(i) $f_{1/3}(u)>1+\frac{2u^2}{3}+\frac{46u^4}{81}$ *for* $u\in(0,1)$;

(ii) $f_{3/5}(u)<1+\frac{2u^2}{5}+\frac{4u^4}{5}$ *for* $u\in(0,0.87]$;

(iii) $f_{2/3}(u)>1+\frac{u^2}{3}$ *for* $u\in(0,1)$;

(iv) $f_{4/5}(u)<1+\frac{3u^2}{10}$ *for* $u\in(0,0.75]$.

Proof. We give the proof of $(i),(ii)$ in details and similar methods for the remaining cases $(iii),(iv)$.

Let us denote

$$\hat{f}_{1/3}(u)=(1+u)^{-2/3}+(1-u)^{-2/3}-\left(1+\frac{2}{3}u^2+\frac{46}{81}u^4\right)\left[(1+u)^{1/3}+(1-u)^{1/3}\right].$$

$$\hat{f}_{3/5}(u)=(1+u)^{-2/5}+(1-u)^{-2/5}-\left(1+\frac{2u^2}{5}+\frac{4u^4}{5}\right)\left[(1+u)^{3/5}+(1-u)^{3/5}\right].$$

Then it suffices to prove $\hat{f}_{1/3}(u)>0$ for $u\in(0,1)$ and $\hat{f}_{3/5}(u)<0$ for $u\in(0,0.87]$.

(i) We first prove $\hat{f}_{1/3}(u)>0$ for $u\in(0,1)$.

By elementary calculations, we obtain

$$\frac{81(1-u)^{2/3}(1+u)^{2/3}}{u}\hat{f}_{1/3}(u) = (1+u)^{2/3}(81-54u+54u^2-46u^3+46u^4) \\ -(1-u)^{2/3}(81+54u+54u^2+46u^3+46u^4),$$

the sign of which is equivalent to

$$\left[(1+u)^{2/3}(81-54u+54u^2-46u^3+46u^4)\right]^3 \\ -\left[(1-u)^{2/3}(81+54u+54u^2+46u^3+46u^4)\right]^3 = 4u^5P_1(u), \quad (2.3)$$

where

$$P_1(u) = 408969+152280u^2+92920u^4-74060u^6-48668u^8.$$

As a special polynomial defined as in Lemma 2.1, it can be easily seen from $P_1(1) = 531441$ that $P_1(u) > 0$ for $u \in (0,1)$. This complete the proof of (i).

(ii) Second, elementary computations lead to

$$\frac{5(1-u)^{2/5}(1+u)^{2/5}}{u}\hat{f}_{3/5}(u) = (1+u)^{2/5}(5-2u+2u^2-4u^3+4u^4) \\ -(1-u)^{2/5}(5+2u+2u^2+4u^3+4u^4). \quad (2.4)$$

In order to determine the sign of (2.4), we only need to consider equivalently as

$$\left[(1+u)^{2/5}(5-2u+2u^2-4u^3+4u^4)\right]^5 \\ -\left[(1-u)^{2/5}(5+2u+2u^2+4u^3+4u^4)\right]^5 = -4u^3P_2(u),$$

where

$$P_2(u) = 4125-1834u^2+3576u^4-5680u^6-6032u^8-1760u^{10} \\ -2688u^{12}+4352u^{14}+1280u^{16}+1536u^{18}.$$

We will divide into two cases to prove $P_2(u) > 0$ for $u \in (0,0.87]$.

(i) For $u \in (0,0.8]$, then it can be easily obtained that $P_2(u) > \hat{P}_2(u) =: 4125 - 1834u^2-5680u^6-6032u^8-1760u^{10}-2688u^{12} \geqslant \hat{P}_2(0.8) > 76$.

(ii) For $u \in (0.8,0.87]$, differentiation yields

$$\frac{dP_2(u)}{du} = -4u\left[917+3576u^2(2u^2-1)+1368u^4+16u^6P_2^*(u)\right], \quad (2.5)$$

where $P_2^*(u) = 754+275u^2+504u^4-952u^6-320u^8-432u^{10}$. This in conjunction with Lemma 2.1 and $P_2^*(0.87) \approx 625.729$ yields $P_2^*(u) > 0$ for $u \in (0,0.87]$. Combining this with (2.5) leads to the conclusion that $P_2(u)$ is strictly decreasing on $(0.8,0.87]$ and so $P_2(u) \geqslant P_2(0.87) \approx 282.609$ for $u \in (0,8,0.87]$.

