Journal of Mathematical Inequalities
Volume 15, Number 4 (2021), 1487–1497 doi:10.7153/jmi-2021-15-102

# FUZZY MEANS AND HGA–TYPE INEQUALITIES

JOANNA NIKODEM AND KAZIMIERZ NIKODEM

(*Communicated by J. Pečarić*)

*Abstract.* The notion of fuzzy means of fuzzy numbers is introduced. Fuzzy counterparts of the arithmetic, geometric and harmonic means are investigated and inequalities between them are presented.

## 1. Introduction

Let $I \subset \mathbb{R}$ be an interval and $n \geqslant 2$ be an integer. A function $M : I^n \to I$ is said to be a mean if

$$\min\{x_1,\dots,x_n\} \leqslant M(x_1,\dots,x_n) \leqslant \max\{x_1,\dots,x_n\}, \tag{1}$$

for all $x_1,\dots,x_n \in I$.

The classical means, such as arithmetic, geometric or harmonic, were known and investigated already in the ancient time. At present the theory of means is a well developed mathematical theory having various applications in mathematics itself as well as in economics, information theory, engineering and natural science. There are books and numerous papers devoted to it (see e.g. [2, 3, 5, 11, 13, 17] and the references given there).

The theory of fuzzy sets, since its introduction over fifty years ago by Zadeh [18], has found wide applications in engineering, economics,information sciences, medicine, etc. (see e.g. [4, 7, 10, 19, 20] and the reference therein). It provides a framework for mathematical modeling of all situations which involve an element of uncertainty or imprecision in their description.

The aim of this paper is to generalize the classical definition of mean (1) to the case of fuzzy numbers. We introduce the general notion of fuzzy means and, as examples, define the arithmetic, geometric and harmonic fuzzy means. We prove that the classical inequalities $H \leqslant G \leqslant A$ between the harmonic, geometric and arithmetic means can be extended on fuzzy means. The notion of quasi-arithmetic fuzzy means is also introduced and investigated.

*Mathematics subject classification* (2020): Primary 26E60; Secondary 26E50, 04A72, 39B62.
*Keywords and phrases*: Means, fuzzy numbers, fuzzy means, inequalities between fuzzy means.


Paper JMI-15-102

## 2. Preliminaries

Let us recall some basic notions and definitions needed in this paper. As usually a fuzzy set $A$ in a space $Z$ is characterized by its membership function $\mu_A : Z \to [0,1]$, where the value $\mu_A(z)$ is the grade of membership of $z$ in $A$. If $A$ is ordinary (crisp) subset of $Z$ its membership function coincides with the characteristic function

$$\mu_A(z) = \chi_A(z) = \begin{cases} 1, \text{ if } z \in A \\ 0, \text{ if } z \notin A. \end{cases}$$

Given a fuzzy set $A$ and a number $\alpha \in (0,1]$ we define the $\alpha$-cat of $A$ by $A_\alpha = \{z \in Z : \mu_A(z) \geqslant \alpha\}$. The set $supp\, A = \{z \in Z : \mu_A(z) > 0\}$ is called the support of $A$.

By a *fuzzy number* $X$ we mean a fuzzy set of the real line $\mathbb{R}$ whose membership function $\mu_X : \mathbb{R} \to [0,1]$ satisfies the conditions:

(i) $\mu_X$ is normalized (i.e. $\mu_X(y) = 1$ for some $y \in \mathbb{R}$);

(ii) $\mu_X$ is quasi-concave, i.e.

$$\mu_X\big(ty_1 + (1-t)y_2\big) \geqslant \min\{\mu_X(y_1), \mu_X(y_2)\}, \text{ for all } t \in [0,1] \text{ and } y_1, y_2 \in \mathbb{R};$$

(iii) $\mu_X$ is upper semi-continuous.

We assume, moreover, that the support of $X$ is bounded.

Denote by $\mathscr{F}(\mathbb{R})$ the family of all fuzzy numbers. We say that $X \in \mathscr{F}(\mathbb{R})$ is positive if $\mu_X(y) = 0$ for all $y < 0$. Using the Zadeh extension principle [19] we define the basic arithmetic and functional operations on fuzzy numbers. In particular, for $X_1, \dots, X_n \in \mathscr{F}(\mathbb{R})$ the sum $X_1 + \dots + X_n$ and the product $X_1 \cdots X_n$ are determined by their membership functions

$$\mu_{X_1+\dots+X_n}(y) = \sup\big\{\min\{\mu_{X_1}(y_1), \dots, \mu_{X_n}(y_n)\} \mid y_1, \dots, y_n \in \mathbb{R},\ y = y_1 + \dots + y_n\big\}$$
$$\mu_{X_1\cdots X_n}(y) = \sup\big\{\min\{\mu_{X_1}(y_1), \dots, \mu_{X_n}(y_n)\} \mid y_1, \dots, y_n \in \mathbb{R},\ y = y_1 \cdots y_n\big\}.$$

If $X \in \mathscr{F}(\mathbb{R})$ and $\lambda \in \mathbb{R},\ \lambda \neq 0$, then $\lambda X$ is defined by

$$\mu_{\lambda X}(y) = \mu_X\left(\frac{1}{\lambda} y\right).$$

It is known that the sum and product of fuzzy numbers, as well as the product of a fuzzy number by a scalar are fuzzy numbers (see e.g. [7, 8]).

