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STRONG CONSISTENCY OF LS ESTIMATORS IN SIMPLE LINEAR EV REGRESSION MODELS WITH WOD ERRORS

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Abstract. For a simple linear errors-in-variables regression model with widely orthant dependent errors, we provide sufficient conditions for the convergence rate in the strong consistency of the least squares estimators. We also provide necessary conditions. Our result improves and extends some results of Liu et al. (J. Math. Ineq., **14** (2020), 771–779).

1. Introduction

Consider the simple linear errors-in-variables (EV) regression model:

$$\eta_k = ax_k + b + \varepsilon_k, \quad \xi_k = x_k + \delta_k, \quad 1 \leqslant k \leqslant n, \tag{1.1}$$

where $a, b, x_1, \cdots, x_n$ are unknown parameters or constants, $(\varepsilon_k, \delta_k)$, $1 \leqslant k \leqslant n$, are random vectors and ξ_k, η_k, $1 \leqslant k \leqslant n$, are observable variables. From (1.1),

$$\eta_k = a\xi_k + b + (\varepsilon_k - a\delta_k), \quad 1 \leqslant k \leqslant n.$$

Then, as a usual regression model of η_k on ξ_k with the errors $\varepsilon_k - a\delta_k$, the least squares (LS) estimators of a and b are given as

$$\hat{a}_n = \frac{\sum_{k=1}^n (\xi_k - \bar{\xi}_n)(\eta_k - \bar{\eta}_n)}{\sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2}, \quad \hat{b}_n = \bar{\eta}_n - \hat{a}_n \bar{\xi}_n,$$

where $\bar{\xi}_n = n^{-1} \sum_{k=1}^n \xi_k$. The notations of $\bar{\eta}_n$, $\bar{\delta}_n$ and $\bar{x}_n$ are defined in the same way.

Based on the above notations, we have

$$\hat{a}_n - a = \frac{\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^n (x_k - \bar{x}_n)(\varepsilon_k - a\delta_k) - a\sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2}{\sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2} \tag{1.2}$$

and

$$\hat{b}_n - b = -\bar{x}_n(\hat{a}_n - a) - (\hat{a}_n - a)\bar{\delta}_n + \bar{\varepsilon}_n - a\bar{\delta}_n. \tag{1.3}$$

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The EV model was proposed by Deaton (1985) to correct the effects of the sampling errors and is somewhat more practical than the ordinary regression model. Fuller (1987) summarized many early works for the EV models. Due to the simple form and wide applicability, the studies for the EV model have attracted much attention for the past three decades. For more details, we refer to Chen et al. (2020), Hu et at. (2017), Lita da Silva (2018, 2020), Liu and Chen (2005), Liu et al. (2020), Miao et al. (2011), Wang et al. (2015), Wang et al. (2018), Wu et al. (2018), Zhang et al. (2019) and so on. In particular, Liu et al. (2020) obtained a necessary and sufficient condition for the convergence rate of the strong consistency for each of the unknown parameters as follows.

THEOREM A. (Liu et al., 2020) *Under the model* (1.1)*, assume that* $\{(\varepsilon,\delta),(\varepsilon_n,\delta_n), n\geqslant 1\}$ *is a sequence of independent and identically distributed random vectors with* $E\varepsilon=E\delta=0,\ 0<E|\varepsilon|^\beta,E|\delta|^\beta<\infty$ *for* $1/p=1/2+1/\beta$*, where* $1\leqslant p<2$*, and* $E\delta\varepsilon\neq aE\delta^2$*. Then*

$$n^{1-1/p}(\hat{a}_n-a)\to 0 \text{ a.s. if and only if } n^{2-1/p}/s_n\to 0,$$

where $s_n=\sum_{k=1}^n(x_k-\bar{x}_n)^2$.

Further, if $\sup_{n\geqslant 1}\min\{n,s_n\}\bar{x}_n^2/s_n^*<\infty$*, then*

$$n^{1-1/p}(\hat{b}_n-b)\to 0 \text{ a.s. if and only if } n^{2-1/p}\bar{x}_n/s_n^*\to 0,$$

where $s_n^*=\max\{n,s_n\}$.

In this paper, we improve and extend Theorem A to widely orthant dependent (WOD) random variables.

