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SOME NEW GENERALIZATIONS OF WEIGHTED DYNAMIC HARDY–KNOPP TYPE INEQUALITIES WITH KERNELS

ESSAM AWWAD

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Abstract. In this paper, we establish some new generalizations of weighted dynamic Hardy-Knopp type inequalities with kernels on time scales and also, some new characterizations of the weights for these inequalities in different spaces. The main results will be proved by using the Hölder inequality, the Jensen inequality and the Minkowski inequality. These inequalities (when $\mathbb{T}=\mathbb{R}$) contain the characterization of Kaijser, Nikolova, Persson and Wedestig. Also, (when $\mathbb{T}=\mathbb{N}$) our results are essentially new.

1. Introduction

In 1920, Hardy [8] proved the discrete inequality

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{i=1}^{n}a(i)\right)^{p}\leqslant\left(\frac{p}{p-1}\right)^{p}\sum_{n=1}^{\infty}a^{p}(n),\quad p>1, \tag{1}$$

where $a(n)\geqslant 0$ for $n\geqslant 1$, $a(n)\in l^{p}(\mathbb{N})$ (i.e. $\sum_{n=1}^{\infty}a^{p}(n)<\infty$) and the constant $(p/(p-1))^{p}$ is the best possible. In [9, Theorem A] Hardy proved the integral version of (1), by using the calculus of variations, which states that for $f\geqslant 0$ and integrable over any finite interval $(0,\lambda)$, where $\lambda\in(0,\infty)$ and $f\in L^{p}(0,\infty)$ and $p>1$, then

$$\int_{0}^{\infty}\left(\frac{1}{\lambda}\int_{0}^{\lambda}f(\tau)d\tau\right)^{p}d\lambda\leqslant\left(\frac{p}{p-1}\right)^{p}\int_{0}^{\infty}f^{p}(\lambda)d\lambda. \tag{2}$$

The constant $(p/(p-1))^{p}$ in (2) is the best possible. Many of generalizations have appeared in the literature, we refer the reader to the papers [7, 11, 12, 13, 19, 24, 25], and the books [17, 18, 20]. In [10] Hardy and Littlewood showed that the inequality (2) holds with reversed sign when $0<p<1$, provided that the integral $\int_{0}^{\lambda}f(t)dt$ is replaced by $\int_{\lambda}^{\infty}f(t)dt$. In particular, it was proved that if $f(\lambda)\geqslant 0$, $\int_{0}^{\infty}f^{p}(\lambda)d\lambda<\infty$, then

$$\int_{0}^{\infty}\left(\frac{1}{\lambda}\int_{\lambda}^{\infty}f(\tau)d\tau\right)^{p}d\lambda>\left(\frac{p}{1-p}\right)^{p}\int_{0}^{\infty}f^{p}(\lambda)d\lambda,\ 0<p<1,$$

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unless $f \equiv 0$. Also, the constant $(p/(1-p))^p$ is the best possible. In 1928, Knopp [16] proved the continuous inequality

$$\int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right)d\zeta \leqslant e\int_0^\infty f(\zeta)d\zeta, \tag{3}$$

where f is a nonnegative and integrable function. The constant e in (3) is the best constant. The inequality (3) is called a Knopp-type inequality. The inequality (3) can be considered as a limit, for p tending to infinity of the classical Hardy integral inequality (2), so for the function $f^{1/p}$, we have

$$\int_0^\infty \left(\frac{1}{\zeta}\int_0^\zeta f^{\frac{1}{p}}(t)dt\right)^p d\zeta \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty f(\zeta)d\zeta.$$

Indeed

$$\lim_{p\to\infty}\left(\frac{1}{\zeta}\int_0^\zeta f^{\frac{1}{p}}(t)dt\right)^p = \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right),$$

while $(p/(p-1))^p \to e$ as $p\to\infty$. If we replace $f(t)$ by $f(t)/t$ in (3), then we have that

$$\begin{aligned}\int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln\frac{f(t)}{t}dt\right)d\zeta &= \int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt - \frac{1}{\zeta}\int_0^\zeta (\ln t)\,dt\right)d\zeta\\ &= \int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt - \ln\zeta + 1\right)d\zeta\\ &= e\int_0^\infty \frac{1}{\zeta}\exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right)d\zeta,\end{aligned}$$

where $\int_0^\zeta (\ln t)\,dt = \zeta\ln\zeta - \zeta$. Therefore we have from (3) by replacing $f(t)$ with $f(t)/t$ that

$$\int_0^\infty \exp\left(\frac{1}{\zeta}\int_0^\zeta \ln f(t)dt\right)\frac{d\zeta}{\zeta} \leqslant \int_0^\infty f(\zeta)\frac{d\zeta}{\zeta}. \tag{4}$$

In 2002, Kaijser et al. [14] generalized (4) with a convex function and proved the general Hardy-Knopp inequality

$$\int_0^\infty \Phi\left(\frac{1}{\zeta}\int_0^\zeta f(t)dt\right)\frac{d\zeta}{\zeta} \leqslant \int_0^\infty \Phi(f(\zeta))\frac{d\zeta}{\zeta}, \tag{5}$$

where Φ is a convex function on $\mathbb{R}^+$ and $f:\mathbb{R}^+\to\mathbb{R}^+$ is a locally integrable positive function. In 2003, Čižmešija et al. [6] proved a generalization of the Hardy-Knopp inequality (5) with two different weighted functions. In particular, it was proved that if $0<b\leqslant\infty$, $u:(0,b)\to\mathbb{R}$ is a nonnegative function such that the function $\zeta\to u(\zeta)/\zeta^2$ is locally integrable on $(0,b)$ and Φ is convex on (a,c), where $-\infty\leqslant a<c\leqslant\infty$, the inequality

$$\int_0^b u(\zeta)\Phi\left(\frac{1}{\zeta}\int_0^\zeta f(t)dt\right)\frac{d\zeta}{\zeta} \leqslant \int_0^b \upsilon(\zeta)\Phi(f(\zeta))\frac{d\zeta}{\zeta},$$

holds for all integrable functions $f:(0,b)\to\mathbb{R}$, such that $f(\zeta)\in(a,c)$ for all $\zeta\in(0,b)$ and the function υ is defined by

$$\upsilon(t):=t\int_t^b \frac{u(\zeta)}{\zeta^2}d\zeta, \quad \text{for} \quad t\in(0,b).$$