(iii) Similarly, by simplifying, it can be easily seen that

$$f_{2/3}(u)-\left(1+\frac{u^2}{3}\right)=\frac{u\left[(1+u)^{1/3}(3-u+u^2)-(1-u)^{1/3}(3+u+u^2)\right]}{3(1-u^2)^{1/3}[(1-u)^{2/3}+(1+u)^{2/3}]} \tag{2.6}$$

and

$$\left[(1+u)^{1/3}(3-u+u^2)\right]^3-\left[(1-u)^{1/3}(3+u+u^2)\right]^3=2u^3(17+9u^2+u^4)>0. \tag{2.7}$$

Thus the inequality (iii) of Lemma 2.5 holds from (2.6) and (2.7).

(iv) Elementary computations lead to

$$f_{4/5}(u)-\left(1+\frac{3u^2}{10}\right)=\frac{u\left[(1+u)^{1/5}(10-3u+3u^2)-(1-u)^{1/5}(10+3u+3u^2)\right]}{10(1-u^2)^{1/5}[(1-u)^{4/5}+(1+u)^{4/5}]} \tag{2.8}$$

and

$$\left[(1+u)^{1/5}(10-3u+3u^2)\right]^5-\left[(1-u)^{1/5}(10+3u+3u^2)\right]^5=-2uP_3(u^2), \tag{2.9}$$

where

$$P_3(x)=50000-33000x-77607x^2-33885x^3-5265x^4-243x^5.$$

Lemma 2.1 and $P_3(0.75^2)\approx 310.579$ enable us to know that $P_3(x)>0$ for $x\in(0,0.75^2]$. Combining this with (2.8) and (2.9) yields the desired inequality of (iv). □

LEMMA 2.6. *Let* $0<p<1$ *and*

$$g_p(u)=\frac{(1-u)^{p-2}-(1+u)^{p-2}}{(1-u)^p+(1+u)^p},\quad h_p(u)=\frac{(1-u)^{2p-2}-(1+u)^{2p-2}}{[(1-u)^p+(1+u)^p]^2}.$$

Then $g_p(u)$ *and* $h_p(u)$ *are strictly increasing on* $(0,1)$.

Proof. First the monotonicity of $g_p(u)$ follows directly from

$$\frac{g_p'(u)}{2}=\frac{(1-u^2)^3[(1-u)^{2p-3}+(1+u)^{2p-3}]+2(1-u^2)^p[1-p+(3-p)u^2]}{(1-u^2)^3\left[(1-u)^p+(1+u)^p\right]^2}>0.$$

Second, we can rewrite

$$h_p(u)=f_p(u)\cdot\hat{h}_p(u), \tag{2.10}$$

where $f_p(u)$ is defined as in Lemma 2.5 and

$$\hat{h}_p(u)=\frac{(1-u)^{p-1}-(1+u)^{p-1}}{(1-u)^p+(1+u)^p}.$$

Differentiation of $f_p(u)$ and $\hat{h}_p(u)$ with $0<p<1$ gives rise to

$$f_p'(u)=\frac{(1-u)^{2p}(1+u)^2\big[1-((1-u)/(1+u))^{2(1-p)}\big]+4(1-p)u(1-u^2)^p}{(1-u^2)^2\big[(1-u)^p+(1+u)^p\big]^2}>0,$$
$$\hat{h}_p'(u)=\frac{(1-u)^{2p}(1+u)^2+2(1-u^2)^p(1-2p+u^2)+(1-u)^2(1+u)^{2p}}{(1-u^2)^2\big[(1-u)^p+(1+u)^p\big]^2}$$
$$>\frac{(1-u)^{2p}(1+u)^2}{(1-u^2)^2\big[(1-u)^p+(1+u)^p\big]^2}\left[1-\left(\frac{1-u}{1+u}\right)^{1-p}\right]^2>0$$

for $u\in(0,1)$. According to this with (2.10), it can be easily seen from $f_p(u)>0$ and $\hat{h}_p(u)>0$ that $h_p(u)$ is strictly increasing on $(0,1)$. □

3. Main results

THEOREM 3.1. *The double inequality*

$$\boldsymbol{M}_{\alpha_1}(a,b)<\boldsymbol{M}_{\tan}(a,b)<\boldsymbol{M}_{\beta_1}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leqslant 1/3$ and $\beta_1\geqslant \log 2/\log(2\tan 1)\approx$ 0.61007.