Given an ordinary (crisp) function $f : \mathbb{R} \to \mathbb{R}$ and a fuzzy number $X$ we define the fuzzy image $f(X)$ by

$$\mu_{f(X)}(y) = \begin{cases} \sup\big\{\mu_X(t) \mid t \in f^{-1}(y)\big\}, & \text{if } f^{-1}(y) \neq \emptyset \\ 0, & \text{othervise.} \end{cases} \tag{2}$$

If $f$ is strictly monotonic on an interval $I \subset \mathbb{R}$, then for every $y \in f(I)$ there exists exactly one $t \in I$ such that $y = f(t)$, and then condition (2) reduces to

$$\mu_{f(X)}(y) = \mu_X(f^{-1}(y)),\ y \in f(I). \tag{3}$$

In particular, applying (3) for $f(t) = 1/t$, $t > 0$ and $f(t) = \sqrt[n]{t}$, $t \geqslant 0$, we define, for a positive fuzzy number $X$, the inverse and the $n$-th root ($n \geqslant 2$) of $X$ putting

$$\mu_{\frac{1}{X}}(y) = \mu_X\left(\frac{1}{y}\right), \ \ y > 0$$
$$\mu_{\sqrt[n]{X}}(y) = \mu_X(y^n), \ \ y \geqslant 0,$$

respectively. From the next proposition it follows that $\frac{1}{X}$ and $\sqrt[n]{X}$ are fuzzy numbers (cf. also [1, 7, 15]).

PROPOSITION 1. Let $X$ be a fuzzy number with $supp\, X \subset I$ and $f : I \to \mathbb{R}$ be a continuous strictly monotonic function. Then $f(X)$ is also a fuzzy number.

*Proof.* Since $\mu_X$ is normalized, we have $\mu_X(y_0) = 1$ for some $y_0 \in \ supp\, X$. Then $\mu_{f(X)}(f(y_0)) = \mu_X(y_0) = 1$ and so $\mu_{f(X)}$ is normalized.

Assume now that $y_1, y_2 \in f(I)$, $t \in [0,1]$ and $\overline{y} = ty_1 + (1-t)y_2$. Since $f$ is strictly monotonic, also $f^{-1}$ is strictly monotonic and hence $f^{-1}(\overline{y})$ is between $f^{-1}(y_1)$ and $f^{-1}(y_2)$. From here, by the quasi-concavity of $\mu_X$, we get

$$\mu_{f(X)}(ty_1 + (1-t)y_2) = \mu_{f(X)}(\overline{y}) = \mu_X(f^{-1}(\overline{y}))$$
$$\geqslant \min\{\mu_X(f^{-1}(y_1)), \mu_X(f^{-1}(y_2))\} = \min\{\mu_{f(X)}(y_1), \mu_{f(X)}(y_2)\},$$

which shows that $\mu_{f(X)}$ is quasi-concave.

Since $f$ is continuous and strictly monotonic, also $f^{-1}$ is continuous. Hence, using the fact that $\mu_X$ is upper semi-continuous, we infer that $\mu_{f(X)} = \mu_X \circ f^{-1}$ is also upper semi-continuous. Thus $f(X)$ is a fuzzy number. $\square$

Since the membership function of any fuzzy number $X$ is quasi-concave and upper semi-continuous, all the $\alpha$-cats of $X$ are closed intervals. Denote

$$\left[\underline{x}_\alpha, \overline{x}_\alpha\right] = X_\alpha = \{y \in \mathbb{R} : \ \mu_X(y) \geqslant \alpha\}, \ \alpha \in (0,1].$$

The inequality relation between two fuzzy numbers is defined by use of their $\alpha$-cats. Let $X, Y \in \mathscr{F}(\mathbb{R})$. We say that $X$ is not greater than $Y$ (and write $X \leqslant Y$) if

$$\underline{x}_\alpha \leqslant \underline{y}_\alpha \ \text{ and } \ \overline{x}_\alpha \leqslant \overline{y}_\alpha$$

for every $\alpha \in (0,1]$ (see e.g. [14]).