A sequence of random variables $\{X_n,n\geqslant 1\}$ is said to be widely upper orthant dependent (WUOD) if for each $n\geqslant 1$, there exists a positive number $g_U(n)$ such that for all real numbers $x_i, 1\leqslant i\leqslant n$,

$$P(X_1>x_1,\cdots,X_n>x_n)\leqslant g_U(n)\prod_{i=1}^n P(X_i>x_i),$$

it is said to be widely lower orthant dependent (WLOD) if for each $n\geqslant 1$, there exists a positive number $g_L(n)$ such that for all real numbers $x_i, 1\leqslant i\leqslant n$,

$$P(X_1\leqslant x_1,\cdots,X_n\leqslant x_n)\leqslant g_L(n)\prod_{i=1}^n P(X_i\leqslant x_i),$$

and it is said to be WOD if it is both WUOD and WLOD. The sequences $\{g_U(n),n\geqslant 1\}$ and $\{g_L(n),n\geqslant 1\}$ are called dominating coefficients (see Wang et al., 2013). If for all $n\geqslant 1,\ g_U(n)=g_L(n)=M$ for some positive constant M, then $\{X_n,n\geqslant 1\}$ is said to be extended negatively dependent (END). In particular, if $M=1$, then $\{X_n,n\geqslant 1\}$ is said to be negatively orthant dependent (NOD) or negatively dependent. Since the class of WOD random variables contains independent random variables, END random variables and NOD random variables as special cases, it is interesting to study the limiting behavior of WOD random variables.

We now state the main results. Some lemmas and the proofs of the main results will be detailed in the next section.

THEOREM 1.1. *Under the model* (1.1), *let* $\{\varepsilon, \varepsilon_n, n \geqslant 1\}$ *and* $\{\delta, \delta_n, n \geqslant 1\}$ *be two sequences of identically distributed WOD random variables with dominating coefficients* $g_L(n)$ *and* $g_U(n)$, $g'_L(n)$ *and* $g'_U(n)$ *for* $n \geqslant 1$, *respectively. Suppose that* $E\varepsilon = E\delta = 0$, $0 < E|\varepsilon|^{2tp/(2t-p)}, E|\delta|^{2tp/(2t-p)} < \infty$ *for some* $1 < p < 2$ *and* $1 \leqslant t < 2p/(4-2p)$, *and there exist a positive function* $g(x)$ *for* $x \geqslant 0$ *and a nonnegative constant* $0 \leqslant \tau < \infty$ *such that* $g(x) = O(x^\tau)$ *and* $\max\{g_L(n), g_U(n), g'_L(n), g'_U(n)\} \leqslant g(n)$ *for* $n \geqslant 1$. *Then the following statements hold:*

(i) If $n^{2-1/t}/s_n = O(1)$ *and* $n^{2-1/p}/s_n \to 0$, *then*

$$n^{1-1/p}(\hat{a}_n - a) \to 0 \ \text{a.s.} \tag{1.4}$$

(ii) If $\sup_{n\geqslant 1} \min\{n, s_n\} n^{1-1/t} \bar{x}_n^2 / s_n^* < \infty$ *and* $n^{2-1/p} \bar{x}_n / s_n^* \to 0$, *then*

$$n^{1-1/p}(\hat{b}_n - b) \to 0 \ \text{a.s.}, \tag{1.5}$$

where $s_n^* = \max\{n, s_n\}$ *is the same as in Theorem A.*

REMARK 1.1. In the proof of Theorem 1.1, ε and δ need finite absolute moments of order greater than 2 (see the proofs of (2.5) and (2.6)). However, $2tp/(2t-p) = 2$ when $p = 1$ and $t = 1$. If the moment conditions of ε and δ are strengthened to $0 < E|\varepsilon|^u, E|\delta|^u < \infty$ for some $u > 2$, then Theorem 1.1 is still valid when $p = 1$ and $t = 1$.

REMARK 1.2. Theorem 1.1 improves and extends the sufficiency parts of Theorem A to WOD random variables. When $t = p$, Theorem 1.1 (i) corresponds to the sufficiency part of the first result of Theorem A. For each fixed $1 < p < 2$, the function $h(t) := 2tp/(2t-p)$ is strictly decreasing on $1 \leqslant t < 2p/(4-2p)$, and hence $h(p) < h(1)$. Since $h(1) = \beta$, where $1/p = 1/2 + 1/\beta$, the moment conditions of Theorem 1.1 are weaker than those of Theorem A. When $t = 1$, Theorem 1.1 (ii) corresponds to the sufficiency part of the second result of Theorem A.

The following theorem is a partial converse of Theorem 1.1.

THEOREM 1.2. *Under the assumptions of Theorem 1.1, further assume that* $\{\varepsilon, \varepsilon_n\}$ *and* $\{\delta, \delta_n\}$ *are independent, and* $aE\delta^2 \neq 0$ *(i.e.,* $E\delta\varepsilon \neq aE\delta^2$*). Then the following statements hold:*

(i) If $1 \leqslant t \leqslant p$, *then* $n^{1-1/p}(\hat{a}_n - a) \to 0$ *a.s. implies* $n^{2-1/p}/s_n \to 0$.