Proof. Since $\boldsymbol{M}_{\tan}(a,b)$ and $\boldsymbol{M}_p(a,b)$ are symmetric and homogenous of degree 1, we may assume that $a>b>0$. Let $u=(a-b)/(a+b)\in(0,1)$ and $p\in\mathbb{R}$ with $p\neq 0$. Then from (1.1) and (1.2) we clearly see that

$$\log[\boldsymbol{M}_{\tan}(a,b)]-\log[\boldsymbol{M}_p(a,b)]=\log\left(\frac{u}{\tan u}\right)-\frac{1}{p}\log\left[\frac{(1+u)^p+(1-u)^p}{2}\right]$$
$$=:\varphi_p(u). \tag{3.1}$$

Simple computations lead to

$$\varphi_p(0^+)=0,\quad \varphi_p(1^-)=\frac{\log 2}{p}-\log(2\tan 1), \tag{3.2}$$
$$u\varphi_p'(u)=f_p(u)-\frac{2u}{\sin(2u)}=:\hat{\varphi}_p(u), \tag{3.3}$$

where $f_p(u)$ is defined as in Lemma 2.5.

We divide the proof into four cases.
Case 1.1. $p=1/3$. Let

$$\rho_1(u)=1+\frac{2u^2}{3}+\frac{46u^4}{81}-\frac{2u}{\sin(2u)}.$$

Then it follows from (2.1) that

$$\rho_1(u) = 1 + \frac{2u^2}{3} + \frac{46u^4}{81} - \left[1 + \sum_{n=1}^{\infty} \frac{2^{2n}(2^{2n}-2)}{(2n)!}|B_{2n}|u^{2n}\right]$$
$$= \frac{104u^4}{405} - \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-2)}{(2n)!}|B_{2n}|u^{2n},$$

which in conjunction with Lemma 2.2 and $\rho_1(1^-) \approx 0.03506$ yields $\rho_1(u) > 0$ for $u \in (0,1)$. According to this with Lemma 2.5 (i), it follows that

$$\hat{\varphi}_{1/3}(u) > \rho_1(u) > 0 \tag{3.4}$$

for $u \in (0,1)$.

Therefore, the inequality

$$\boldsymbol{M}_{\tan}(a,b) > \boldsymbol{M}_{1/3}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ follows from (3.1)–(3.4).

Case 1.2. $p = \sigma =: \log 2/\log(2\tan 1)$. Then from (3.2) we clearly see that

$$\varphi_\sigma(0^+) = 0, \quad \varphi_\sigma(1^-) = 0. \tag{3.5}$$

Note that

$$f_p(u) = \frac{1}{L_{p-1}(1-u,1+u)},$$

where $L_q(a,b)$ is the q^{th} Lehmer mean [15]. It is well-known that $L_q(a,b)$ is strictly increasing for $q > 0$ with fixed $a,b > 0$ with $a \neq b$. This in conjunction with (3.3) and Lemma 2.5 (ii) together with $\sigma > 3/5$ gives

$$\hat{\varphi}_\sigma(u) < \hat{\varphi}_{3/5}(u) < 1 + \frac{2u^2}{5} + \frac{4u^4}{5} - \frac{2u}{\sin(2u)} =: \rho_2(u^2) \tag{3.6}$$

for $u \in (0, 0.87]$, where

$$\rho_2(x) = 1 + \frac{2x}{5} + \frac{4x^2}{5} - \frac{2\sqrt{x}}{\sin(2\sqrt{x})}.$$

By Lemma 2.3, $\rho_2(x)$ has the power series expansion

$$\rho_2(x) = \frac{22x}{45}\left(x - \frac{6}{11}\right) - \sum_{n=3}^{\infty} \frac{2^{2n}(2^{2n}-2)}{(2n)!}|B_{2n}|x^n.$$

