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SOME SIMPSON TYPE FRACTIONAL INTEGRAL INEQUALITIES FOR FUNCTIONS OF BOUNDED VARIATION

MOHAMED DOUBBI BOUNOUA AND CHUNTAO YIN

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Abstract. In this paper, we focus on the Simpson type fractional integral inequalities. The approximate schemes of these integral inequalities are derived as well.

1. Introduction

It is a well known truth that the integral inequality plays an important role in the theory of differential and integral equations. Indeed, this importance seems to have increased during the last several decades. Moreover, the study of fractional order integral inequality is also of great importance in the theory of existence and uniqueness for fractional differential equations [2, 12, 13, 14, 15, 16, 17, 18, 27, 29]. The Simpson type integral inequality is one of the fundamental results in numerical integration inequalities, it has attracted considerable attention as it is very important and remarkable in numerical analysis and the study of convex and non-convex differentiable mappings [3, 4, 5, 6, 7, 21, 22]. This type of inequality has been extended and generalized to the case of fractional order by many researchers [8, 23].

The classical Simpson's integral inequality is considered as follows

$$\left|\int_a^b f(t)\mathrm{d}t - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_\infty, \tag{1}$$

where $f^{(4)}$ exists and is bounded on (a,b), with

$$\left\|f^{(4)}\right\|_\infty := \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty.$$

In [25], the authors established an approximate scheme of the integral $\int_a^b f(t)\mathrm{d}t$ related to the Simpson's inequality,

$$A_s(f,I_n) := \sum_{i=0}^{n-1}\frac{l_i}{3}\left[\frac{f(t_i)+f(t_{i+1})}{2}+2f\left(\frac{t_i+t_{i+1}}{2}\right)\right],$$

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where $I_n : a = t_0 < t_1 < \ldots < t_n = b$ is a partition of the interval $[a,b]$. The error estimate is given by

$$R_s(f, I_n) = \int_a^b f(t)\mathrm{d}t - A_s(f, I_n),$$

and satisfies the inequality

$$|R_s(f, I_n)| \leqslant \frac{1}{2880} \left\| f^{(4)} \right\|_\infty \sum_{i=0}^{n-1} l_i^5,$$

where $l_i = t_{i+1} - t_i$, $i = 0, 1, \ldots, n-1$. Plenty of novel Simpson type inequalities for convex and non-convex functions have been refined and extended by many mathematicians [24, 25, 26, 28]. In [6], Dragomir gave another inequality which can be achieved without conditions on $f^{(4)}$,

$$\left| \int_a^b f(t)\mathrm{d}t - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \frac{1}{3}(b-a)\bigvee_a^b(f), \tag{2}$$

where $\bigvee_a^b(f)$ denotes the total variation of f on $[a,b]$.

This paper aims to build the Simpson integral inequalities with Riemann-Liouville and Hadamard fractional integrals, and the results are extended to Caputo and Caputo-Hadamard cases as well. Further, the approximate schemes of these integral inequalities are also derived.

2. Preliminaries

In this section, we begin with some fundamental definitions of fractional integrals and derivatives.

DEFINITION 1. ([16]) The Riemann-Liouville integral of order $\alpha > 0$ is defined by

$${}_{RL}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-u)^{\alpha-1} f(u)\mathrm{d}u, \;\; t > a,$$

where $\Gamma(\alpha)$ is the Gamma function.

DEFINITION 2. ([16]) The Riemann-Liouville derivative of order α is defined by

$$\begin{aligned} {}_{RL}\mathrm{D}_{a,t}^{\alpha} f(t) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left({}_{RL}\mathrm{D}_{a,t}^{-(n-\alpha)} f(t)\right) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t (t-u)^{n-\alpha-1} f(u)\mathrm{d}u, \;\; t > a, \end{aligned}$$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

DEFINITION 3. ([16]) The Caputo derivative of order α is defined by

$$
\begin{aligned}
{}_{C}\mathrm{D}_{a,t}^{\alpha} f(t) &= {}_{RL}\mathrm{D}_{a,t}^{-(n-\alpha)} f^{(n)}(t) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-u)^{n-\alpha-1} f^{(n)}(u)\mathrm{d}u, \ \ t > a,
\end{aligned}
$$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

DEFINITION 4. ([11, 12]) The Hadamard integral of order $\alpha > 0$ is defined by

$$
{}_{H}\mathrm{D}_{a,t}^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{u}\right)^{\alpha-1} f(u) \frac{\mathrm{d}u}{u}, \ \ t > a > 0.
$$

DEFINITION 5. ([11, 12]) The Hadamard derivative of order α is defined as

$$
\begin{aligned}
{}_{H}\mathrm{D}_{a,t}^{\alpha} f(t) &= \left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \left({}_{H}\mathrm{D}_{a,t}^{-(n-\alpha)} f(t)\right) \\
&= \frac{1}{\Gamma(n-\alpha)} \left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_a^t \left(\log \frac{t}{u}\right)^{n-\alpha-1} f(u) \frac{\mathrm{d}u}{u}, \ \ t > a > 0,
\end{aligned}
$$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