(ii) If $\sup_{n\geqslant 1} \min\{n, s_n\} n^{1-1/t} \bar{x}_n^2 / s_n^* < \infty$, *then* $n^{1-1/p}(\hat{b}_n - b) \to 0$ *a.s. implies* $n^{2-1/p} \bar{x}_n / s_n^* \to 0$.

REMARK 1.3. If $1 \leqslant t \leqslant p$, condition $n^{2-1/t}/s_n = O(1)$ of Theorem 1.1 (i) follows from $n^{2-1/p}/s_n \to 0$ (i.e., condition $n^{2-1/t}/s_n = O(1)$ can be deleted). Hence Theorem 1.2 (i) is the converse of Theorem 1.1 (i) when $1 \leqslant t \leqslant p$. Theorem 1.2, of course, requires an additional condition that $\{\varepsilon_n\}$ and $\{\delta_n\}$ are independent.

2. Lemmas and proofs

To prove the main results, we need the following lemmas. The first one provides a strong law of large numbers for weighted sums of WOD random variables.

LEMMA 2.1. (Yi et al., 2020) *Let* $1 \leqslant p < 2$ *and* $\alpha, \beta > 0$ *with* $1/p = 1/\alpha + 1/\beta$. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of identically distributed WOD random variables with dominating coefficients* $g_L(n)$ *and* $g_U(n)$ *for* $n \geqslant 1$. *Suppose that there exist a positive function* $g(x)$ *for* $x \geqslant 0$ *and a nonnegative constant* $0 \leqslant \tau < \infty$ *such that* $g(x) = O(x^\tau)$ *and* $\max\{g_L(n), g_U(n)\} \leqslant g(n)$ *for* $n \geqslant 1$. *Let* $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying*

$$\sum_{k=1}^{n} |a_{nk}|^\alpha = O(n).$$

If $EX = 0$ *and* $E|X|^\beta < \infty$, *then*

$$n^{-1/p} \sum_{k=1}^{n} a_{nk} X_k \to 0 \quad a.s.$$

If all weights a_{nk} in Lemma 2.1 have the same value, then $\sum_{k=1}^{n} |a_{nk}|^\alpha = O(n)$ for any $\alpha > 0$. Hence, the following lemma follows easily from Lemma 2.1.

LEMMA 2.2. *Let* $1 \leqslant p < 2$. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of identically distributed WOD random variables with dominating coefficients* $g_L(n)$ *and* $g_U(n)$ *for* $n \geqslant 1$. *Suppose that there exist a positive function* $g(x)$ *for* $x \geqslant 0$ *and a nonnegative constant* $0 \leqslant \tau < \infty$ *such that* $g(x) = O(x^\tau)$ *and* $\max\{g_L(n), g_U(n)\} \leqslant g(n)$ *for* $n \geqslant 1$. *If* $EX = 0$ *and* $E|X|^\beta < \infty$ *for some* $\beta > p$, *then*

$$n^{-1/p} \sum_{k=1}^{n} X_k \to 0 \quad a.s.$$

REMARK 2.1. Lemma 2.2 can be obtained by Theorem 2.2 of Lita da Silva (2020). In fact, the result of Lita da Silva (2020) improves and generalizes Lemma 2.2.

To prove Theorem 1.2, we need the following lemma.

LEMMA 2.3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of nonnegative WOD random variables with dominating coefficients* $g_L(n)$ *and* $g_U(n)$ *for* $n \geqslant 1$, *and let* $\{Y_n, n \geqslant 1\}$ *be a sequence of nonnegative WOD random variables with dominating coefficients* $g'_L(n)$ *and* $g'_U(n)$ *for* $n \geqslant 1$. *Assume that* $\{X_n, n \geqslant 1\}$ *and* $\{Y_n, n \geqslant 1\}$ *are independent. Then* $\{X_n Y_n, n \geqslant 1\}$ *is a sequence of WOD random variables with dominating coefficients* $g_L(n) g'_L(n)$ *and* $g_U(n) g'_U(n)$ *for* $n \geqslant 1$.

Proof. The proof is similar to those of Lemma 1 in Chen et al. (2019) and Lemma 2.2 in Lang et al. (2021). For the sake of completeness, we give the proof here.