## 3. Fuzzy means

In this section we introduce the general definition of a fuzzy mean and consider arithmetic, geometric and harmonic fuzzy means. Given a set $A \subset \mathbb{R}$ we denote by $\operatorname{conv} A$ the convex hull of $A$. Assume that $n \geqslant 2$.

DEFINITION 1. We say that a function $\widetilde{M} : \mathscr{F}(\mathbb{R})^n \to \mathscr{F}(\mathbb{R})$ is a *fuzzy mean* if

$$\begin{aligned}&\mu_{\widetilde{M}(X_1,\dots,X_n)}(y)\\ &\leqslant \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y \in \operatorname{conv}\{y_1,\dots,y_n\}\big\}\end{aligned} \tag{4}$$

for all $X_1,\dots,X_n \in \mathscr{F}(\mathbb{R})$ and $y \in \mathbb{R}$.

PROPOSITION 2. If $\widetilde{M}$ is a fuzzy mean, then

$$\mathit{supp}\ \widetilde{M}(X_1,\dots,X_n) \subset \operatorname{conv}(\mathit{supp}\ X_1 \cup \dots \cup \mathit{supp}\ X_n) \tag{5}$$

for all $X_1,\dots,X_n \in \mathscr{F}(\mathbb{R})$. If $X_1,\dots,X_n \in \mathbb{R}$ (are ordinary numbers), then

$$\mathit{supp}\ \widetilde{M}(X_1,\dots,X_n) \subset \big[\min\{X_1,\dots,X_n\}, \max\{X_1,\dots,X_n\}\big]. \tag{6}$$

*Proof.* Take $y \in \mathit{supp}\ \widetilde{M}(X_1,\dots,X_n)$. Then

$$\mu_{\widetilde{M}(X_1,\dots,X_n)}(y) > 0. \tag{7}$$

Suppose, contrary to our claim, that $y \notin \operatorname{conv}(\mathit{supp}\ X_1 \cup \dots \cup \mathit{supp}\ X_n)$. Then for all $y_1,\dots,y_n \in \mathbb{R}$ with $y \in \operatorname{conv}\{y_1,\dots,y_n\}$ there exists an $i \in \{1,\dots,n\}$ such that $y_i \notin \mathit{supp}\ X_i$. Hence $\mu_{X_i}(y_i) = 0$ and, consequently,

$$\begin{aligned}\mu_{\widetilde{M}(X_1,\dots,X_n)}(y) &\leqslant \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y \in \operatorname{conv}\{y_1,\dots,y_n\}\big\}\\ &= 0,\end{aligned}$$

which contradicts (7) and proves (5).

If $X_1,\dots,X_n \in \mathbb{R}$, then $\mathit{supp}\ X_i = \{x_i\}$, $i = 1,\dots,n$. From here

$$\operatorname{conv}(\mathit{supp}\, X_1 \cup \dots \cup \mathit{supp}\, X_n) = \operatorname{conv}\{X_1,\dots X_n\} = \big[\min\{X_1,\dots,X_n\}, \max\{X_1,\dots,X_n\}\big],$$

which proves (6). □

PROPOSITION 3. Every ordinary mean $M : \mathbb{R}^n \to \mathbb{R}$ is a fuzzy mean.

*Proof.* Assume that $M$ is a mean and fix $X_1,\dots,X_n \in \mathbb{R}$. Then

$$\mu_{M(X_1,\dots,X_n)}(y) = \chi_{M(X_1,\dots,X_n)}(y).$$

Hence, if $y \neq M(X_1,\dots,X_n)$, then $\mu_{M(X_1,\dots,X_n)}(y) = 0$ and condition (4) in Definition 1 is satisfied. If $y = M(X_1,\dots,X_n)$, then by the definition of means

$$y \in \big[\min\{X_1,\dots,X_n\}, \max\{X_1,\dots,X_n\}\big] = \operatorname{conv}\{X_1,\dots,X_n\}.$$

Taking $y_i = X_i$, $i = 1,\dots,n$, we get

$$y \in \operatorname{conv}\{y_1,\dots,y_n\} \text{ and } \mu_{X_i}(y_i) = 1,\ i = 1,\dots,n.$$

Hence

$$\sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y \in \operatorname{conv}\{y_1,\dots,y_n\}\big\} = 1$$

which shows that condition (4) is also satisfied. □

REMARK 1. Let $S(\mathbb{R})$ be the family of all nonempty subsets of $\mathbb{R}$. Following [16] a map $\mathbb{M}:\mathbb{R}^n \to S(\mathbb{R})$ is called a set-valued mean if

$$\mathbb{M}(X_1,\dots,X_n) \subset \operatorname{conv}\{X_1,\dots,X_n\},$$

for all $X_1,\dots,X_n \in \mathbb{R}$. Note that such set-valued means satisfy also condition (4) in Definition 1. Indeed, assume that $X_1,\dots,X_n \in \mathbb{R}$. If $y \notin \mathbb{M}(X_1,\dots,X_n)$, then $\mu_{\mathbb{M}(X_1,\dots,X_n)} = 0$ and (4) is satisfied trivially. If $y \in \mathbb{M}(X_1,\dots,X_n)$, then taking $y_i = X_i$, $i=1,\dots,n$, we get

$$\sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y \in \operatorname{conv}\{y_1,\dots,y_n\}\big\} = 1$$

and (4) is also satisfied.