DEFINITION 6. ([10]) The Caputo-Hadamard derivative of order α is defined as

$$
\begin{aligned}
{}_{CH}\mathrm{D}_{a,t}^{\alpha} f(t) &= {}_{H}\mathrm{D}_{a,t}^{-(n-\alpha)} \left(\left(t \frac{\mathrm{d}}{\mathrm{d}t}\right)^n f(t)\right) \\
&= \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\log \frac{t}{u}\right)^{n-\alpha-1} \left(u \frac{\mathrm{d}}{\mathrm{d}u}\right)^n f(u) \frac{\mathrm{d}u}{u}, \ \ t > a > 0,
\end{aligned}
$$

where $n-1 < \alpha < n \in \mathbb{Z}^+$.

3. Main results

In this section, we state and prove the Simpson type fractional integral inequalities. Throughout this paper, we always denote ${}_{RL}\mathrm{D}_{a,t}^{-q} f(t)|_{t=b}$ by ${}_{RL}\mathrm{D}_{a,t}^{-q} f(b)$.

THEOREM 1. *Let $g : [a,b] \longrightarrow \mathbb{R}$ be a positive and continuous function and f be a mapping of bounded variation on $[a,b]$. For each $x \in [a,b]$, set*

$$
h(x) = \int_a^x (b-t)^{q-1} g(t)\mathrm{d}t. \tag{3}
$$

If $x \in \left[\frac{h(b)}{6}, \frac{5h(b)}{6}\right]$ and $q \geqslant 0$, then

$$
\begin{aligned}
&\left| {}_{RL}\mathrm{D}_{a,t}^{-q}(fg)(b) - \frac{{}_{RL}\mathrm{D}_{a,t}^{-q} g(b)}{3} \left[\frac{f(a)+f(b)}{2} + 2f(h^{-1}(x))\right] \right| \\
&\leqslant \left(\frac{{}_{RL}\mathrm{D}_{a,t}^{-q} g(b)}{3} + \left| \frac{x}{\Gamma(q)} - \frac{{}_{RL}\mathrm{D}_{a,t}^{-q} g(b)}{2} \right| \right) \bigvee_a^b (f),
\end{aligned} \tag{4}
$$

where $\bigvee_a^b(f)$ *denotes the total variation of* f *on the interval* $[a,b]$.

Proof. For $x \in \left[\frac{h(b)}{6}, \frac{5h(b)}{6}\right]$, set

$$w(t) := \begin{cases} h(t) - \frac{h(b)}{6}, & t \in [a, h^{-1}(x)), \\ h(t) - \frac{5h(b)}{6}, & t \in [h^{-1}(x), b]. \end{cases}$$

Using integration by parts, we obtain

$$\begin{aligned} \int_a^b w(t)f'(t)\mathrm{d}t &= \int_a^{h^{-1}(x)} w(t)f'(t)\mathrm{d}t + \int_{h^{-1}(x)}^b w(t)f'(t)\mathrm{d}t \\ &= \left(h(t) - \frac{h(b)}{6}\right) f(t)\,|_a^{h^{-1}(x)} - \int_a^{h^{-1}(x)} (b-t)^{q-1}g(t)f(t)\mathrm{d}t \\ &\quad + \left(h(t) - \frac{5h(b)}{6}\right) f(t)\,|_{h^{-1}(x)}^b - \int_{h^{-1}(x)}^b (b-t)^{q-1}g(t)f(t)\mathrm{d}t \\ &= \left(x - \frac{h(b)}{6}\right) f(h^{-1}(x)) + \frac{h(b)f(a)}{6} - \int_a^{h^{-1}(x)} (b-t)^{q-1}g(t)f(t)\mathrm{d}t \\ &\quad + \frac{h(b)}{6} f(b) - \left(x - \frac{5h(b)}{6}\right) f(h^{-1}(x)) - \int_{h^{-1}(x)}^b (b-t)^{q-1}g(t)f(t)\mathrm{d}t \\ &= \frac{h(b)}{3}\left[\frac{f(a)+f(b)}{2} + 2f(h^{-1}(x))\right] - \int_a^b (b-t)^{q-1}g(t)f(t)\mathrm{d}t. \end{aligned}$$

Then one has

$$\begin{aligned} &\frac{1}{\Gamma(q)} \int_a^b w(t)f'(t)\mathrm{d}t \\ &= \frac{{}_{RL}\mathrm{D}_{a,t}^{-q} g(b)}{3}\left[\frac{f(a)+f(b)}{2} + 2f(h^{-1}(x))\right] - {}_{RL}\mathrm{D}_{a,t}^{-q}(fg)(b). \end{aligned} \tag{5}$$