If $z_1,\cdots,z_n$ are all nonnegative, we have that

$$\begin{aligned}
&P(X_1Y_1 \leqslant z_1,\cdots,X_nY_n \leqslant z_n)\\
&=\int\cdots\int I(x_1y_1 \leqslant z_1,\cdots,x_ny_n \leqslant z_n)\ dF_{X_1,\cdots,X_n,Y_1,\cdots,Y_n}(x_1,\cdots,x_n,y_1,\cdots,y_n)\\
&=\int\cdots\int I(x_1y_1 \leqslant z_1,\cdots,x_ny_n \leqslant z_n)\ dF_{X_1,\cdots,X_n}(x_1,\cdots,x_n)dF_{Y_1,\cdots,Y_n}(y_1,\cdots,y_n)\\
&\qquad \text{(by independence of } \{X_n\} \text{ and } \{Y_n\})\\
&=\int\cdots\int P(x_1Y_1 \leqslant z_1,\cdots,x_nY_n \leqslant z_n)\ dF_{X_1,\cdots,X_n}(x_1,\cdots,x_n)\\
&\leqslant g_L'(n)\int\cdots\int P(x_1Y_1 \leqslant z_1)\cdots P(x_nY_n \leqslant z_n)\ dF_{X_1,\cdots,X_n}(x_1,\cdots,x_n)\ \ \text{(by WOD of } \{Y_n\})\\
&=g_L'(n)E\left[F_{Y_1}(z_1/X_1)\cdots F_{Y_n}(z_n/X_n)\right].
\end{aligned}$$

Since $F_{Y_i}(z_i/\cdot)$ is nonincreasing for each $1 \leqslant i \leqslant n$, we have by Proposition 1.1 of Wang et al. (2013) that $\{F_{Y_1}(z_1/X_1),\cdots,F_{Y_n}(z_n/X_n)\}$ is a sequence of WOD with dominating coefficients $g_U(i)$ and $g_L(i)$ for $1 \leqslant i \leqslant n$, and hence

$$E\left[F_{Y_1}(z_1/X_1)\cdots F_{Y_n}(z_n/X_n)\right] \leqslant g_L(n)E\left[F_{Y_1}(z_1/X_1)\right]\cdots E\left[F_{Y_n}(z_n/X_n)\right].$$

On the other hand,

$$\begin{aligned}
&E\left[F_{Y_1}(z_1/X_1)\right]\cdots E\left[F_{Y_n}(z_n/X_n)\right]\\
&=\iint I(x_1y_1 \leqslant z_1)\ dF_{Y_1}(y_1)dF_{X_1}(x_1)\cdots\iint I(x_ny_n \leqslant z_n)\ dF_{Y_n}(y_n)dF_{X_n}(x_n)\\
&=\iint I(x_1y_1 \leqslant z_1)\ dF_{X_1,Y_1}(x_1,y_1)\cdots\iint I(x_ny_n \leqslant z_n)\ dF_{X_n,Y_n}(x_n,y_n)\\
&\qquad \text{(by independence of } \{X_n\} \text{ and } \{Y_n\})\\
&=P(X_1Y_1 \leqslant z_1)\cdots P(X_nY_n \leqslant z_n).
\end{aligned}$$

It follows that

$$P(X_1Y_1 \leqslant z_1,\cdots,X_nY_n \leqslant z_n) \leqslant g_L'(n)g_L(n)P(X_1Y_1 \leqslant z_1)\cdots P(X_nY_n \leqslant z_n).$$

Otherwise, we have that

$$P(X_1Y_1 \leqslant z_1,\cdots,X_nY_n \leqslant z_n) = g_L'(n)g_L(n)P(X_1Y_1 \leqslant z_1)\cdots P(X_nY_n \leqslant z_n) = 0.$$

Similarly, we also have that

$$P(X_1Y_1 > z_1,\cdots,X_nY_n > z_n) \leqslant g_U'(n)g_U(n)P(X_1Y_1 > z_1)\cdots P(X_nY_n > z_n).$$

Therefore, $X_1Y_1,\cdots,X_nY_n$ are WOD with dominating coefficients $g_L'(i)g_L(i)$ and $g_U'(i)g_U(i)$ for $1 \leqslant i \leqslant n$. $\square$

Proof of Theorem 1.1. The idea of the proof is similar to that of Liu et al. (2020), but some details should be changed. It is important to observe at the outset that ε and

δ have finite absolute moments of order greater than 2, since $2tp/(2t-p) > 2$ when $1 < p < 2$ and $1 \leqslant t < 2p/(4-2p)$.

(i) Assume that $n^{2-1/p}/s_n \to 0$ and $n^{2-1/t}/s_n = O(1)$. From (1.2), it suffices to prove that

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k \to 0 \quad \text{a.s.}, \tag{2.1}$$

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)(\varepsilon_k - a\delta_k) \to 0 \ \text{ a.s.}, \tag{2.2}$$

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 \to 0 \ \text{ a.s.}, \tag{2.3}$$

$$s_n^{-1} \sum_{k=1}^{n} (\xi_k - \bar{\xi}_n)^2 \to 1 \ \text{ a.s.} \tag{2.4}$$