By this, it can be easily seen that $\rho_2(x) < 0$ for $x \in (0, 6/11]$ and $\rho_2'''(x) < 0$, and so $\rho_2'(x)$ is strictly concave on $(0,1)$. Differentiation yields

$$\rho_2'(x) = \frac{2}{5}(1+4x) + \left[\frac{2}{\tan(2\sqrt{x})} - \frac{1}{\sqrt{x}}\right]\frac{1}{\sin(2\sqrt{x})}.$$

By numerical calculations, we have $\rho_2'(6/11) \approx 0.10153$ and $\rho_2'(0.87^2) \approx 0.09833$. According to this with the concavity of $\rho_2'(x)$, it can be easily obtained that $\rho_2'(x) > \min\{\rho_2'(6/11), \rho_2'(0.87^2)\} > 0$, and so $\rho_2(x)$ is strictly increasing on $(6/11, 0.87^2]$. Thus $\rho_2(x) \leqslant \rho_2(0.87^2) \approx -0.00413 < 0$ for $x \in (6/11, 0.87^2]$.

Combining this with (3.6), it follows that

$$\hat{\varphi}_\sigma(u) < 0 \tag{3.7}$$

for $u \in (0, 0.87]$.

On the other hand, differentiation of $\hat{\varphi}_p(u)$ gives

$$\begin{aligned}\hat{\varphi}_p'(u) &= f_p'(u) - \left[\frac{2u}{\sin(2u)}\right]' \\ &= (1-p)g_p(u) + ph_p(u) - \frac{2[\sin(2u) - 2u\cos(2u)]}{\sin^2(2u)},\end{aligned} \tag{3.8}$$

where $g_p(u)$ and $h_p(u)$ are defined as in Lemma 2.6.

Further, Lemma 2.3 enables us to know that

$$\frac{2[\sin(2u) - 2u\cos(2u)]}{\sin^2(2u)} = \left[\frac{2u}{\sin(2u)}\right]' = \sum_{n=1}^{\infty} \frac{n2^{2n+2}(2^{2n-1}-1)}{(2n)!}|B_{2n}|u^{2n-1}$$

is strictly increasing on $(0,1)$. This in conjunction with (3.8) and Lemma 2.6 yields

$$\hat{\varphi}_\sigma'(u) > (1-\sigma)g_\sigma(0.87) + \sigma h_\sigma(0.87) - \frac{2(\sin 2 - 2\cos 2)}{\sin^2 2} \approx 0.337643 > 0$$

for $u \in (0.87, 1)$. Combining this with $\hat{\varphi}_\sigma(0.87) \approx -0.05444$ and $\hat{\varphi}_\sigma(1^-) = \infty$, it follows from (3.3) and (3.7) that there exists $u_1 \in (0.87, 1)$ such that $\varphi_p(u)$ is strictly decreasing on $(0, u_1)$ and strictly increasing on $(u_1, 1)$.

Therefore, the inequality

$$\boldsymbol{M}_{\tan}(a,b) < \boldsymbol{M}_\sigma(a,b)$$

holds for all $a, b > 0$ with $a \neq b$ follows from (3.1) and (3.5) together with the piece-wise monotonicity of $\varphi_p(u)$.

Case 1.3. $p > 1/3$. Let $u = (a-b)/(a+b) \to 0^+$. Then making use of (3.1) and Taylor's formula we obtain

$$\begin{aligned}&\log[\boldsymbol{M}_{\tan}(a,b)] - \log[\boldsymbol{M}_p(a,b)] \\ &\quad = \log\left(\frac{u}{\tan u}\right) - \frac{1}{p}\log\left[\frac{(1+u)^p + (1-u)^p}{2}\right] = \frac{1}{2}\left(\frac{1}{3} - p\right)u^2 + o(u^2).\end{aligned} \tag{3.9}$$

Equation (3.9) implies that there exists small enough $\varepsilon_1 > 0$ such that

$$\boldsymbol{M}_{\tan}(a,b) < \boldsymbol{M}_p(a,b)$$

for all $a,b>0$ with $(a-b)/(a+b)\in(0,\varepsilon_1)$.

Case 1.4. $p<\log 2/\log(2\tan 1)$. Then it follows from (3.2) that

$$\varphi_p(1^-)>0. \tag{3.10}$$

Equation (3.1) and inequality (3.10) lead to the conclusion that there exists small enough $\varepsilon_2>0$ such that

$$\boldsymbol{M}_{\tan}(a,b)>\boldsymbol{M}_p(a,b)$$

for all $a,b>0$ with $(a-b)/(a+b)\in(1-\varepsilon_2,1)$.