From [1], for a continuous function $\mu : [a,b] \longrightarrow \mathbb{R}$ and a bounded variation function $\nu : [a,b] \longrightarrow \mathbb{R}$, the following inequality holds

$$\left| \int_a^b \mu(t)\nu'(t)\mathrm{d}t \right| \leqslant \sup_{t\in[a,b]} |\mu(t)| \bigvee_a^b(\nu). \tag{6}$$

Using (5) and (6), we find

$$\begin{aligned}&\left|{}_{RL}\mathrm{D}_{a,t}^{-q}(fg)(b)-\frac{{}_{RL}\mathrm{D}_{a,t}^{-q}g(b)}{3}\left[\frac{f(a)+f(b)}{2}+2f(h^{-1}(x))\right]\right|\\&\leqslant\frac{1}{\Gamma(q)}\sup_{t\in[a,b]}|w(t)|\bigvee_a^b(f).\end{aligned}\tag{7}$$

Because of the fact that $w_{[a,h^{-1}(x)]}(t)=h(t)-\frac{h(b)}{6}$ is increasing on $[a,h^{-1}(x))$ and $w_{[h^{-1}(x),b]}(t)=h(t)-\frac{5h(b)}{6}$ is also increasing on $[h^{-1}(x),b]$, we have

$$\sup_{t\in[a,b]}|w(t)|=\max\left\{|w(a)|,\left|\lim_{t\to\left(h^{-1}(x)\right)^-}w(t)\right|,\left|w(h^{-1}(x))\right|,|w(b)|\right\},$$

that is

$$\sup_{t\in[a,b]}|w(t)|=\max\left\{\frac{h(b)}{6},x-\frac{h(b)}{6},\frac{5h(b)}{6}-x\right\}.$$

Now we use the fact that $\max\{A,B\}=\frac{1}{2}(A+B+|A-B|)$, which leads to

$$\sup_{t\in[a,b]}|w(t)|=\frac{h(b)}{3}+\left|x-\frac{h(b)}{2}\right|.$$

It is evident that

$$\frac{1}{\Gamma(q)}\sup_{t\in[a,b]}|w(t)|=\frac{{}_{RL}\mathrm{D}_{a,t}^{-q}g(b)}{3}+\left|\frac{x}{\Gamma(q)}-\frac{{}_{RL}\mathrm{D}_{a,t}^{-q}g(b)}{2}\right|.\tag{8}$$

By (7) and (8), we get

$$\begin{aligned}&\left|{}_{RL}\mathrm{D}_{a,t}^{-q}(fg)(b)-\frac{{}_{RL}\mathrm{D}_{a,t}^{-q}g(b)}{3}\left[\frac{f(a)+f(b)}{2}+2f(h^{-1}(x))\right]\right|\\&\leqslant\left(\frac{{}_{RL}\mathrm{D}_{a,t}^{-q}g(b)}{3}+\left|\frac{x}{\Gamma(q)}-\frac{{}_{RL}\mathrm{D}_{a,t}^{-q}g(b)}{2}\right|\right)\bigvee_a^b(f).\end{aligned}$$

Thus, the proof is completed. □

COROLLARY 1. *Let $f:[a,b]\longmapsto\mathbb{R}$ be a mapping bounded variation. Then the following inequality holds for $0\leqslant q\leqslant 1$.*

$$\begin{aligned}&\left|{}_{RL}\mathrm{D}_{a,t}^{-q}f(b)-\frac{(b-a)^q}{3\Gamma(q+1)}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{b+a}{2}\right)\right]\right|\\&\leqslant\frac{(b-a)^q}{\Gamma(q+1)}\left(\frac{1}{2^q}-\frac{1}{6}\right)\bigvee_a^b(f).\end{aligned}$$

Proof. If we choose $g(t)=1$ in Theorem 1, we have $h(t)=\frac{1}{q}[(b-a)^q-(b-t)^q]$, and for $x=\left(\frac{2^q-1}{q2^q}\right)(b-a)^q$, we have $t=h^{-1}(x)=h^{-1}\left(\frac{2^q-1}{q2^q}(b-a)^q\right)=\frac{b+a}{2}$. Then inequality (4) becomes

$$\begin{aligned}
&\left|{}_{RL}\mathrm{D}_{a,t}^{-q}f(b)-\frac{(b-a)^q}{3\Gamma(q+1)}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{b+a}{2}\right)\right]\right|\\
&\leqslant\left(\frac{(b-a)^q}{3\Gamma(q+1)}+\left|\left(\frac{2^q-1}{q2^q}\right)\frac{(b-a)^q}{\Gamma(q)}-\frac{(b-a)^q}{2\Gamma(q+1)}\right|\right)\bigvee_a^b(f)\\
&=\left[\frac{(b-a)^q}{3\Gamma(q+1)}+\frac{(b-a)^q}{\Gamma(q+1)}\left|\frac{2^q-1}{2^q}-\frac{1}{2}\right|\right]\bigvee_a^b(f)\\
&=\frac{(b-a)^q}{\Gamma(q+1)}\left(\frac{1}{3}+\left|\frac{1}{2}-\frac{1}{2^q}\right|\right)\bigvee_a^b(f).
\end{aligned}$$