By Proposition 1.1 in Wang et al. (2013) or Lemma 2.3 in Chen and Sung (2019), $\{(\varepsilon_n^+)^2, n \geqslant 1\}$ is a sequence of WOD random variables with dominating coefficients $g_L(n)$ and $g_U(n)$ for $n \geqslant 1$, and $\{(\varepsilon_n^-)^2, n \geqslant 1\}$ is a sequence of WOD random variables with dominating coefficients $g_U(n)$ and $g_L(n)$ for $n \geqslant 1$. Then by Lemma 2.2,

$$n^{-1} \sum_{k=1}^{n} (\varepsilon_k^+)^2 \to E(\varepsilon^+)^2 \ \text{ a.s. and } n^{-1} \sum_{k=1}^{n} (\varepsilon_k^-)^2 \to E(\varepsilon^-)^2 \ \text{ a.s.},$$

which follow that

$$n^{-1} \sum_{k=1}^{n} \varepsilon_k^2 = n^{-1} \sum_{k=1}^{n} (\varepsilon_k^+)^2 + n^{-1} \sum_{k=1}^{n} (\varepsilon_k^-)^2 \to E\varepsilon^2 \ \text{ a.s.} \tag{2.5}$$

Similarly, we have

$$n^{-1} \sum_{k=1}^{n} \delta_k^2 \to E\delta^2 \ \text{ a.s.} \tag{2.6}$$

By the Hölder inequality, (2.5), and (2.6),

$$\limsup_{n\to\infty} n^{-1} \left| \sum_{k=1}^{n} \varepsilon_k \delta_k \right| \leqslant \limsup_{n\to\infty} \left(n^{-1} \sum_{k=1}^{n} \varepsilon_k^2 \right)^{1/2} \left(n^{-1} \sum_{k=1}^{n} \delta_k^2 \right)^{1/2} = \sqrt{E\varepsilon^2 E\delta^2} \ \text{ a.s.},$$

which, together with the facts that $\bar{\varepsilon}_n \to 0$ a.s. and $\bar{\delta}_n \to 0$ a.s. from Lemma 2.2, implies that

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k = \frac{n^{2-1/p}}{s_n} \cdot \left(\frac{1}{n} \sum_{k=1}^{n} \varepsilon_k \delta_k - \bar{\varepsilon}_n \bar{\delta}_n \right) \to 0 \ \text{ a.s.},$$

i.e., (2.1) holds.

By (2.6) and $\bar{\delta}_n \to 0$ a.s.,

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 = \frac{n^{2-1/p}}{s_n} \cdot \left(\frac{1}{n} \sum_{k=1}^{n} \delta_k^2 - \bar{\delta}_n^2 \right) \to 0 \cdot (E\delta^2 - 0) = 0 \text{ a.s.},$$

i.e., (2.3) holds.

Set $a_{nk} = n(x_k - \bar{x}_n)/s_n$ for $n \geqslant 1$ and $1 \leqslant k \leqslant n$. Then

$$\sup_{n\geqslant 1} n^{-1} \sum_{k=1}^{n} |a_{nk}|^{2t} = \sup_{n\geqslant 1} \frac{n^{2t-1} \sum_{k=1}^{n} |x_k - \bar{x}_n|^{2t}}{s_n^{2t}} \leqslant \sup_{n\geqslant 1} \frac{n^{2t-1} s_n^t}{s_n^{2t}} = \left(\sup_{n\geqslant 1} \frac{n^{2-1/t}}{s_n} \right)^t < \infty.$$

Therefore by Lemma 2.1 with $\alpha = 2t$ and $\beta = 2tp/(2t-p)$,

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)\varepsilon_k = n^{-1/p} \sum_{k=1}^{n} a_{nk}\varepsilon_k \to 0 \text{ a.s.} \tag{2.7}$$

and

$$s_n^{-1} \cdot n^{1-1/p} \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k = n^{-1/p} \sum_{k=1}^{n} a_{nk}\delta_k \to 0 \text{ a.s.} \tag{2.8}$$

Then (2.2) holds from (2.7) and (2.8).

Noting that

$$s_n^{-1} \sum_{k=1}^{n} (\xi_k - \bar{\xi}_n)^2 = 1 + 2s_n^{-1} \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k + s_n^{-1} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2,$$

(2.4) holds by (2.3) and (2.8). Hence the proof of (i) is completed.