The classical arithmetic, geometric and harmonic means defined by

$$\begin{aligned} A(y_1,\dots,y_n) &= \frac{y_1+\dots+y_n}{n}, \quad y_1,\dots,y_n \in \mathbb{R}, \\ G(y_1,\dots,y_n) &= \sqrt[n]{y_1\cdots y_n}, \quad y_1,\dots,y_n \geqslant 0, \\ H(y_1,\dots,y_n) &= \frac{n}{\frac{1}{y_1}+\dots+\frac{1}{y_n}}, \quad y_1,\dots,y_n > 0, \end{aligned}$$

can be extended in a natural way on fuzzy numbers (see [6, 12, 9] for some related results).

Given fuzzy numbers $X_1,\dots,X_n \in \mathscr{F}(\mathbb{R})$, we define their arithmetic fuzzy mean by

$$\widetilde{A}(X_1,\dots,X_n) = \frac{1}{n}\big(X_1+\dots+X_n\big).$$

PROPOSITION 4. The membership function of the arithmetic fuzzy mean of $X_1,\dots,X_n \in \mathscr{F}(\mathbb{R})$ is of the form

$$\begin{aligned} &\mu_{\widetilde{A}(X_1,\dots,X_n)}(y) \\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y = A(y_1,\dots,y_n)\big\}. \end{aligned} \tag{8}$$

*Proof.* By the definitions of the sum of fuzzy numbers and the multiplication of fuzzy numbers by a scalar we obtain

$$\begin{aligned} &\mu_{\widetilde{A}(X_1,\dots,X_n)}(y) = \mu_{\frac{1}{n}(X_1+\dots+X_n)}(y) = \mu_{X_1+\dots+X_n}(ny) \\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y_1+\dots+y_n = ny\big\} \\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y = A(y_1,\dots,y_n)\big\}. \quad \square \end{aligned}$$

REMARK 2. From (8), using the fact that $A(y_1,\dots,y_n) \in \operatorname{conv}\{y_1,\dots,y_n\}$, we obtain

$$\mu_{\widetilde{A}(X_1,\dots,X_n)}(y) \leqslant \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n \in \mathbb{R},\ y \in \operatorname{conv}\{y_1,\dots,y_n\}\big\},$$

which shows that $\widetilde{A}:\mathscr{F}(\mathbb{R})^n \to \mathscr{F}(\mathbb{R})$ is a fuzzy mean in the sense of Definition 1. If $X_1,\ldots,X_n \in \mathbb{R}$ (are ordinary numbers), then $\widetilde{A}(X_1,\ldots,X_n)$ coincides with the classical arithmetic mean $A(X_1,\ldots,X_n)$. Indeed, if $y = A(X_1,\ldots,X_n)$, then taking in (8) $y_i = X_i$ we get $\mu_{\widetilde{A}(X_1,\ldots,X_n)}(y) = 1$. On the other hand, if $y \neq A(X_1,\ldots,X_n)$, then for each $y_1,\ldots,y_n \in \mathbb{R}$ with $y = A(y_1,\ldots,y_n)$ there exists an $i \in \{1,\ldots,n\}$ such that $y_i \neq x_i$, and hence $\mu_{\widetilde{A}(X_1,\ldots,X_n)}(y) = 0$. Thus $\mu_{\widetilde{A}(X_1,\ldots,X_n)}(y) = \chi_{A(X_1,\ldots,X_n)}(y)$ and so $\widetilde{A}(X_1,\ldots,X_n) = A(X_1,\ldots,X_n)$, $X_1,\ldots,X_n \in \mathbb{R}$.