In 2005, Kaijser et al. [15] applied Jensen's inequality for convex functions and Fubini's theorem and established an interesting generalization of Hardy's type inequalitiy (2). In particular, they proved that if $0<b\leqslant\infty$, $u:(0,b)\to\mathbb{R}$ and $k:(0,b)\times(0,b)\to\mathbb{R}$ are non-negative functions, such that $0<K(t):=\int_0^t k(t,\theta)d\theta<\infty$, $t\in(0,b)$ and

$$\upsilon(\zeta):=\zeta\int_\zeta^b u(t)\frac{k(t,\zeta)}{K(t)}\frac{dt}{t}<\infty, \quad \zeta\in(0,b),$$

then

$$\int_0^b u(\zeta)\Phi(A_kf(\zeta))\frac{d\zeta}{\zeta}\leqslant\int_0^b \upsilon(\zeta)\Phi(f(\zeta))\frac{d\zeta}{\zeta}, \tag{6}$$

where Φ is a convex function on an interval $I\subseteq\mathbb{R}$, $f:(0,b)\to\mathbb{R}$ is a function with values in I, and

$$A_kf(\zeta):=\frac{1}{K(\zeta)}\int_0^\zeta k(\zeta,\theta)f(\theta)d\theta,\ K(\zeta)=\int_0^\zeta k(\zeta,\theta)d\theta,\ \ \zeta\in(0,b).$$

Also, in [15] it is proved that if $1<p\leqslant q<\infty$, $s\in(1,p)$ and $0<b<\infty$. Furthermore assume that Φ is a convex and strictly monotone function on (a,c), $-\infty<a<c<\infty$ and assumed that the general Hardy operator A_k defined as following

$$A_kf(\zeta)=\frac{1}{K(\zeta)}\int_0^\zeta k(\zeta,\theta)f(\theta)d\theta, \quad K(\zeta)=\int_0^\zeta k(\zeta,\theta)d\theta,$$

where $k:\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$ is a nonnegative kernel and assume that $u(\zeta)$ and $v(\zeta)$ are nonnegative weighted functions. Then the inequality

$$\left(\int_0^b[\Phi(A_kf(\zeta))]^q u(\zeta)\frac{d\zeta}{\zeta}\right)^{\frac{1}{q}}\leqslant C\left[\int_0^b\Phi^p(f(\zeta))v(\zeta)\frac{d\zeta}{\zeta}\right]^{\frac{1}{p}}, \tag{7}$$

holds for all the nonnegative functions $f(\zeta)$, $a<f(\zeta)<c$, $\zeta\in[0,b]$ and $C>0$, if

$$A(s):=\sup_{0<\theta<b}[V(\theta)]^{\frac{s-1}{p}}\left(\int_\theta^b\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^q[V(\zeta)]^{\frac{q(p-s)}{p}}u(\zeta)\frac{d\zeta}{\zeta}\right)^{\frac{1}{q}}<\infty,$$

where

$$V(\theta)=\int_0^\theta[v(t)]^{\frac{-1}{p-1}}t^{\frac{1}{p-1}}dt.$$

In this paper we consider dynamic inequalities on time scales; see [1, 2]. In [23] Saker et al. proved that the dynamic inequality

$$\left(\int_a^b u(\zeta)\left(\int_a^{\sigma(\zeta)}f(t)\Delta t\right)^q\Delta\zeta\right)^{1/q}\leqslant C\left(\int_a^b \upsilon(\zeta)f^p(\zeta)\Delta\zeta\right)^{1/p},\ 1<p\leqslant q<\infty, \tag{8}$$

holds for all nonnegative rd-continuous functions f on $[a,b]_{\mathbb{T}}$ with $a,\ b\in\mathbb{T}$ if and only if the following condition holds

$$B=\sup_{a<\zeta<b}\left(\int_{\zeta}^{b}u(t)\Delta t\right)^{1/q}\left(\int_{a}^{\sigma(\zeta)}\upsilon^{1-p^{*}}(t)\Delta t\right)^{1/p^{*}}<\infty,\quad p^{*}=\frac{p}{p-1}. \tag{9}$$

Moreover, the estimate for the constant $C>0$ in (8) is given by

$$B\leqslant C\leqslant\left(1+\frac{q}{p^{*}}\right)^{1/q}\left(1+\frac{p^{*}}{q}\right)^{1/p^{*}}B.$$

The functions u and v which are nonnegative rd-continuous functions are called the weighted functions and the condition (9) gives the characterization of these two functions which leads to the validation of the inequality (8). Also in [23] they proved that if $1<p\leqslant q<\infty,\ f\in C_{rd}\left([a,b]_{\mathbb{T}},\mathbb{R}\right)$ is a nonnegative function and $u,\ v$ are positive rd-continuous functions on $(a,b)_{\mathbb{T}}$, then the inequality

$$\left(\int_{a}^{b}u(\zeta)\left(\int_{\zeta}^{b}f(t)\Delta t\right)^{q}\Delta\zeta\right)^{1/q}\leqslant C\left(\int_{a}^{b}\upsilon(\zeta)f^{p}(\zeta)\Delta\zeta\right)^{1/p}, \tag{10}$$

holds, if and only if

$$B_{1}=\sup_{a<\zeta<b}\left(\int_{a}^{\sigma(\zeta)}u(t)\Delta t\right)^{1/q}\left(\int_{\zeta}^{b}\upsilon^{1-p^{*}}(t)\Delta t\right)^{1/p^{*}}<\infty,\ p^{*}=\frac{p}{p-1}.$$