(ii) By Lemma 2.2,

$$n^{1-1/p}\bar{\varepsilon}_n \to 0 \text{ a.s. and } n^{1-1/p}\bar{\delta}_n \to 0 \text{ a.s.}$$

Hence, to prove (1.5), it suffices by (1.3) to show that

$$\limsup_{n\to\infty} |\hat{a}_n - a| < \infty \text{ a.s.} \tag{2.9}$$

and

$$n^{1-1/p} \cdot \bar{x}_n(\hat{a}_n - a) \to 0 \text{ a.s.} \tag{2.10}$$

It is clear that

$$\frac{1}{s_n^*} \sum_{k=1}^{n} (\xi_k - \bar{\xi}_n)^2 = \frac{s_n}{s_n^*} + \frac{2}{s_n^*} \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*} \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2. \tag{2.11}$$

Observing that

$$\sup_{n\geqslant 1} \frac{1}{n} \sum_{k=1}^{n} (n(x_k - \bar{x}_n)/s_n^*)^2 = \sup_{n\geqslant 1} \frac{ns_n}{(s_n^*)^2} \leqslant 1,$$

we have by Lemma 2.1 with $p=1$ and $\alpha=\beta=2$ that

$$\frac{1}{s_n^*}\sum_{k=1}^{n}(x_k-\bar{x}_n)\delta_k=\frac{1}{n}\sum_{k=1}^{n}[n(x_k-\bar{x}_n)/s_n^*]\delta_k\to 0 \text{ a.s.} \tag{2.12}$$

By (2.6) and Lemma 2.2,

$$\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2=\frac{1}{n}\sum_{k=1}^{n}\delta_k^2-\bar{\delta}_n^2\to E\delta^2 \text{ a.s.,} \tag{2.13}$$

which, together with the definition of s_n^*, implies that

$$\begin{aligned}\min\{1,E\delta^2\}&\leqslant\liminf_{n\to\infty}\left(\frac{s_n}{s_n^*}+\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2\right)\\&\leqslant\limsup_{n\to\infty}\left(\frac{s_n}{s_n^*}+\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2\right)\leqslant 1+E\delta^2 \text{ a.s.}\end{aligned} \tag{2.14}$$

It follows by (2.11), (2.12), and (2.14) that

$$\min\{1,E\delta^2\}\leqslant\liminf_{n\to\infty}\frac{1}{s_n^*}\sum_{k=1}^{n}(\xi_k-\bar{\xi}_n)^2\leqslant\limsup_{n\to\infty}\frac{1}{s_n^*}\sum_{k=1}^{n}(\xi_k-\bar{\xi}_n)^2\leqslant 1+E\delta^2 \text{ a.s.} \tag{2.15}$$

By the Hölder inequality, (2.5), (2.6), and Lemma 2.2,

$$\begin{aligned}\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k\right|&\leqslant\limsup_{n\to\infty}\left(\sqrt{\frac{1}{n}\sum_{k=1}^{n}\delta_k^2\cdot\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^2}+|\bar{\delta}_n\bar{\varepsilon}_n|\right)\\&=\sqrt{E\varepsilon^2E\delta^2} \text{ a.s.,}\end{aligned} \tag{2.16}$$

and hence

$$\limsup_{n\to\infty}\left|\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k\right|=\limsup_{n\to\infty}\left|\frac{n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k\right|\leqslant\sqrt{E\varepsilon^2E\delta^2} \text{ a.s.} \tag{2.17}$$

By the same argument as (2.12),

$$\frac{1}{s_n^*}\sum_{k=1}^{n}(x_k-\bar{x}_n)(\varepsilon_k-a\delta_k)\to 0 \text{ a.s.} \tag{2.18}$$

Therefore, we have by (1.2), (2.13), (2.15), (2.17), and (2.18) that

$$\limsup_{n\to\infty}|\hat{a}_n-a|\leqslant\frac{\sqrt{E\varepsilon^2E\delta^2}+|a|E\delta^2}{\min\{1,E\delta^2\}} \text{ a.s.,}$$

since

$$\limsup_{n\to\infty}\frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2\leqslant\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2=E\delta^2 \text{ a.s.}$$

Hence (2.9) holds.

By (2.16) and (2.6),

$$n^{1-1/p}\cdot\frac{\bar{x}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k=\frac{n^{2-1/p}\bar{x}_n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)\varepsilon_k\to 0 \text{ a.s.} \tag{2.19}$$

and

$$n^{1-1/p}\cdot\frac{\bar{x}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2=\frac{n^{2-1/p}\bar{x}_n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k-\bar{\delta}_n)^2\to 0 \text{ a.s.} \tag{2.20}$$