Given positive fuzzy numbers $X_1,\ldots,X_n \in \mathscr{F}(\mathbb{R})$, we define their geometric fuzzy mean by

$$\widetilde{G}(X_1,\ldots,X_n) = \sqrt[n]{X_1\cdots X_n}.$$

PROPOSITION 5. The membership function of the geometric fuzzy mean of positive $X_1,\ldots,X_n \in \mathscr{F}(\mathbb{R})$ is of the form

$$\begin{aligned}&\mu_{\widetilde{G}(X_1,\ldots,X_n)}(y)\\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\ldots,\mu_{X_n}(y_n)\} \mid y_1,\ldots,y_n \geqslant 0,\ y = G(y_1,\ldots,y_n)\big\}.\end{aligned} \tag{9}$$

*Proof.* By the definition of the $n$-th root of fuzzy numbers we obtain

$$\begin{aligned}&\mu_{\widetilde{G}(X_1,\ldots,X_n)}(y) = \mu_{\sqrt[n]{X_1\cdots X_n}}(y) = \mu_{X_1\cdots X_n}(y^n)\\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\ldots,\mu_{X_n}(y_n)\} \mid y_1,\ldots,y_n \geqslant 0,\ y_1\cdots y_n = y^n\big\}\\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\ldots,\mu_{X_n}(y_n)\} \mid y_1,\ldots,y_n \geqslant 0,\ y = G(y_1,\ldots,y_n)\big\}. \quad \square\end{aligned}$$

REMARK 3. From (9), using the fact that $G(y_1,\ldots,y_n) \in \operatorname{conv}\{y_1,\ldots,y_n\}$, we obtain

$$\mu_{\widetilde{G}(X_1,\ldots,X_n)}(y) \leqslant \sup\big\{\min\{\mu_{X_1}(y_1),\ldots,\mu_{X_n}(y_n)\} \mid y_1,\ldots,y_n \geqslant 0,\ y \in \operatorname{conv}\{y_1,\ldots,y_n\}\big\},$$

which shows that $\widetilde{G}$ is a fuzzy mean in the sense of Definition 1. Note also that for ordinary numbers $X_1,\ldots,X_n \geqslant 0$ we have $\mu_{\widetilde{G}(X_1,\ldots,X_n)} = \chi_{G(X_1,\ldots,X_n)}$ and so $\widetilde{G}$ coincides with the classical geometric mean in this case.

Now, given positive fuzzy numbers $X_1,\ldots,X_n \in \mathscr{F}(\mathbb{R})$, we define their harmonic fuzzy mean by

$$\widetilde{H}(X_1,\ldots,X_n) = \frac{n}{\frac{1}{X_1} + \cdots + \frac{1}{X_n}}.$$

PROPOSITION 6. The membership function of the harmonic fuzzy mean of positive $X_1,\ldots,X_n \in \mathscr{F}(\mathbb{R})$ is of the form

$$\begin{aligned}&\mu_{\widetilde{H}(X_1,\ldots,X_n)}(y)\\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\ldots,\mu_{X_n}(y_n)\} \mid y_1,\ldots,y_n > 0,\ y = H(y_1,\ldots,y_n)\big\}.\end{aligned} \tag{10}$$

*Proof.* By the definitions of the inverse, sum and scalar multiplication of fuzzy numbers we get

$$\mu_{\widetilde{H}(X_1,\dots,X_n)}(y) = \mu_{\frac{n}{\frac{1}{X_1}+\dots+\frac{1}{X_n}}}(y) = \mu_{\frac{1}{\frac{1}{X_1}+\dots+\frac{1}{X_n}}}\left(\frac{1}{n}y\right) = \mu_{\frac{1}{X_1}+\dots+\frac{1}{X_n}}\left(\frac{n}{y}\right)$$
$$= \sup\left\{\min\{\mu_{\frac{1}{X_1}}(s_1),\dots,\mu_{\frac{1}{X_n}}(s_n)\} \mid s_1,\dots,s_n > 0,\ s_1+\dots+s_n = \frac{n}{y}\right\}$$
$$= \sup\left\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n > 0,\ y = H(y_1,\dots,y_n)\right\}. \quad \square$$

REMARK 4. From (10), using the fact that $H(y_1,\dots,y_n) \in \operatorname{conv}\{y_1,\dots,y_n\}$, we obtain

$$\mu_{\widetilde{H}(X_1,\dots,X_n)}(y) \leqslant \sup\left\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n > 0,\ y \in \operatorname{conv}\{y_1,\dots,y_n\}\right\},$$

which shows that $\widetilde{H}$ is a fuzzy mean in the sense of Definition 1. Note also that for ordinary numbers $X_1,\dots,X_n > 0$ we have $\mu_{\widetilde{H}(X_1,\dots,X_n)} = \chi_{H(X_1,\dots,X_n)}$ and so $\widetilde{H}$ coincides with the classical harmonic mean in this case.

## 4. The $\widetilde{H} \leqslant \widetilde{G} \leqslant \widetilde{A}$ inequalities

In this section we prove that the classical inequalities between the harmonic, geometric and arithmetic means can be extended to the corresponding fuzzy means.