Moreover, the estimate for the constant $C>0$ in (10) is given by

$$B_{1}\leqslant C\leqslant\left(1+\frac{q}{p^{*}}\right)^{1/q}\left(1+\frac{p^{*}}{q}\right)^{1/p^{*}}B_{1}.$$

In [21] Özkan et al. proved that if $0\leqslant a<b\leqslant\infty,\ u\in C_{rd}\left([a,b),\mathbb{R}\right)$ is a nonnegative function such that the delta integral $\int_{t}^{b}\frac{u(\zeta)}{(\zeta-a)(\sigma(\zeta)-a)}\Delta\zeta$ exists as a finite number, and the function v is defined by

$$v(t):=(t-a)\int_{t}^{b}\frac{u(\zeta)}{(\zeta-a)(\sigma(\zeta)-a)}\Delta\zeta,\quad t\in[a,b).$$

Furthermore if $\Phi:(c,d)\rightarrow\mathbb{R}$ is continuous and convex, where $c,\ d\in\mathbb{R}$, then the inequality

$$\int_{a}^{b}u(\zeta)\Phi\left(\frac{1}{\sigma(\zeta)-a}\int_{a}^{\sigma(\zeta)}f(t)\Delta t\right)\frac{\Delta\zeta}{\zeta-a}\leqslant\int_{a}^{b}v(\zeta)\Phi\left(f(\zeta)\right)\frac{\Delta\zeta}{\zeta-a}, \tag{11}$$

holds for all delta integrable functions $f\in C_{rd}\left([a,b),\mathbb{R}\right)$ such that $f(\zeta)\in(c,d)$.

Also, they proved that if $u\in C_{rd}\left([b,\infty),\mathbb{R}\right)$ is a nonnegative function and the function v is defined by

$$v(t):=\frac{1}{t}\int_{b}^{t}u(\zeta)\Delta\zeta,\quad t\in[b,\infty).$$

Furthermore if $\Phi : (c,d) \to \mathbb{R}$ is a continuous and convex function, where $c,\ d \in \mathbb{R}$, then the inequality

$$\int_a^b u(\zeta)\Phi\left(\frac{1}{\sigma(\zeta)-a}\int_a^{\sigma(\zeta)} f(t)\Delta t\right)\frac{\Delta\zeta}{\zeta-a} \leqslant \int_a^b v(\zeta)\Phi(f(\zeta))\frac{\Delta\zeta}{\zeta-a},$$

holds for all delta integrable functions $f \in C_{rd}([b,\infty),\mathbb{R})$ such that $f(\zeta) \in (c,d)$ for all $\zeta \in [b,\infty)$. In 2009, Özkan and Yildirim [22] proved the time scale version of (6) and proved that if $k(\zeta,\theta) \in C_{rd}([a,b)\times[a,b),\mathbb{R})$ and $u \in C_{rd}([a,b),\mathbb{R})$ are nonnegative functions and the function v is defined by

$$v(t) = (t-a)\int_t^b \frac{k(\zeta,t)}{K(\sigma(\zeta),\zeta)}u(\zeta)\frac{\Delta\zeta}{\zeta-a}, \quad t \in [a,b).$$

Furthermore if $\Phi : (c,d) \to \mathbb{R}$ is a continuous and convex function, where $c,\ d \in \mathbb{R}$, then the inequality

$$\int_a^b u(\zeta)\Phi(A_k f(\sigma(\zeta),\zeta))\frac{\Delta\zeta}{\zeta-a} \leqslant \int_a^b v(\zeta)\Phi(f(\zeta))\frac{\Delta\zeta}{\zeta-a}, \tag{12}$$

holds for all delta integrable functions $f \in C_{rd}([a,b),\mathbb{R})$ such that $f(\zeta) \in (c,d)$, where

$$A_k f(t,s) := \frac{1}{K(t,s)}\int_a^t k(s,\theta)f(\theta)\Delta\theta, \quad K(t,s) := \int_a^t k(s,\theta)\Delta\theta.$$

Also, the book [2] by Agarwal, O'Regan and Saker contains some dynamic inequalities of Hardy-Knopp type on time scales.

Our aim in this paper is to establish some new characterizations of the weights for dynamic inequalities of Hardy-Knopp type with kernels and generalize the inequality (6) on time scales with formula

$$\int_a^b \phi^t\left(A_k f(\sigma(\zeta),\zeta)\right)\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta \leqslant \left(\frac{B}{A}\right)^t\left(\int_a^b \phi(f(\theta))\frac{v(\theta)}{\sigma(\theta)-a}\Delta\theta\right)^t, \ t \geqslant 1.$$

The paper is organized as follows. In Section 2, we present some preliminaries concerning the theory of time scales and some basic theorems needed in Section 3 where we prove the main results. Our main results (when $\mathbb{T}=\mathbb{R}$) our results give the characterizations of inequality (6) proved by Kaijser, Nikolova, Persson and Wedestig. Also, (when $\mathbb{T}=\mathbb{N}$) our results are essentially new. As a special case of our results on time scales, we can get the inequality (12) proved by Özkan and Yildirim [22].

2. Preliminaries and basic lemmas

For completeness, we recall the following concepts related to the notion of time scales. For more details of time scale analysis we refer the reader to the two books by Bohner and Peterson [4], [5] which summarize and organize much of the time scale calculus. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

The derivative of the product fg and the quotient f/g (where $gg^\sigma \neq 0$) of two differentiable functions f and g are given by

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma, \quad \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}. \tag{13}$$

In this paper, we will refer to the (delta) integral which is defined as follows: If $G^\Delta(t) = g(t)$, then $\int_a^t g(s)\Delta s := G(t) - G(a)$. It can be shown (see [4]) that if $g \in C_{rd}(\mathbb{T},\mathbb{R})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^\Delta(t) = g(t)$, $t \in \mathbb{T}$. An improper integral is defined by $\int_a^\infty g(t)\Delta t = \lim_{b\to\infty} \int_a^b g(t)\Delta t$, and the integration by parts formula on time scales is given by

$$\int_a^b u(t)v^\Delta(t)\Delta t = [u(t)v(t)]_a^b - \int_a^b u^\Delta(t)v^\sigma(t)\Delta t. \tag{14}$$