Therefore, Theorem 3.1 follows easily from Cases 1.1-1.4 and the monotonicity of the function $p\mapsto \boldsymbol{M}_p(a,b)$. □

THEOREM 3.2. *The double inequality*

$$\boldsymbol{M}_{\alpha_2}(a,b)<\boldsymbol{M}_{\sinh}(a,b)<\boldsymbol{M}_{\beta_2}(a,b)$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\alpha_2\leqslant 2/3$ and $\beta_2\geqslant \log 2/\log(2\sinh 1)\approx$ 0.81109.

Proof. Since $\boldsymbol{M}_{\sinh}(a,b)$ and $\boldsymbol{M}_p(a,b)$ are symmetric and homogenous of degree 1, we may assume that $a>b>0$. Let $u=(a-b)/(a+b)\in(0,1)$ and $p\in\mathbb{R}$ with $p\neq 0$. Then (1.1) and (1.3) lead to

$$\log[\boldsymbol{M}_{\sinh}(a,b)]-\log[\boldsymbol{M}_p(a,b)]=\log\left(\frac{u}{\sinh u}\right)-\frac{1}{p}\log\left[\frac{(1+u)^p+(1-u)^p}{2}\right]. \tag{3.11}$$

Let

$$\phi_p(u)=:\log\left(\frac{u}{\sinh u}\right)-\frac{1}{p}\log\left[\frac{(1+u)^p+(1-u)^p}{2}\right].$$

Then simple computations lead to

$$\phi_p(0^+)=0,\quad \phi_p(1^-)=\frac{\log 2}{p}-\log(2\sinh 1), \tag{3.12}$$

$$u\phi_p'(u)=f_p(u)-\frac{u}{\tanh u}=:\hat{\phi}_p(u), \tag{3.13}$$

where $f_p(u)$ is defined as in Lemma 2.5.

We divide the proof into four cases.

Case 2.1. $p=2/3$. By differentiation, it can be easily proved that

$$u\cosh u-\sinh u>0 \tag{3.14}$$

for $0<u<1$, which follows from $(u\cosh u-\sinh u)'=u\sinh u>0$.

Let

$$\eta_1(u)=\left(1+\frac{u^2}{3}\right)\sinh u-u\cosh u.$$

Then it follows from (3.14) that

$$\eta_1'(u) = \frac{1}{3}u(u\cosh u - \sinh u) > 0,$$

which in conjunction with $\eta_1(0) = 0$ gives $\eta_1(u) > 0$ for $0 < u < 1$.

According to this with Lemma 2.5 (iii), it follows that

$$\hat{\phi}_{2/3}(u) > \frac{\eta_1(u)}{\sinh u} > 0 \tag{3.15}$$

for $u \in (0,1)$.

Therefore, the inequality

$$\boldsymbol{M}_{\sinh}(a,b) > \boldsymbol{M}_{2/3}(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ follows from (3.11)–(3.13) and (3.15).

Case 2.2. $p = \tau =: \log 2/\log(2\sinh 1)$. Then from (3.12) we clearly see that

$$\phi_\tau(0^+) = 0, \quad \phi_\tau(1^-) = 0. \tag{3.16}$$

As shown in Case 1.2 of Theorem 3.1, $f_p(u)$ is strictly decreasing for $p \in \mathbb{R}$. This in conjunction with Lemma 2.5 (iv) and $\tau > 4/5$ yields

$$\hat{\phi}_\tau(u) < \hat{\phi}_{4/5}(u) < 1 + \frac{3u^2}{10} - \frac{u}{\tanh u} =: \eta_2(u) \tag{3.17}$$

for $u \in (0, 0.75]$.