THEOREM 5. *For every $n \geqslant 2$ and all $X_1,\dots,X_n \in \mathscr{F}(\mathbb{R})$,*

$$\widetilde{H}(X_1,\dots,X_n) \leqslant \widetilde{G}(X_1,\dots,X_n) \leqslant \widetilde{A}(X_1,\dots,X_n).$$

*Proof.* Assume, for simplicity, that $n = 2$ and fix positive $X_1, X_2 \in \mathscr{F}(\mathbb{R})$. To prove that $\widetilde{H}(X_1,X_2) \leqslant \widetilde{G}(X_1,X_2)$ fix any $\alpha \in (0,1]$ and consider the $\alpha$-cuts

$$[\underline{h}_\alpha, \overline{h}_\alpha] = \{y \in \mathbb{R} :\ \mu_{\widetilde{H}(X_1,X_2)}(y) \geqslant \alpha\},$$

$$[\underline{g}_\alpha, \overline{g}_\alpha] = \{y \in \mathbb{R} :\ \mu_{\widetilde{G}(X_1,X_2)}(y) \geqslant \alpha\}.$$

We have to prove that

$$\underline{h}_\alpha \leqslant \underline{g}_\alpha \quad \text{and} \quad \overline{h}_\alpha \leqslant \overline{g}_\alpha. \tag{11}$$

To show the left inequality in (11) we consider two cases.

Clearly $\mu_{\widetilde{G}(X_1,X_2)}(\underline{g}_\alpha) \geqslant \alpha$. Assume first that $\mu_{\widetilde{G}(X_1,X_2)}(\underline{g}_\alpha) > \alpha$. By the form of the membership function of $\widetilde{G}(X_1,X_2)$ (see (9)) there exist $y_1, y_2 > 0$ such that $\underline{g}_\alpha = G(y_1,y_2)$ and $\mu_{X_1}(y_1) > \alpha$, $\mu_{X_2}(y_2) > \alpha$. Put $z = H(y_1,y_2)$. Then $z \leqslant \underline{g}_\alpha$ (because $H(y_1,y_2) \leqslant G(y_1,y_2)$) and, by (10),

$$\mu_{\widetilde{H}(X_1,X_2)}(z) = \sup\left\{\min\{\mu_{X_1}(z_1),\mu_{X_2}(z_2)\} \mid z_1,z_2 > 0,\ z = H(z_1,z_2)\right\}$$
$$\geqslant \min\{\mu_{X_1}(y_1),\mu_{X_2}(y_2)\} > \alpha.$$

Hence
$$\underline{h}_\alpha = \min\{y \in \mathbb{R} : \mu_{\widetilde{H}(X_1,X_2)}(y) \geqslant \alpha\} \leqslant z \leqslant \underline{g}_\alpha.$$
Now assume that $\mu_{\widetilde{G}(X_1,X_2)}(\underline{g}_\alpha) = \alpha$. By the form of $\mu_{\widetilde{G}(X_1,X_2)}$ and the definition of supremum there exist two sequences $(y_{1,n}),(y_{2,n})$ of positive numbers such that $\underline{g}_\alpha = G(y_{1,n},y_{2,n})$ for all $n \in \mathbb{N}$, and
$$\lim_{n\to\infty} \min\{\mu_{X_1}(y_{1,n}),\mu_{X_2}(y_{2,n})\} = \alpha. \tag{12}$$
Define $z_n = H(y_{1,n},y_{2,n})$, $n \in \mathbb{N}$. Then $z_n \leqslant \underline{g}_\alpha$ and
$$\begin{aligned}\mu_{\widetilde{H}(X_1,X_2)}(z_n) &= \sup\big\{\min\{\mu_{X_1}(z_{1,n}),\mu_{X_2}(z_{2,n})\} \mid z_{1,n},z_{2,n} > 0, z_n = H(z_{1,n},z_{2,n})\big\} \\ &\geqslant \min\{\mu_{X_1}(y_{1,n}),\mu_{X_2}(y_{2,n})\}.\end{aligned} \tag{13}$$
Since $z_n \in (0,\underline{g}_\alpha]$, $n \in \mathbb{N}$, the sequence $(z_n)$ is bounded. Therefore it contains a convergent subsequence. Without loss of generality we may assume that $z_n \to z$. Then, $z \leqslant \underline{g}_\alpha$ and by the upper semi-continuity of $\mu_{\widetilde{G}(X_1,X_2)}$, (13), and (12), we get
$$\begin{aligned}\mu_{\widetilde{H}(X_1,X_2)}(z) &\geqslant \limsup_{n\to\infty} \mu_{\widetilde{H}(X_1,X_2)}(z_n) \\ &\geqslant \limsup_{n\to\infty} \min\{\mu_{X_1}(y_{1,n}),\mu_{X_2}(y_{2,n})\} = \alpha.\end{aligned}$$
Hence
$$\underline{h}_\alpha = \min\{y \in \mathbb{R} : \mu_{\widetilde{H}(X_1,X_2)}(y) \geqslant \alpha\} \leqslant z \leqslant \underline{g}_\alpha,$$
which finish the proof of the left inequality in (11).