The time scales chain rule (see [4, Theorem 1.87]) is given by

$$(g\circ\delta)^\Delta(t) = g'(\delta(d))\,\delta^\Delta(t), \text{ where } d \in [t,\sigma(t)], \tag{15}$$

where it is assumed that $g : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and $\delta : \mathbb{T} \to \mathbb{R}$ is delta differentiable. A simple consequence of Keller's chain rule [4, Theorem 1.90] is given by

$$(\zeta^\gamma(t))^\Delta = \gamma \int_0^1 [h\zeta^\sigma(t) + (1-h)\zeta(t)]^{\gamma-1}\, dh\, \zeta^\Delta(t). \tag{16}$$

The Hölder inequality, see [4, Theorem 6.13], on time scales is given by

$$\int_a^b |f(\lambda)g(\lambda)|\Delta\lambda \leqslant \left[\int_a^b |f(\lambda)|^\gamma \Delta\lambda\right]^{\frac{1}{\gamma}} \left[\int_a^b |g(\lambda)|^\nu \Delta\lambda\right]^{\frac{1}{\nu}}, \tag{17}$$

where $a,\ b \in \mathbb{T}$, $f,\ g \in \mathrm{C}_{rd}(\mathbb{I},\mathbb{R})$, $\gamma > 1$ and $\frac{1}{\gamma} + \frac{1}{\nu} = 1$. The special case $\gamma = \nu = 2$ in (17) yields the time scales Cauchy-Schwarz inequality.

THEOREM 1. (Jensen's inequality) *Assume that $\mathbb{T}$ is a time scale with $a,\ b \in \mathbb{T}$ and $c,\ d \in \mathbb{R}$. If $h \in \mathrm{C}_{rd}([a,b]_\mathbb{T},\mathbb{R})$, $g : [a,b]_\mathbb{T} \to (c,d)$ is rd-continuous and $\Phi : (c,d) \to \mathbb{R}$ is continuous and convex, then*

$$\Phi\left(\frac{1}{\int_a^b h(s)\Delta s}\int_a^b h(t)g(t)\Delta t\right) \leqslant \frac{1}{\int_a^b h(s)\Delta s}\int_a^b h(t)\Phi(g(t))\Delta t. \tag{18}$$

The direction of the inequality (18) will be reversed if Φ is a concave function.

Let $(\Omega,\mathscr{M},\mu_\Delta)$ and $(\Lambda,\mathscr{L},\lambda_\Delta)$ be finite dimensional time scale measure spaces. We define the product measure space $(\Omega\times\Lambda,\mathscr{M}\times\mathscr{L},\mu_\Delta\times\lambda_\Delta)$, where $\mathscr{M}\times\mathscr{L}$ is the product $\sigma-$algebra generated by $\{E\times F : E\in\mathscr{M}, F\in\mathscr{L}\}$ and $(\mu_\Delta\times\lambda_\Delta)(E\times F) = \mu_\Delta(E)\lambda_\Delta(F)$.

THEOREM 2. (Minkowski's inequality [3]) *Let u, v and f be nonnegative functions on Ω, Λ and $\Omega\times\Lambda$, respectively. If $\alpha \geqslant 1$, then*

$$\left(\int_{\Omega} u(\zeta)\left(\int_{\Lambda} f(\zeta,\theta)v(\theta)\Delta\theta\right)^{\alpha}\Delta\zeta\right)^{\frac{1}{\alpha}} \leqslant \int_{\Lambda} v(\theta)\left(\int_{\Omega} f^{\alpha}(\zeta,\theta)u(\zeta)\Delta\zeta\right)^{\frac{1}{\alpha}}\Delta\theta. \tag{19}$$

3. Main results

Throughout the paper, we will assume that the functions (without mentioning) are nonnegative rd-continuous functions on $[a,b]_{\mathbb{T}}$ and the integrals considered are assumed to exist (finite i.e. convergent). We define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}} := [a,b]\cap\mathbb{T}$. Also, we define the general Hardy operator A_k as following

$$A_k f(\zeta,s) := \frac{1}{K(\zeta,s)}\int_a^{\zeta} k(s,\theta)f(\theta)\Delta\theta, \quad K(\zeta,s) := \int_a^{\zeta} k(s,\theta)\Delta\theta,$$

where $\zeta,\ s > a$ and $f \in C_{rd}([a,b]_{\mathbb{T}},\mathbb{R})$ and $k(s,\theta)\in C_{rd}([a,b]_{\mathbb{T}}\times[a,b]_{\mathbb{T}},\mathbb{R})$ are delta integrable and nonnegative functions.

Now, we are ready to state and prove our main results. We begin the time scale version of the inequality (6).

THEOREM 3. *Assume that $\mathbb{T}$ is a time scale with $a,\ b\in\mathbb{T}$, $t\geqslant 1$ and u, v are nonnegative weighted functions such that*

$$v(\theta) = (\sigma(\theta)-a)\left(\int_{\theta}^{b}\left(\frac{k(\zeta,\theta)}{K(\sigma(\zeta),\zeta)}\right)^{t}\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\right)^{\frac{1}{t}}. \tag{20}$$

Furthermore assume that ϕ,ψ are nonnegative functions on (c,d), $-\infty < c < d < \infty$ and ψ is a convex function such that

$$A\psi(\zeta) \leqslant \phi(\zeta) \leqslant B\psi(\zeta), \;\; c<\zeta<d, \tag{21}$$

where A, B are constants, then

$$\int_a^b \phi^t\left(A_k f(\sigma(\zeta),\zeta)\right)\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta \leqslant \left(\frac{B}{A}\right)^{t}\left(\int_a^b \phi(f(\theta))\frac{v(\theta)}{\sigma(\theta)-a}\Delta\theta\right)^{t}, \tag{22}$$

holds for the nonnegative function f.