Noting that

$$\begin{aligned}\sup_{n\geqslant 1}\frac{1}{n}\sum_{k=1}^{n}\left|\frac{n\bar{x}_n(x_k-\bar{x}_n)}{s_n^*}\right|^{2t}&=\sup_{n\geqslant 1}\frac{n^{2t-1}|\bar{x}_n|^{2t}\sum_{k=1}^{n}|x_k-\bar{x}_n|^{2t}}{s_n^{*2t}}\\ &\leqslant\sup_{n\geqslant 1}\frac{n^{2t-1}|\bar{x}_n|^{2t}s_n^t}{s_n^{*2t}}=\left(\sup_{n\geqslant 1}\min\{n,s_n\}n^{1-1/t}\bar{x}_n^2/s_n^*\right)^t<\infty,\end{aligned}$$

we have by Lemma 2.1 with $\alpha=2t$ and $\beta=2tp/(2t-p)$ that

$$n^{1-1/p}\cdot\frac{\bar{x}_n}{s_n^*}\sum_{k=1}^{n}(x_k-\bar{x}_n)(\varepsilon_k-a\delta_k)=\frac{1}{n^{1/p}}\sum_{k=1}^{n}\frac{n\bar{x}_n(x_k-\bar{x}_n)}{s_n^*}(\varepsilon_k-a\delta_k)\to 0 \text{ a.s.} \tag{2.21}$$

Then (2.10) follows from (1.2), (2.15), and (2.19)–(2.21). The proof of (ii) is completed. □

Proof of Theorem 1.2. We first show that

$$\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k\delta_k\to 0 \quad \text{a.s.} \tag{2.22}$$

We can rewrite $n^{-1}\sum_{k=1}^{n}\varepsilon_k\delta_k$ as

$$\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k\delta_k=\frac{1}{n}\sum_{k=1}^{n}(\varepsilon_k^+\delta_k^+-\varepsilon_k^+\delta_k^--\varepsilon_k^-\delta_k^++\varepsilon_k^-\delta_k^-).$$

By Proposition 1.1 in Wang et al. (2013) or Lemma 2.3 in Chen and Sung (2019), $\{\varepsilon_n^+,n\geqslant 1\}$ and $\{\delta_n^+,n\geqslant 1\}$ are sequences of WOD random variables with dominating coefficients $g_L(n)$ and $g_U(n)$, $g'_L(n)$ and $g'_U(n)$ for $n\geqslant 1$, respectively. It follows by Lemma 2.3 that $\{\varepsilon_n^+\delta_n^+,n\geqslant 1\}$ is still a sequence of WOD random variables with dominating coefficients $g_L(n)g'_L(n)$ and $g_U(n)g'_U(n)$. Then by Lemma 2.2,

$$\frac{1}{n}\sum_{k=1}^{n}\varepsilon_k^+\delta_k^+\to E\varepsilon^+\delta^+ \quad a.s.$$

Similarly, $n^{-1}\sum_{k=1}^n \varepsilon_k^+\delta_k^- \to E\varepsilon^+\delta^-$ a.s., $n^{-1}\sum_{k=1}^n \varepsilon_k^-\delta_k^+ \to E\varepsilon^-\delta^+$ a.s., and $n^{-1}\sum_{k=1}^n \varepsilon_k^-\delta_k^- \to E\varepsilon^-\delta^-$ a.s. Hence,

$$\frac{1}{n}\sum_{k=1}^n \varepsilon_k\delta_k \to E\varepsilon^+\delta^+ - E\varepsilon^+\delta^- - E\varepsilon^-\delta^+ + E\varepsilon^-\delta^- = E\varepsilon\delta = E\varepsilon\cdot E\delta = 0,$$

i.e., (2.22) holds.

(i) Suppose that $n^{2-1/p}/s_n$ does not converge to 0. Taking a subsequence when necessary, we may assume that $n^{2-1/p}/s_n \to c \in (0,\infty]$.

Noting that

$$\sup_{n\geqslant 1}\frac{1}{n}\sum_{k=1}^n \left|\frac{n^{1/(2t)}(x_k-\bar{x}_n)}{s_n^{1/2}}\right|^{2t} = \sup_{n\geqslant 1}\frac{\sum_{k=1}^n |x_k-\bar{x}_n|^{2t}}{s_n^t} \leqslant 1,$$

we have by Lemma 2.1 with $\alpha = 2t$ and $\beta = 2tp/(2t-p)$ that

$$\frac{n^{-1/p+1/(2t)}}{s_n^{1/2}}\sum_{k=1}^n (x_k-\bar{x}_n)(\varepsilon_k - a\delta_k) = \frac{1}{n^{1/p}}\sum_{k=1}^n \frac{n^{1/(2t)}(x_k-\bar{x}_n)}{s_n^{1/2}}(\varepsilon_k - a\delta_k) \to 0 \text{ a.s.}$$