To prove the right inequality in (11) we also consider two cases.

Clearly $\mu_{\widetilde{H}(X_1,X_2)}(\overline{h}_\alpha) \geqslant \alpha$. Assume first that $\mu_{\widetilde{G}(X_1,X_2)}(\overline{h}_\alpha) > \alpha$. By the form of the membership function of $\widetilde{H}(X_1,X_2)$ (see (10)) there exist $y_1,y_2 > 0$ such that $\overline{h}_\alpha = H(y_1,y_2)$ and $\mu_{X_1}(y_1) > \alpha$, $\mu_{X_2}(y_2) > \alpha$. Put $z = G(y_1,y_2)$. Then $z \geqslant \overline{h}_\alpha$ and, by (9),
$$\begin{aligned}\mu_{\widetilde{G}(X_1,X_2)}(z) &= \sup\big\{\min\{\mu_{X_1}(z_1),\mu_{X_2}(z_2)\} \mid z_1,z_2 \geqslant 0,\ z = G(z_1,z_2)\big\} \\ &\geqslant \min\{\mu_{X_1}(y_1),\mu_{X_2}(y_2)\} > \alpha.\end{aligned}$$
Consequently,
$$\overline{g}_\alpha = \max\{y \in \mathbb{R} : \mu_{\widetilde{G}(X_1,X_2)}(y) \geqslant \alpha\} \geqslant z \geqslant \overline{h}_\alpha.$$
Now assume that $\mu_{\widetilde{G}(X_1,X_2)}(\overline{h}_\alpha) = \alpha$. By the form of $\mu_{\widetilde{H}(X_1,X_2)}$ there exist two sequences $(y_{1,n}),(y_{2,n})$ of positive numbers such that $\overline{h}_\alpha = H(y_{1,n},y_{2,n})$ for all $n \in \mathbb{N}$, and
$$\lim_{n\to\infty} \min\{\mu_{X_1}(y_{1,n}),\mu_{X_2}(y_{2,n})\} = \alpha. \tag{14}$$
Define $z_n = G(y_{1,n},y_{2,n})$, $n \in \mathbb{N}$. Then $z_n \geqslant \overline{h}_\alpha$ and
$$\begin{aligned}\mu_{\widetilde{G}(X_1,X_2)}(z_n) &= \sup\big\{\min\{\mu_{X_1}(z_{1,n}),\mu_{X_2}(z_{2,n})\} \mid z_{1,n},z_{2,n} \geqslant 0, z_n = G(z_{1,n},z_{2,n})\big\} \\ &\geqslant \min\{\mu_{X_1}(y_{1,n}),\mu_{X_2}(y_{2,n})\}.\end{aligned} \tag{15}$$

From (14) it follows that $y_{1,n} \in supp\, X_1$ and $y_{2,n} \in supp\, X_2$ for sufficiently large $n \in \mathbb{N}$. Since, by our assumption, the supports of $X_1$ and $X_2$ are bounded, the sequence $(z_n)$ is bounded and, consequently it contains a convergent subsequence. We may assume that $z_n \to z$. Then, $z \geqslant \overline{h}_\alpha$ and by the upper semi-continuity of $\mu_{\widetilde{G}(X_1,X_2)}$, (15), and (14), we get

$$\begin{aligned}\mu_{\widetilde{G}(X_1,X_2)}(z) &\geqslant \limsup_{n\to\infty} \mu_{\widetilde{G}(X_1,X_2)}(z_n)\\ &\geqslant \limsup_{n\to\infty} \min\{\mu_{X_1}(y_{1,n}), \mu_{X_2}(y_{2,n})\} = \alpha.\end{aligned}$$

Hence

$$\overline{g}_\alpha = \max\{y \in \mathbb{R} :\ \mu_{\widetilde{G}(X_1,X_2)}(y) \geqslant \alpha\} \geqslant z \geqslant \overline{h}_\alpha,$$

which finishes the proof of the right inequality in (11). Thus for every $\alpha \in (0,1]$ both inequalities in (11) are satisfied, which means that $\widetilde{H}(X_1,X_2) \leqslant \widetilde{G}(X_1,X_2)$.