Proof. Using (21) and Applying Jensen's inequality (where ψ is convex), we obtain

$$\begin{aligned}
&\int_a^b \phi^t\left(A_k f(\sigma(\zeta),\zeta)\right) u(\zeta)\frac{\Delta\zeta}{\sigma(\zeta)-a}\\
&=\int_a^b \phi^t\left(\frac{1}{K(\sigma(\zeta),\zeta)}\int_a^{\sigma(\zeta)} k(\zeta,\theta)f(\theta)\Delta\theta\right)\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\\
&\leqslant B^t\int_a^b \psi^t\left(\frac{1}{K(\sigma(\zeta),\zeta)}\int_a^{\sigma(\zeta)} k(\zeta,\theta)f(\theta)\Delta\theta\right)\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\\
&\leqslant B^t\int_a^b \frac{1}{K^t(\sigma(\zeta),\zeta)}\left(\int_a^{\sigma(\zeta)} k(\zeta,\theta)\psi(f(\theta))\Delta\theta\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta. \qquad (23)
\end{aligned}$$

Applying Minkowski's inequality on the term

$$\int_a^b \frac{1}{K^t(\sigma(\zeta),\zeta)}\left(\int_a^{\sigma(\zeta)} k(\zeta,\theta)\psi(f(\theta))\Delta\theta\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta,$$

with $t\geqslant 1$, we see that

$$\begin{aligned}
&\left(\int_a^b \frac{1}{K^t(\sigma(\zeta),\zeta)}\left(\int_a^{\sigma(\zeta)} k(\zeta,\theta)\psi(f(\theta))\Delta\theta\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\right)^{\frac{1}{t}}\\
&\leqslant \int_a^b \psi(f(\theta))\left(\int_\theta^b\left(\frac{k(\zeta,\theta)}{K(\sigma(\zeta),\zeta)}\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\right)^{\frac{1}{t}}\Delta\theta,
\end{aligned}$$

then

$$\begin{aligned}
&\int_a^b \frac{1}{K^t(\sigma(\zeta),\zeta)}\left(\int_a^{\sigma(\zeta)} k(\zeta,\theta)\psi(f(\theta))\Delta\theta\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\\
&\leqslant\left[\int_a^b \psi(f(\theta))\left(\int_\theta^b\left(\frac{k(\zeta,\theta)}{K(\sigma(\zeta),\zeta)}\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\right)^{\frac{1}{t}}\Delta\theta\right]^t\\
&=\left[\int_a^b \psi(f(\theta))\frac{1}{\sigma(\theta)-a}(\sigma(\theta)-a)\left(\int_\theta^b\left(\frac{k(\zeta,\theta)}{K(\sigma(\zeta),\zeta)}\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\right)^{\frac{1}{t}}\Delta\theta\right]^t. \qquad (24)
\end{aligned}$$

Substituting (24) into (23), we have from (20) that

$$\begin{aligned}
&\int_a^b \phi^t\left(A_k f(\sigma(\zeta),\zeta)\right) u(\zeta)\frac{\Delta\zeta}{\sigma(\zeta)-a}\\
&\leqslant B^t\left[\int_a^b \psi(f(\theta))\frac{1}{\sigma(\theta)-a}(\sigma(\theta)-a)\left(\int_\theta^b\left(\frac{k(\zeta,\theta)}{K(\sigma(\zeta),\zeta)}\right)^t\frac{u(\zeta)}{\sigma(\zeta)-a}\Delta\zeta\right)^{\frac{1}{t}}\Delta\theta\right]^t\\
&=B^t\left[\int_a^b \psi(f(\theta))\frac{1}{\sigma(\theta)-a}v(\theta)\Delta\theta\right]^t,
\end{aligned}$$

and then by using (21), we observe that

$$\int_a^b \phi^t \left(A_k f\left(\sigma(\zeta),\zeta\right)\right) u(\zeta) \frac{\Delta\zeta}{\sigma(\zeta)-a} \leqslant \left(\frac{B}{A}\right)^t \left[\int_a^b \phi(f(\theta)) \frac{1}{\sigma(\theta)-a} v(\theta)\Delta\theta\right]^t,$$

which is the desired inequality (22). The proof is complete. □

REMARK 1. As a special case of Theorem 3, when $(t=1,\ A=B)$ and by replacing $u(\zeta)$ and $v(\zeta)$ with $(\sigma(\zeta)-a)u(\zeta)/(\zeta-a)$ and $(\sigma(\zeta)-a)v(\zeta)/(\zeta-a)$, respectively, we obtain (12) proved by Özkan and Yildirim [22]

REMARK 2. When $\mathbb{T}=\mathbb{R}$, $a=0$, $\sigma(\zeta)=\zeta$, $t=1$ and $A=B$, we get the inequality (6) proved by Kaijser et al. [15].

REMARK 3. When $\mathbb{T}=\mathbb{N}$, $a=1$, $\sigma(n)=n+1$, the inequality (22) reduces to the discrete inequality

$$\begin{aligned}&\sum_{n=1}^{N} \phi^t \left(\frac{1}{\sum_{m=1}^{n} k(n,m)} \sum_{m=1}^{n} k(n,m) f(m)\right) \frac{u(n)}{n} \\ &\leqslant \left(\frac{B}{A}\right)^t \left[\sum_{n=1}^{N} \phi(f(n)) \frac{v(n)}{n}\right]^t, \quad \text{for } t \geqslant 1 \text{ and } N \in \mathbb{N}.\end{aligned}$$

In the following theorem, we characterize the weighted functions for dynamic inequalities with kernels in different spaces.