On the other hand, for $0<q<1$, we have $\frac{1}{2}<\frac{1}{2^q}$, which implies

$$\begin{aligned}
&\left|{}_{RL}\mathrm{D}_{a,t}^{-q}f(b)-\frac{(b-a)^q}{3\Gamma(q+1)}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{b+a}{2}\right)\right]\right|\\
&\leqslant\frac{(b-a)^q}{\Gamma(q+1)}\left(\frac{1}{2^q}-\frac{1}{6}\right)\bigvee_a^b(f).
\end{aligned}$$

The proof is thus completed. □

THEOREM 2. *Let $F\in C([a,b])$ be positive function and be a mapping of bounded variation. Let*

$$h(t)=(n+1)\int_a^t(b-u)^{-p}F'(u)\mathrm{d}s,\ \ t\in[a,b],\ \ n\in\mathbb{N},\ \ 0\leqslant p\leqslant 1.$$

Then

$$\begin{aligned}
&\left|\frac{{}_C\mathrm{D}_{a,t}^{p}F^{n+1}(b)}{n+1}-\frac{{}_C\mathrm{D}_{a,t}^{p}F(b)}{3}\left[\frac{F^n(a)+F^n(b)}{2}+2F^n(h^{-1}(x))\right]\right|\\
&\leqslant\left(\frac{{}_C\mathrm{D}_{a,t}^{p}F(b)}{3}+\left|\frac{x}{\Gamma(1-p)}-\frac{{}_C\mathrm{D}_{a,t}^{p}F(b)}{2}\right|\right)\bigvee_a^b(F^n),
\end{aligned}$$

where $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$.

Proof. Using Theorem 1, let $q=1-p$, $f(t)=(n+1)(F(t))^n$ and $g(t)=F'(t)$, which implies

$$h(t)=(n+1)\int_a^t(b-u)^{-p}F'(u)\mathrm{d}u.$$

One has

$$\begin{aligned}&\Bigg|{}_{RL}\mathrm{D}_{a,t}^{-(1-p)}(n+1)(F^nF')(b)-\frac{{}_{RL}\mathrm{D}_{a,t}^{-(1-p)}F'(b)}{3}\\&\times\left[\frac{(n+1)F^n(a)+(n+1)F^n(b)}{2}+2(n+1)F^n(h^{-1}(x))\right]\Bigg|\\&\leqslant(n+1)\left(\frac{{}_{RL}\mathrm{D}_{a,t}^{-(1-p)}F'(b)}{3}+\left|\frac{x}{\Gamma(1-p)}-\frac{{}_{RL}\mathrm{D}_{a,t}^{-(1-p)}F'(b)}{2}\right|\right)\bigvee_a^b(F^n).\end{aligned}$$

Then

$$\begin{aligned}&\left|\frac{{}_{RL}\mathrm{D}_{a,t}^{1-p}(F^{n+1})'(b)}{n+1}-\frac{{}_{RL}\mathrm{D}_{a,t}^{1-p}F'(b)}{3}\left[\frac{F^n(a)+F^n(b)}{2}+2F^n(h^{-1}(x))\right]\right|\\&\leqslant\left(\frac{{}_{RL}\mathrm{D}_{a,t}^{1-p}F'(b)}{3}+\left|\frac{x}{\Gamma(1-p)}-\frac{{}_{RL}\mathrm{D}_{a,t}^{1-p}F'(b)}{2}\right|\right)\bigvee_a^b(F^n).\end{aligned}$$

Thus

$$\begin{aligned}&\left|\frac{{}_{C}\mathrm{D}_{a,t}^{p}F^{n+1}(b)}{n+1}-\frac{{}_{C}\mathrm{D}_{a,t}^{p}F(b)}{3}\left[\frac{F^n(a)+F^n(b)}{2}+2F^n(h^{-1}(x))\right]\right|\\&\leqslant\left(\frac{{}_{C}\mathrm{D}_{a,t}^{p}F(b)}{3}+\left|\frac{x}{\Gamma(1-p)}-\frac{{}_{C}\mathrm{D}_{a,t}^{p}F(b)}{2}\right|\right)\bigvee_a^b(F^n).\end{aligned}$$

The proof is finished. □

THEOREM 3. *Let $g:[a,b]\longrightarrow\mathbb{R}$ be positive and continuous function and f be a mapping of bounded variation. Set*

$$h(t)=\int_a^t\left(\log\frac{b}{u}\right)^{q-1}g(u)\frac{\mathrm{d}u}{u},\ \ t\in[a,b]. \tag{9}$$

Then the inequality

$$\begin{aligned}&\left|{}_{H}\mathrm{D}_{a,t}^{-q}(fg)(b)-\frac{{}_{H}\mathrm{D}_{a,t}^{-q}g(b)}{3}\left[\frac{f(a)+f(b)}{2}+2f(h^{-1}(x))\right]\right|\\&\leqslant\left(\frac{{}_{H}\mathrm{D}_{a,t}^{-q}g(b)}{3}+\left|\frac{x}{\Gamma(q)}-\frac{{}_{H}\mathrm{D}_{a,t}^{-q}g(b)}{2}\right|\right)\bigvee_a^b(f)\end{aligned} \tag{10}$$

holds for $x\in\left[\frac{h(b)}{6},\frac{5h(b)}{6}\right]$ and $q\geqslant 0$.