The proof of the inequality $\widetilde{G}(X_1,X_2) \leqslant \widetilde{A}(X_1,X_2)$ is quite analogous and therefore we omit its details. For any fixed $\alpha \in (0,1]$ we consider the $\alpha$-cuts

$$\left[\underline{g}_\alpha, \overline{g}_\alpha\right] = \{y \in \mathbb{R} :\ \mu_{\widetilde{G}(X_1,X_2)}(y) \geqslant \alpha\},$$

$$\left[\underline{a}_\alpha, \overline{a}_\alpha\right] = \{y \in \mathbb{R} :\ \mu_{\widetilde{A}(X_1,X_2)}(y) \geqslant \alpha\}$$

and, in a similarly way as previously, we prove that

$$\underline{g}_\alpha \leqslant \underline{a}_\alpha \quad \text{and} \quad \overline{g}_\alpha \leqslant \overline{a}_\alpha.$$

This shows that $\widetilde{G}(X_1,X_2) \leqslant \widetilde{A}(X_1,X_2)$ and completes the whole proof. □

## 5. Quasi-arithmetic fuzzy means

In this section we introduce the notion of quasi-arithmetic fuzzy mean which is a joint generalization of arithmetic, geometric and harmonic fuzzy means. Let $I \subset \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a continuous strictly monotonic function. The function $A_f : I^n \to I$ defined by

$$A_f(x_1,\ldots,x_n) = f^{-1}\Big(\frac{f(x_1)+\cdots+f(x_n)}{n}\Big)$$

is the classical quasi-arithmetic mean generated by $f$ (see e.g. [13] and the references therein). In an analogous way we can define the quasi-arithmetic fuzzy mean generated by $f$ putting, for $X_1,\ldots,X_n \in \mathscr{F}(\mathbb{R})$,

$$\widetilde{A_f}(X_1,\ldots,X_n) = f^{-1}\Big(\frac{f(X_1)+\cdots+f(X_n)}{n}\Big).$$

PROPOSITION 7. The membership function of the quasi-arithmetic fuzzy mean of $X_1,\dots,X_n \in \mathscr{F}(\mathbb{R})$ is of the form

$$\begin{aligned}&\mu_{\widetilde{A_f}(X_1,\dots,X_n)}(y) \\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n\in\mathbb{R},\ y=A_f(y_1,\dots,y_n)\big\}.\end{aligned} \tag{16}$$

*Proof.* By the definitions of operations on fuzzy numbers we have

$$\begin{aligned}&\mu_{\widetilde{A_f}(X_1,\dots,X_n)}(y) = \mu_{f^{-1}\left(\frac{f(X_1)+\dots+f(X_n)}{n}\right)}(y) = \mu_{\frac{f(X_1)+\dots+f(X_n)}{n}}(f(y)) \\ &= \sup\Big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n\in\mathbb{R},\ \frac{f(y_1)+\dots+f(y_n)}{n} = f(y)\Big\} \\ &= \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n\in\mathbb{R},\ y=A_f(y_1,\dots,y_n)\big\}. \quad \square\end{aligned}$$

REMARK 6. By Proposition 1 it follows tyhat $\widetilde{A_f}(X_1,\dots,X_n)$ is a fuzzy number. From (16), using the fact that $A_f(y_1,\dots,y_n)\in \mathrm{conv}\{y_1,\dots,y_n\}$, we obtain

$$\mu_{\widetilde{A_f}(X_1,\dots,X_n)}(y) \leqslant \sup\big\{\min\{\mu_{X_1}(y_1),\dots,\mu_{X_n}(y_n)\} \mid y_1,\dots,y_n\in\mathbb{R},\ y\in\mathrm{conv}\{y_1,\dots,y_n\}\big\}.$$

This shows that $\widetilde{A_f}$ is a fuzzy mean in the sense of Definition 1. For ordinary numbers $X_1,\dots,X_n \geqslant 0$ we have $\mu_{\widetilde{A_f}(X_1,\dots,X_n)} = \chi_{A_f(X_1,\dots,X_n)}$ and so $\widetilde{A_f}$ coincides with the classical quasi-arithmetic mean in this case. Note also that for positive fuzzy numbers the arithmetic, geometric and harmonic fuzzy means are quasi-arithmetic fuzzy means generated by the functions $f(x)=x$, $f(x)=\ln x$ and $f(x)=1/x$, respectively.

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(Received August 26, 2020)

*Joanna Nikodem*
*University of Bielsko-Biala*
*Dept. of Computer Science and Automatics*
*ul. Willowa 2, 43-309 Bielsko-Biała, Poland*
*e-mail:* jnikodem@ath.bielsko.pl

*Kazimierz Nikodem*
*University of Bielsko-Biala*
*Dept. of Mathematics, ul. Willowa 2*
*43-309 Bielsko-Biała, Poland*
*e-mail:* knikodem@ath.bielsko.pl