THEOREM 4. *Assume that $\mathbb{T}$ is a time scale with $a,\ b\in\mathbb{T}$, $0<p,\ s<1$ such that $0<p<s<\infty$ and $1<q<\infty$. Also, we assume that ϕ is a nonnegative and convex function on (c,d), $-\infty<c<d<\infty$, and $u,\ v$ are nonnegative weighted functions. Then the inequality*

$$\begin{aligned}&\left(\int_a^b \left[\phi\left(A_k f\left(\sigma(\zeta),\zeta\right)\right)\right]^{\frac{q}{p}} u(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta)-a)}\right)^{\frac{p}{q}} \\ &\leqslant C\left[\int_a^b \phi^{\frac{1}{p}}(f(\theta)) \frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}} \Delta\theta\right]^p,\end{aligned} \tag{25}$$

holds for the nonnegative function f and $C>0$, if

$$A(s)=\sup_{\theta\in[a,b]_{\mathbb{T}}} [V^\sigma(\theta)]^{\frac{1-s}{p}} \left(\int_\theta^b k^{\frac{q}{p}}(\zeta,\theta) \left(\frac{[V^\sigma(\zeta)]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\sigma(\zeta)-a)} \Delta\zeta\right)^{\frac{1}{q}} < \infty,$$

where

$$V(\theta)=\int_a^\theta [v(t)]^{\frac{-1}{1-p}} (\sigma(t)-a)^{\frac{1}{1-p}} \Delta t.$$

Proof. By applying Jensen's inequality, we get that

$$\begin{aligned}
&\int_a^b \left[\phi\left(A_k f(\sigma(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}\\
&=\int_a^b \left[\phi\left(\frac{1}{K(\sigma(\zeta),\zeta)}\int_a^{\sigma(\zeta)} k(\zeta,\theta)f(\theta)\Delta\theta\right)\right]^{\frac{q}{p}} \frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta\\
&\leqslant \int_a^b \left(\frac{1}{K(\sigma(\zeta),\zeta)}\int_a^{\sigma(\zeta)} k(\zeta,\theta)\phi(f(\theta))\Delta\theta\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta\\
&=\int_a^b \frac{1}{K^{\frac{q}{p}}(\sigma(\zeta),\zeta)}\left(\int_a^{\sigma(\zeta)} k(\zeta,\theta)\phi(f(\theta))\Delta\theta\right)^{\frac{q}{p}} \frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta. \qquad (26)
\end{aligned}$$

Define a function g such that

$$\phi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}=\phi(g(\theta)), \qquad (27)$$

and then we have that

$$\begin{aligned}
&\int_a^{\sigma(\zeta)} k(\zeta,\theta)\phi(f(\theta))\Delta\theta\\
&=\int_a^{\sigma(\zeta)} k(\zeta,\theta)\phi^p(g(\theta))\left[V^\sigma(\theta)\right]^{1-s}\left[V^\sigma(\theta)\right]^{s-1}\left[v(\theta)\right]^{-1}(\sigma(\theta)-a)\Delta\theta. \qquad (28)
\end{aligned}$$

By applying the Hölder inequality (17) with $\gamma=1/p>1$ and $\nu=1/(1-p)$, (where $0<p<1$) on the term

$$\int_a^{\sigma(\zeta)} k(\zeta,\theta)\phi^p(g(\theta))\left[V^\sigma(\theta)\right]^{1-s}\left[V^\sigma(\theta)\right]^{s-1}\left[v(\theta)\right]^{-1}(\sigma(\theta)-a)\Delta\theta,$$

we obtain that

$$\begin{aligned}
&\int_a^{\sigma(\zeta)} k(\zeta,\theta)\phi^p(g(\theta))\left[V^\sigma(\theta)\right]^{1-s}\left[V^\sigma(\theta)\right]^{s-1}\left[v(\theta)\right]^{-1}(\sigma(\theta)-a)\Delta\theta\\
&\leqslant \left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\phi(g(\theta))\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^p\\
&\quad\times\left(\int_a^{\sigma(\zeta)}\left[V^\sigma(\theta)\right]^{\frac{s-1}{1-p}}\left[v(\theta)\right]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta\right)^{1-p}. \qquad (29)
\end{aligned}$$

Substituting (29) into (28), we see that

$$\begin{aligned}
&\int_a^{\sigma(\zeta)} k(\zeta,\theta)\phi(f(\theta))\Delta\theta\\
&\leqslant \left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\phi(g(\theta))\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^p\\
&\quad\times\left(\int_a^{\sigma(\zeta)}\left[V^\sigma(\theta)\right]^{\frac{s-1}{1-p}}\left[v(\theta)\right]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta\right)^{1-p},
\end{aligned}$$

and then we have from (26) that

$$\begin{aligned}
&\int_a^b [\phi(A_k f(\sigma(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)} \\
&\leqslant \int_a^b \frac{u(\zeta)}{(\sigma(\zeta)-a)K^{\frac{q}{p}}(\sigma(\zeta),\zeta)}\left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\phi(g(\theta))[V^{\sigma}(\theta)]^{\frac{1-s}{p}}\Delta\theta\right)^q \\
&\quad \times\left(\int_a^{\sigma(\zeta)} [V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta\right)^{\frac{q(1-p)}{p}}\Delta\zeta. \qquad (30)
\end{aligned}$$

Since

$$V(\theta)=\int_a^{\theta}[v(t)]^{\frac{-1}{1-p}}(\sigma(t)-a)^{\frac{1}{1-p}}\Delta t,$$

then

$$V^{\Delta}(\theta)=[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}>0. \qquad (31)$$

Thus the function V is increasing. Applying the chain rule formula (15) on the term $V^{1+\frac{s-1}{1-p}}(\theta)$, we observe that

$$\left[V^{1+\frac{s-1}{1-p}}(\theta)\right]^{\Delta}=\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta}=\left(\frac{s-p}{1-p}\right)V^{\frac{s-1}{1-p}}(\zeta)V^{\Delta}(\theta), \qquad (32)$$

where $\zeta\in[\theta,\sigma(\theta)]$. Thus by substituting (31) into (32), we see that

$$\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta}=\left(\frac{s-p}{1-p}\right)V^{\frac{s-1}{1-p}}(\zeta)[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}. \qquad (33)$$