Making use of (2.2) and Lemma 2.4, we obtain

$$\begin{aligned}
\eta_2(u) &= -\left[\frac{u^2}{45}\left(\frac{3}{2} - u^2\right) + \sum_{n=3}^{\infty} \frac{2^{2n}}{(2n)!} B_{2n} u^{2n}\right] \\
&< -\sum_{k=1}^{\infty} \left[\frac{2^{4k+2}}{(4k+2)!}|B_{4k+2}| - \frac{2^{4k+4}}{(4k+4)!}|B_{4k+4}|u^2\right] u^{4k+2} \\
&< -\sum_{k=1}^{\infty} \frac{2^{4k+4}|B_{4k+2}|}{(4k+4)!}\left[\frac{(4k+3)(4k+4)}{4} - \left|\frac{B_{4k+4}}{B_{4k+2}}\right|\right] u^{4k+2} \\
&< -\sum_{k=1}^{\infty} \frac{2^{4k+2}|B_{4k+2}|[(\pi^2-1)2^{4k+4} - (\pi^2-4)]}{\pi^2(2^{4k+4}-1)(4k+2)!} u^{4k+2} < 0.
\end{aligned} \tag{3.18}$$

According to (3.17) and (3.18), it follows that

$$\hat{\phi}_\tau(u) < 0 \tag{3.19}$$

for $u \in (0, 0.75]$.

On the other hand, twice differentiation with (3.14) yields

$$\left(\frac{u}{\tanh u}\right)'' = \frac{2(u\cosh u - \sinh u)}{\sinh^3 u} > 0,$$

which yields $(u/\tanh u)'$ is strictly increasing on $(0,1)$.

By the monotonicity of $(u/\tanh u)'$, it can be obtained from Lemma 2.6 that

$$\begin{aligned}\hat{\phi}_\tau'(u) = f_\tau'(u) - \left(\frac{u}{\tanh u}\right)' &= (1-\tau)g_\tau(u) + \tau h_\tau(u) - \frac{\sinh(2u)-2u}{2\sinh^2 u} \\ &> (1-\tau)g_\tau(0.75) + \tau h_\tau(0.75) - \frac{\sinh 2 - 2}{2\sinh^2 1} \approx 0.07448 > 0\end{aligned}$$

for $u \in (0.75, 1)$. Combining this, $\hat{\phi}_\tau(0.75) \approx -0.02299$ and $\hat{\phi}_\tau(1^-) = \infty$ imply that there exists $u_2 \in (0.75,1)$ such that $\hat{\phi}_\tau(u) < 0$ for $u \in (0.75, u_2)$ and $\hat{\phi}_\tau(u) > 0$ for $u \in (u_2, 1)$. Further, (3.13) and (3.19) make us to know that $\phi_\tau(u)$ is strictly decreasing on $(0, u_2)$ and strictly increasing on $(u_2, 1)$.

Therefore, the inequality

$$\boldsymbol{M}_{\sinh}(a,b) < \boldsymbol{M}_\tau(a,b)$$

holds for all $a,b > 0$ with $a \neq b$ follows from (3.11) and (3.16) together with the piecewise monotonicity of $\phi_p(u)$.

Case 2.3. $p > 2/3$. Let $u = (a-b)/(a+b) \to 0^+$. Then utilizing the Taylor formula, (3.1) makes us to obtain

$$\begin{aligned}&\log[\boldsymbol{M}_{\sinh}(a,b)] - \log[\boldsymbol{M}_p(a,b)] \\ &\quad = \log\left(\frac{u}{\sinh u}\right) - \frac{1}{p}\log\left[\frac{(1+u)^p + (1-u)^p}{2}\right] = \frac{1}{2}\left(\frac{2}{3} - p\right)u^2 + o(u^2). \quad (3.20)\end{aligned}$$

Equation (3.20) implies that there exists small enough $\varepsilon_3 > 0$ such that

$$\boldsymbol{M}_{\sinh}(a,b) < \boldsymbol{M}_p(a,b)$$

for all $a,b > 0$ with $(a-b)/(a+b) \in (0, \varepsilon_3)$.

Case 2.4. $p < \log 2/\log(2\sinh 1)$. Then it follows from (3.12) that

$$\phi_p(1^-) > 0. \quad (3.21)$$

Equation (3.1) and inequality (3.21) lead to the conclusion that there exists small enough $\varepsilon_4 > 0$ such that

$$\boldsymbol{M}_{\sinh}(a,b) > \boldsymbol{M}_p(a,b)$$

for all $a,b > 0$ with $(a-b)/(a+b) \in (1-\varepsilon_4, 1)$.

Therefore, Theorem 3.2 follows easily from Cases 2.1-2.4 and the monotonicity of the function $p \mapsto \boldsymbol{M}_p(a,b)$. □

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