Proof. The proof is similar to Theorem 1, so the details are omitted here. □

COROLLARY 2. *Let $f : [a,b] \longmapsto \mathbb{R}$ be a mapping bounded variation, then the following inequality holds for $0 \leqslant q \leqslant 1$.*

$$\left| {}_H\mathrm{D}_{a,t}^{-q} f(b) - \frac{\left(\log\frac{b}{a}\right)^q}{3\Gamma(q+1)}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{b+a}{2}\right)\right]\right| \leqslant \frac{1}{\Gamma(q+1)}\left[\frac{\left(\log\frac{b}{a}\right)^q}{3} + \frac{1}{2}\left|\left(\log\frac{(a+b)^2}{4ab}\right)^q\right|\right]\bigvee_a^b(f).$$

Proof. If we choose $g(t) = 1$ in Theorem 3, we get $h(t) = \frac{1}{q}\left[\left(\log\frac{b}{a}\right)^q - \left(\log\frac{b}{t}\right)^q\right]$, and for $x = \frac{1}{q}\left(\left(\log\frac{b}{a}\right)^q - \left(\log\frac{2b}{a+b}\right)^q\right)$, we have

$$t = h^{-1}(x) = h^{-1}\left[\frac{1}{q}\left(\left(\log\frac{b}{a}\right)^q - \left(\log\frac{2b}{a+b}\right)^q\right)\right] = \frac{b+a}{2}.$$

By inequality (10), one has

$$\begin{aligned}
&\left| {}_H\mathrm{D}_{a,t}^{-q} f(b) - \frac{\left(\log\frac{b}{a}\right)^q}{3\Gamma(q+1)}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{b+a}{2}\right)\right]\right| \\
&\leqslant \left(\frac{\left(\log\frac{b}{a}\right)^q}{3\Gamma(q+1)} + \left|\frac{1}{\Gamma(q+1)}\left(\left(\log\frac{b}{a}\right)^q - \left(\log\frac{2b}{a+b}\right)^q\right) - \frac{\left(\log\frac{b}{a}\right)^q}{2\Gamma(q+1)}\right|\right)\bigvee_a^b(f) \\
&= \left[\frac{\left(\log\frac{b}{a}\right)^q}{3\Gamma(q+1)} + \frac{1}{\Gamma(q+1)}\left|\frac{1}{2}\left(\log\frac{b}{a}\right)^q - \left(\log\frac{2b}{a+b}\right)^q\right|\right]\bigvee_a^b(f).
\end{aligned}$$

On the other hand, for $0 < q < 1$ and $B \geqslant A \geqslant 0$, we have $B^q - A^q \leqslant (B-A)^q$, which implies

$$\begin{aligned}
&\left| {}_H\mathrm{D}_{a,t}^{-q} f(b) - \frac{\left(\log\frac{b}{a}\right)^q}{3\Gamma(q+1)}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{b+a}{2}\right)\right]\right| \\
&\leqslant \left[\frac{\left(\log\frac{b}{a}\right)^q}{3\Gamma(q+1)} + \frac{1}{\Gamma(q+1)}\left|\frac{1}{2}\left(\log\frac{a+b}{2a}\right)^q - \frac{1}{2}\left(\log\frac{2b}{a+b}\right)^q\right|\right]\bigvee_a^b(f) \\
&\leqslant \left[\frac{\left(\log\frac{b}{a}\right)^q}{3\Gamma(q+1)} + \frac{1}{2\Gamma(q+1)}\left|\left(\log\frac{(a+b)^2}{4ab}\right)^q\right|\right]\bigvee_a^b(f).
\end{aligned}$$

So the proof is finished. □

COROLLARY 3. *Let $F \in C([a,b])$ be positive function and be a mapping of bounded variation. Let*

$$h(t) = (n+1)\int_a^t \left(\log\frac{b}{u}\right)^{-p} F'(u)\mathrm{d}u, \ \ t \in [a,b], \ \ n \in \mathbb{N}, \ \ 0 \leqslant p \leqslant 1.$$