Since $\zeta\leqslant\sigma(\theta)$ and V is increasing, we have that

$$V(\zeta)\leqslant V^{\sigma}(\theta).$$

Using the facts that $0<s,\ p<1$ and $p<s<\infty$, where $(s-1)/(1-p)<0$ and $s-p>0$, we see that

$$V^{\frac{s-1}{1-p}}(\zeta)\geqslant[V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}. \qquad (34)$$

Substituting (34) into (33), we get (note $(s-p)/(1-p)>0$) that

$$\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta}\geqslant\left(\frac{s-p}{1-p}\right)[V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}},$$

and then

$$\int_a^{\sigma(\zeta)}\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta}\Delta\theta\geqslant\left(\frac{s-p}{1-p}\right)\int_a^{\sigma(\zeta)}[V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta.$$

Thus

$$\begin{aligned}&\int_a^{\sigma(\zeta)}\left[V^{\sigma}(\theta)\right]^{\frac{s-1}{1-p}}\left[v(\theta)\right]^{\frac{-1}{1-p}}\left(\sigma(\theta)-a\right)^{\frac{1}{1-p}}\Delta\theta\\&\leqslant\left(\frac{1-p}{s-p}\right)\int_a^{\sigma(\zeta)}\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta}\Delta\theta\\&=\left(\frac{1-p}{s-p}\right)\left[V^{\sigma}(\zeta)\right]^{\frac{s-p}{1-p}}.\end{aligned}\tag{35}$$

Substituting (35) into (30), we have

$$\begin{aligned}&\int_a^b\left[\phi\left(A_kf\left(\sigma(\zeta),\zeta\right)\right)\right]^{\frac{q}{p}}u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}\\&\leqslant\left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}}\int_a^b\left(\int_a^{\sigma(\zeta)}k^{\frac{1}{p}}(\zeta,\theta)\phi(g(\theta))\left[V^{\sigma}(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q\\&\quad\times\left(\frac{\left[V^{\sigma}(\zeta)\right]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta.\end{aligned}\tag{36}$$

From (36) and the definition of g in (27), we obtain

$$\begin{aligned}&\int_a^b\left[\phi\left(A_kf\left(\sigma(\zeta),\zeta\right)\right)\right]^{\frac{q}{p}}u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}\\&\leqslant\left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}}\int_a^b\left(\int_a^{\sigma(\zeta)}k^{\frac{1}{p}}(\zeta,\theta)\phi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\left[V^{\sigma}(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q\\&\quad\times\left(\frac{\left[V^{\sigma}(\zeta)\right]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta.\end{aligned}\tag{37}$$

Applying the Minkowski inequality on the term

$$\begin{aligned}&\int_a^b\left(\int_a^{\sigma(\zeta)}k^{\frac{1}{p}}(\zeta,\theta)\phi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\left[V^{\sigma}(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q\\&\quad\times\left(\frac{\left[V^{\sigma}(\zeta)\right]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta,\end{aligned}$$

with $q>1$, we observe that

$$\begin{aligned}&\int_a^b\left(\int_a^{\sigma(\zeta)}k^{\frac{1}{p}}(\zeta,\theta)\phi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\left[V^{\sigma}(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q\\&\quad\times\left(\frac{\left[V^{\sigma}(\zeta)\right]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta\end{aligned}$$

$$\leqslant \left[\int_a^b \phi^{\frac{1}{p}}(f(\theta)) \frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}} \right.$$
$$\left. \times [V^\sigma(\theta)]^{\frac{1-s}{p}} \left(\int_\theta^b k^{\frac{q}{p}}(\zeta,\theta) \left(\frac{[V^\sigma(\zeta)]^{s-p}}{K(\sigma(\zeta),\zeta)} \right)^{\frac{q}{p}} \frac{u(\zeta)}{(\sigma(\zeta)-a)} \Delta\zeta \right)^{\frac{1}{q}} \Delta\theta \right]^q . \qquad (38)$$

Substituting (38) into (37), we get

$$\int_a^b [\phi(A_k f(\sigma(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta)-a)}$$
$$\leqslant \left(\frac{1-p}{s-p} \right)^{\frac{q(1-p)}{p}} \left[\int_a^b \phi^{\frac{1}{p}}(f(\theta)) \frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}} \right.$$
$$\left. \times [V^\sigma(\theta)]^{\frac{1-s}{p}} \left(\int_\theta^b k^{\frac{q}{p}}(\zeta,\theta) \left(\frac{[V^\sigma(\zeta)]^{s-p}}{K(\sigma(\zeta),\zeta)} \right)^{\frac{q}{p}} \frac{u(\zeta)}{(\sigma(\zeta)-a)} \Delta\zeta \right)^{\frac{1}{q}} \Delta\theta \right]^q$$
$$\leqslant \left(\frac{1-p}{s-p} \right)^{\frac{q(1-p)}{p}} A^q(s) \left[\int_a^b \phi^{\frac{1}{p}}(f(\theta)) \frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}} \Delta\theta \right]^q ,$$

and then

$$\left(\int_a^b [\phi(A_k f(\sigma(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta)-a)} \right)^{\frac{p}{q}}$$
$$\leqslant \left(\frac{1-p}{s-p} \right)^{1-p} A^p(s) \left[\int_a^b \phi^{\frac{1}{p}}(f(\theta)) \frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}} \Delta\theta \right]^p ,$$

which is the desired inequality (25) with $C = \left(\frac{1-p}{s-p}\right)^{1-p} A^p(s)$. The proof is complete. □