From the proof of Theorem 1.1, $\bar{\varepsilon}_n \to 0$ a.s., $\bar{\delta}_n \to 0$ a.s., and $n^{-1}\sum_{k=1}^n \delta_k^2 \to E\delta^2$ a.s. Then we obtain by (2.22), $1 \leqslant t \leqslant p$, and $n^{2-1/p}/s_n \to c \in (0,\infty]$ that

$$\begin{aligned}
&n^{-1}\left\{\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^n (x_k-\bar{x}_n)(\varepsilon_k - a\delta_k) - a\sum_{k=1}^n (\delta_k-\bar{\delta}_n)^2\right\}\\
&= n^{-1}\sum_{k=1}^n \delta_k\varepsilon_k - \bar{\delta}_n\bar{\varepsilon}_n + \frac{1}{n^{1/(2t)-1/(2p)}}\cdot\frac{s_n^{1/2}}{n^{1-1/(2p)}}\cdot\frac{n^{-1/p+1/(2t)}}{s_n^{1/2}}\sum_{k=1}^n (x_k-\bar{x}_n)(\varepsilon_k - a\delta_k)\\
&\quad - an^{-1}\sum_{k=1}^n \delta_k^2 + a\bar{\delta}_n^{\,2}\\
&\to -aE\delta^2 \text{ a.s.}
\end{aligned} \tag{2.23}$$

and

$$\begin{aligned}
&n^{-2+1/p}\sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2\\
&= n^{-2+1/p}\left\{s_n + 2\sum_{k=1}^n (x_k-\bar{x}_n)\delta_k + \sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2\right\}\\
&= n^{-2+1/p}s_n + 2n^{-1+1/p}\cdot n^{-1}\sum_{k=1}^n (x_k-\bar{x}_n)\delta_k + n^{-1+1/p}\cdot n^{-1}\sum_{k=1}^n \delta_k^2 - n^{-1+1/p}\bar{\delta}_n^{\,2}\\
&\to \begin{cases} c^{-1} \text{ a.s.} & \text{if } 0 < c < \infty,\\ 0 \text{ a.s.} & \text{if } c = \infty,\end{cases}
\end{aligned} \tag{2.24}$$

since $n^{-1}\sum_{k=1}^n (x_k-\bar{x}_n)\delta_k \to 0$ a.s. by the same argument as (2.23).

Therefore, we have by (1.2), (2.23), and (2.24) that

$$n^{1-1/p}(\hat{a}_n - a) \to \begin{cases} -caE\delta^2 \text{ a.s.} & \text{if } 0 < c < \infty, \\ -\infty \text{ a.s.} & \text{if } c = \infty,\ a > 0, \\ \infty \text{ a.s.} & \text{if } c = \infty,\ a < 0, \end{cases}$$

which contradicts that $n^{1-1/p}(\hat{a}_n - a) \to 0$ a.s.

(ii) We prove that $n^{1-1/p}(\hat{b}_n - b) \to 0$ a.s. if and only if $n^{2-1/p}\bar{x}_n/s_n^* \to 0$, under condition $\sup_{n\geqslant 1} \min\{n, s_n\} n^{1-1/t}\bar{x}_n^2/s_n^* < \infty$. From the proof of Theorem 1.1 (ii), $\limsup_{n\to\infty} |\hat{a}_n - a| < \infty$ a.s., $n^{1-1/p}\bar{\varepsilon}_n \to 0$ a.s., and $n^{1-1/p}\bar{\delta}_n \to 0$ a.s. It follows by (1.3) that

$$n^{1-1/p}(\hat{b}_n - b) \to 0 \text{ a.s.} \quad \Longleftrightarrow \quad n^{1-1/p}\bar{x}_n(\hat{a}_n - a) \to 0 \text{ a.s.} \tag{2.25}$$

By (1.2), (2.15), and (2.21),

$$\begin{aligned} & n^{1-1/p}\bar{x}_n(\hat{a}_n - a) \to 0 \text{ a.s.} \\ \Longleftrightarrow\ & n^{1-1/p}\frac{\bar{x}_n}{s_n^*}\left\{\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^n (x_k - \bar{x}_n)(\varepsilon_k - a\delta_k) - a\sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2\right\} \to 0 \text{ a.s.} \\ \Longleftrightarrow\ & n^{1-1/p}\frac{\bar{x}_n}{s_n^*}\left\{\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k - a\sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2\right\} \to 0 \text{ a.s.} \end{aligned} \tag{2.26}$$

We have by (2.22), together with $\bar{\varepsilon}_n \to 0$ a.s., $\bar{\delta}_n \to 0$ a.s., and $n^{-1}\sum_{k=1}^n \delta_k^2 \to E\delta^2$ a.s., that

$$n^{-1}\left\{\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k - a\sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2\right\} \to -aE\delta^2 \text{ a.s.}$$

Hence,

$$\begin{aligned} & n^{1-1/p}\frac{\bar{x}_n}{s_n^*}\left\{\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k - a\sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2\right\} \to 0 \text{ a.s.} \\ & \Longleftrightarrow n^{2-1/p}\bar{x}_n/s_n^* \to 0. \end{aligned} \tag{2.27}$$

Combining (2.25)–(2.27), we obtain the result. □

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