Then

$$\left|\frac{{}_{CH}\mathrm{D}^{p}_{a,t}F^{n+1}(b)}{n+1}-\frac{{}_{CH}\mathrm{D}^{p}_{a,t}F(b)}{3}\left[\frac{F^{n}(a)+F^{n}(b)}{2}+2F^{n}(h^{-1}(x))\right]\right|$$
$$\leqslant\left(\frac{{}_{CH}\mathrm{D}^{p}_{a,t}F(b)}{3}+\left|\frac{x}{\Gamma(1-p)}-\frac{{}_{CH}\mathrm{D}^{p}_{a,t}F(b)}{2}\right|\right)\bigvee_{a}^{b}(F^{n}).$$

Proof. If we apply Theorem 3 for $q=1-p$, $f(t)=(n+1)(F(t))^{n}$ and $g(t)=tF'(t)$, which implies

$$h(t)=(n+1)\int_{a}^{t}\left(\log\frac{b}{u}\right)^{-p}F'(u)\mathrm{d}u.$$

Thus, we find the desired inequality. □

Next, we give the approximate schemes of the proposed integral inequalities. For convenience, we make the following hypothesis.

Hypothesis $[H]$: Assume that $G:[a,b]\longrightarrow\mathbb{R}$ be a positive and continuous function, $f:[a,b]\longrightarrow\mathbb{R}$ be a mapping of bounded variation. Let $I_n: a=x_0<x_1<\ldots<x_n=b$ be a partition of $[a,b]$, $\xi_i\in\left[\frac{h_i(x_{i+1})}{6},\frac{5h_i(x_{i+1})}{6}\right]$, and $m_i=h_i(x_{i+1})=\int_{x_i}^{x_{i+1}}(b-t)^{q-1}G(t)\mathrm{d}t$, where $i=0,1,\ldots,n-1$.

THEOREM 4. *Let f,G,h_i and m_i be defined as Hypothesis $[H]$.*
Let $M=\max\limits_{0\leqslant i\leqslant n-1}\{m_i\}$. When $0\leqslant q\leqslant 1$, the following inequality holds

$$\left|\Gamma(q)_{RL}\mathrm{D}^{-q}_{a,t}(fG)(b)-\sum_{i=0}^{n-1}\frac{m_i}{3}\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f(h_i^{-1}(\xi_i))\right)\right|\leqslant\frac{2}{3}M\bigvee_{a}^{b}(f).$$

Proof. Set

$$g_i(t)=\frac{(b-t)^{q-1}}{(x_{i+1}-t)^{q-1}}G(t),\ \ i=0,1,\ldots,n,$$

and

$$h_i(t)=\int_{x_i}^{t}(x_{i+1}-u)^{q-1}g_i(u)\mathrm{d}u,\ \ i=0,1,\ldots,n.$$

By Theorem 1, for h_i and $[x_i,x_{i+1}]$, we find

$$\left|{}_{RL}\mathrm{D}^{-q}_{x_i,t}(fg_i)(x_{i+1})-\frac{{}_{RL}\mathrm{D}^{-q}_{x_i,t}g_i(x_{i+1})}{3}\left[\frac{f(x_i)+f(x_{i+1})}{2}+2f(h_i^{-1}(\xi_i))\right]\right|$$
$$\leqslant\left(\frac{{}_{RL}\mathrm{D}^{-q}_{x_i,t}g_i(x_{i+1})}{3}+\left|\frac{\xi_i}{\Gamma(q)}-\frac{{}_{RL}\mathrm{D}^{-q}_{x_i,t}g_i}{2}\right|\right)\bigvee_{x_i}^{x_{i+1}}(f).$$

Thus

$$\left|\frac{1}{\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}f(t)G(t)\mathrm{d}t - \frac{1}{3\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}G(t)\mathrm{d}t\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f(h_i^{-1}(\xi_i))\right)\right|$$
$$\leqslant\left(\frac{1}{3\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}G(t)\mathrm{d}t+\left|\frac{\xi_i}{\Gamma(q)}-\frac{1}{2\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}G(t)\mathrm{d}t\right|\right)\bigvee_{x_i}^{x_{i+1}}(f).$$

Now we use the fact that $|A+B|\leqslant|A|+|B|$, which yields that

$$\left|\sum_{i=0}^{n-1}\frac{1}{\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}f(t)G(t)\mathrm{d}t - \sum_{i=0}^{n-1}\frac{1}{3\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}G(t)\mathrm{d}t\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f(h_i^{-1}(\xi_i))\right)\right|$$
$$\leqslant\sum_{i=0}^{n-1}\left(\frac{1}{3\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}G(t)\mathrm{d}t+\left|\frac{\xi_i}{\Gamma(q)}-\frac{1}{2\Gamma(q)}\int_{x_i}^{x_{i+1}}(b-t)^{q-1}G(t)\mathrm{d}t\right|\right)\bigvee_{x_i}^{x_{i+1}}(f).$$

Then we get

$$\left|\frac{1}{\Gamma(q)}\int_a^b(b-t)^{q-1}f(t)G(t)\mathrm{d}t-\sum_{i=0}^{n-1}\frac{m_i}{3\Gamma(q)}\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f(h_i^{-1}(\xi_i))\right)\right|$$
$$\leqslant\sum_{i=0}^{n-1}\left(\frac{m_i}{3\Gamma(q)}+\left|\frac{\xi_i}{\Gamma(q)}-\frac{mi}{2\Gamma(q)}\right|\right)\bigvee_{x_i}^{x_{i+1}}(f).$$

Therefore

$$\left|{}_{RL}\mathrm{D}_{a,t}^{-q}(fG)(b)-\sum_{i=0}^{n-1}\frac{m_i}{3\Gamma(q)}\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f(h_i^{-1}(\xi_i))\right)\right|$$
$$\leqslant\frac{1}{\Gamma(q)}\max_{i=0,\dots,n-1}\left(\frac{m_i}{3}+\left|\xi_i-\frac{m_i}{2}\right|\right)\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)$$
$$\leqslant\frac{1}{\Gamma(q)}\left(\frac{m_i}{3}+\max_{i=0,\dots,n-1}\left|\xi_i-\frac{m_i}{2}\right|\right)\bigvee_a^b(f).$$

Due to $M=\max\limits_{0\leqslant i\leqslant n-1}\{m_i\}$ and $\xi_i\in\left[\frac{m_i}{6},\frac{5m_i}{6}\right]$, it is easy to obtain that

$$\left|\Gamma(q)_{RL}\mathrm{D}_{a,t}^{-q}(fG)(b)-\sum_{i=0}^{n-1}\frac{m_i}{3}\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f\left(h_i^{-1}(\xi_i)\right)\right)\right|$$
$$\leqslant\frac{2}{3}M\bigvee_a^b(f).$$

This ends the proof. $\square$

THEOREM 5. *Let* $T(x) = \int_a^x (b-u)^{q-1} G(u)\mathrm{d}u,\ x \in [a,b]$. *Suppose that*

$$x_i = T^{-1}\left(\frac{i\Gamma(q)_{RL}\mathrm{D}_{a,t}^{-q}G(b)}{n}\right), \quad i = 0, 1, \ldots, n.$$

If $h_i(x) = \int_{x_i}^x (b-u)^{-q} G(u)\mathrm{d}u$ *and* $\xi_i = \frac{2^{\frac{1}{n}} h_i(x_{i+1})}{3}$, $i = 0, 1, \ldots, n-1$. *Then*

$$\begin{aligned} {}_{RL}\mathrm{D}_{a,t}^{-q}(fG)(b) = & \frac{{}_{RL}\mathrm{D}_{a,t}^{-q}(G)(b)}{3n} \sum_{i=0}^{n-1}\left[\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(h_i^{-1}\left(\frac{2^{\frac{1}{n}} h_i(x_{i+1})}{3}\right)\right)\right] \\ & + S(I_n, f, G, \xi, q), \end{aligned}$$

where

$$|S(I_n, f, G, \xi, q)| \leqslant \frac{(5 - 2^{\frac{1}{n}+1})_{RL}\mathrm{D}_{a,t}^{-q}(G)(b)}{6n\Gamma(q)} \bigvee_a^b (f).$$

Proof. In fact

$$\begin{aligned} & \left|\Gamma(q)_{RL}\mathrm{D}_{a,t}^{-q}(fG)(b) - \sum_{i=0}^{n-1} \frac{m_i}{3}\left(\frac{f(x_i)+f(x_{i+1})}{2} + 2f\left(h_i^{-1}(\xi_i)\right)\right)\right| \\ & \leqslant \sum_{i=0}^{n-1}\left(\frac{m_i}{3\Gamma(q)} + \left|\frac{\xi_i}{\Gamma(q)} - \frac{mi}{2\Gamma(q)}\right|\right) \bigvee_{x_i}^{x_{i+1}}(f). \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \sum_{i=0}^{n-1}\left(\frac{m_i}{3\Gamma(q)} + \left|\frac{\xi_i}{\Gamma(q)} - \frac{mi}{2\Gamma(q)}\right|\right) \bigvee_{x_i}^{x_{i+1}}(f) &= \frac{1}{\Gamma(q)} \sum_{i=0}^{n-1}\left(\frac{1}{3} + \left|\frac{2^{\frac{1}{n}}}{3} - \frac{1}{2}\right|\right) m_i \bigvee_{x_i}^{x_{i+1}}(f) \\ &= \frac{5 - 2^{\frac{1}{n}+1}}{6\Gamma(q)} \sum_{i=0}^{n-1} m_i \bigvee_{x_i}^{x_{i+1}}(f). \end{aligned}$$

It is evident that

$$\begin{aligned} m_i = h_i(x_{i+1}) &= T(x_{i+1}) - T(x_i) \\ &= \frac{(i+1)\Gamma(q)_{RL}\mathrm{D}_{a,t}^{-q}G(b)}{n} - \frac{i\Gamma(q)_{RL}\mathrm{D}_{a,t}^{-q}G(b)}{n} \\ &= \frac{1}{n}\Gamma(q)_{RL}\mathrm{D}_{a,t}^{-q}G(b). \end{aligned}$$

So the following estimation holds

$$\begin{aligned}
&\left|{}_{RL}\mathrm{D}_{a,t}^{-q}(fG)(b)-\frac{{}_{RL}\mathrm{D}_{a,t}^{-q}(G)(b)}{3n}\sum_{i=0}^{n-1}\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f\left(h_i^{-1}\left(\frac{2^{\frac{1}{n}}h_i(x_{i+1})}{3}\right)\right)\right)\right|\\
&\leqslant \frac{(5-2^{\frac{1}{n}+1})_{RL}\mathrm{D}_{a,t}^{-q}(G)(b)}{6n\Gamma(q)}\sum_{i=0}^{n-1}\bigvee_{x_i}^{x_{i+1}}(f)\\
&=\frac{(5-2^{\frac{1}{n}+1})_{RL}\mathrm{D}_{a,t}^{-q}(G)(b)}{6n\Gamma(q)}\bigvee_a^b(f).
\end{aligned}$$

This completes the proof. □

Similarly, we have the following assertions.

THEOREM 6. *Suppose that $g:[a,b]\longrightarrow\mathbb{R}$ is positive and continuous function and f is a monotonic mapping of bounded variation. Let*

$$h(t)=\int_a^t\left(\log\frac{b}{u}\right)^{q-1}g(u)\frac{\mathrm{d}u}{u},\ \ t\in[a,b].$$

Then

$$\begin{aligned}
&\left|\Gamma(q)_H\mathrm{D}_{a,t}^{-q}(fG)(b)-\sum_{i=0}^{n-1}\frac{m_i}{3}\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f\left(h_i^{-1}(\xi_i)\right)\right)\right|\\
&\leqslant\frac{2}{3}M|f(b)-f(a)|.
\end{aligned}$$

THEOREM 7. *Let f and G be defined as Hypothesis $[H]$. If $0\leqslant q\leqslant 1$, $M^H=\max\limits_{0\leqslant i\leqslant n-1}\{m_i^H\}$, where $m_i^H=h_i(x_{i+1})$ and $h_i(x)=\int_{x_i}^x\log\frac{b}{u}G(u)\frac{\mathrm{d}u}{u}$. The following inequality holds*

$$\begin{aligned}
&\left|\Gamma(q)_H\mathrm{D}_{a,t}^{-q}(fG)(b)-\sum_{i=0}^{n-1}\frac{m_i^H}{3}\left(\frac{f(x_i)+f(x_{i+1})}{2}+2f(h_i^{-1}(\xi_i))\right)\right|\\
&\leqslant\frac{2}{3}M^H\bigvee_a^b(f).
\end{aligned}$$

THEOREM 8. *Let $T(x)=\int_a^x\left(\log\frac{b}{u}\right)^{q-1}G(u)\frac{\mathrm{d}u}{u}$, $x\in[a,b]$. Suppose that*

$$x_i=T^{-1}\left(\frac{i\Gamma(q)_H\mathrm{D}_{a,t}^{-q}G(b)}{n}\right),\ \ i=0,1,\dots,n.$$

If $h_i(x) = \int_{x_i}^{x} \left(\log \frac{b}{u}\right)^{-q} G(u) \frac{\mathrm{d}u}{u}$ *and* $\xi_i = \frac{2^{\frac{1}{n}} h_i(x_{i+1})}{3}$, $i = 0, 1, \ldots, n-1$. *Then*

$$
\begin{aligned}
{}_H\mathrm{D}_{a,t}^{-q}(fG)(b) = & \frac{{}_H\mathrm{D}_{a,}^{-q}(G)(b)}{3n} \sum_{i=0}^{n-1} \left[\frac{f(x_i) + f(x_{i+1})}{2} + 2f\left(h_i^{-1}\left(\frac{2^{\frac{1}{n}} h_i(x_{i+1})}{3} \right) \right) \right] \\
& + S_H\left(I_n, f, G, \xi, q\right),
\end{aligned}
$$

where

$$
|S_H(I_n, f, G, \xi, q)| \leqslant \frac{(5 - 2^{\frac{1}{n}+1})_H\mathrm{D}_{a,t}^{-q}(G)(b)}{6n\Gamma(q)} \bigvee_a^b (f).
$$

4. Conclusion

In the present paper, some classical Simpson integral inequalities are extended to fractional order cases. The results involving a mapping of bounded variation function have been also generated. The approximate schemes of the proposed integral inequalities are studied as well.

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Mohamed Doubbi Bounoua
Department of Mathematics
Shanghai University
Shanghai 200444, China
e-mail: doubbibounoua.mohamed@yahoo.fr

Chuntao Yin
Department of Mathematics
Shanghai University
Shanghai 200444, China
and
Department of Mathematics and Physics
Shijiazhuang Tiedao University
Shijiazhuang 050043, China
e-mail: yct@stdu.edu.cn