THEOREM 5. *Assume that $\mathbb{T}$ is a time scale with $a,\ b \in \mathbb{T},\ 0<p,\ s<1$ such that $0<p<s<\infty$ and $1<q<\infty$. Also, we assume that ϕ,ψ are nonnegative functions on (c,d), $-\infty<c<d<\infty$ and ψ is a convex function such that*

$$A\psi \leqslant \phi \leqslant B\psi, \qquad (39)$$

where A, B are constants and u, v are nonnegative weighted functions. Then the inequality

$$\left(\int_a^b [\phi(A_k f(\sigma(\zeta),\zeta))]^{\frac{q}{p}} u(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta)-a)} \right)^{\frac{p}{q}}$$
$$\leqslant C \left[\int_a^b \phi^{\frac{1}{p}}(f(\theta)) \frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}} \Delta\theta \right]^p , \qquad (40)$$

holds for the nonnegative function f *and* $C>0$, *if*

$$D(s)=\sup_{\theta\in[a,b]_{\mathbb{T}}}[V^{\sigma}(\theta)]^{\frac{1-s}{p}}\left(\int_{\theta}^{b}k^{\frac{q}{p}}(\zeta,\theta)\left(\frac{[V^{\sigma}(\zeta)]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta\right)^{\frac{1}{q}}<\infty. \tag{41}$$

Proof. From (39) and by applying the Jensen inequality, we see that

$$\begin{aligned}
&\int_{a}^{b}[\phi(A_kf(\sigma(\zeta),\zeta))]^{\frac{q}{p}}u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}\\
&\leqslant B^{\frac{q}{p}}\int_{a}^{b}[\psi(A_kf(\sigma(\zeta),\zeta))]^{\frac{q}{p}}u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}\\
&=B^{\frac{q}{p}}\int_{a}^{b}\left[\psi\left(\frac{1}{K(\sigma(\zeta),\zeta)}\int_{a}^{\sigma(\zeta)}k(\zeta,\theta)f(\theta)\Delta\theta\right)\right]^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta\\
&\leqslant B^{\frac{q}{p}}\int_{a}^{b}\left(\frac{1}{K(\sigma(\zeta),\zeta)}\int_{a}^{\sigma(\zeta)}k(\zeta,\theta)\psi(f(\theta))\Delta\theta\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta\\
&=B^{\frac{q}{p}}\int_{a}^{b}\frac{1}{K^{\frac{q}{p}}(\sigma(\zeta),\zeta)}J^{\frac{q}{p}}(\zeta)\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta,
\end{aligned} \tag{42}$$

where

$$J(\zeta)=\int_{a}^{\sigma(\zeta)}k(\zeta,\theta)\psi(f(\theta))\Delta\theta. \tag{43}$$

Define a function g such that

$$\psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}=\psi(g(\theta)). \tag{44}$$

Substituting (44) into (43), we obtain

$$J(\zeta)=\int_{a}^{\sigma(\zeta)}k(\zeta,\theta)\psi^{p}(g(\theta))[v(\theta)]^{-1}(\sigma(\theta)-a)\Delta\theta.$$

Note that

$$J(\zeta)=\int_{a}^{\sigma(\zeta)}k(\zeta,\theta)\psi^{p}(g(\theta))[V^{\sigma}(\theta)]^{1-s}[V^{\sigma}(\theta)]^{s-1}[v(\theta)]^{-1}(\sigma(\theta)-a)\Delta\theta. \tag{45}$$

By applying the Hölder inequality (17) with $\gamma=1/p>1$ and $\nu=1/(1-p)$, (where $0<p<1$) on the term

$$\int_{a}^{\sigma(\zeta)}k(\zeta,\theta)\psi^{p}(g(\theta))[V^{\sigma}(\theta)]^{1-s}[V^{\sigma}(\theta)]^{s-1}[v(\theta)]^{-1}(\sigma(\theta)-a)\Delta\theta,$$

we see that

$$\begin{aligned}&\int_a^{\sigma(\zeta)} k(\zeta,\theta)\psi^p(g(\theta))\left[V^\sigma(\theta)\right]^{1-s}\left[V^\sigma(\theta)\right]^{s-1}[v(\theta)]^{-1}(\sigma(\theta)-a)\Delta\theta\\ &\leqslant\left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\psi(g(\theta))\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^p\\ &\quad\times\left(\int_a^{\sigma(\zeta)}\left[V^\sigma(\theta)\right]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta\right)^{1-p}. \end{aligned} \tag{46}$$

Substituting (46) into (45), we observe that

$$\begin{aligned}J(\zeta)\leqslant&\left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\psi(g(\theta))\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^p\\ &\times\left(\int_a^{\sigma(\zeta)}\left[V^\sigma(\theta)\right]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta\right)^{1-p}. \end{aligned} \tag{47}$$

Substituting (47) into (42), we have that

$$\begin{aligned}&\int_a^b\left[\phi\left(A_k f(\sigma(\zeta),\zeta)\right)\right]^{\frac{q}{p}}u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}\\ &\leqslant B^{\frac{q}{p}}\int_a^b\frac{1}{K^{\frac{q}{p}}(\sigma(\zeta),\zeta)}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\psi(g(\theta))\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q\\ &\quad\times\left(\int_a^{\sigma(\zeta)}\left[V^\sigma(\theta)\right]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta\right)^{\frac{q(1-p)}{p}}\Delta\zeta. \end{aligned} \tag{48}$$

Since

$$V(\theta)=\int_a^\theta[v(t)]^{\frac{-1}{1-p}}(\sigma(t)-a)^{\frac{1}{1-p}}\Delta t,$$

then

$$V^\Delta(\theta)=[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}>0. \tag{49}$$

Therefore the function V is increasing. By applying the chain rule formula (15) on the term $V^{1+\frac{s-1}{1-p}}(\theta)$, we obtain

$$\left[V^{1+\frac{s-1}{1-p}}(\theta)\right]^\Delta=\left[V^{\frac{s-p}{1-p}}(\theta)\right]^\Delta=\left(\frac{s-p}{1-p}\right)V^{\frac{s-1}{1-p}}(\zeta)V^\Delta(\theta), \tag{50}$$

where $\zeta\in[\theta,\sigma(\theta)]$. Thus by substituting (49) into (50), we see that

$$\left[V^{\frac{s-p}{1-p}}(\theta)\right]^\Delta=\left(\frac{s-p}{1-p}\right)V^{\frac{s-1}{1-p}}(\zeta)[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}. \tag{51}$$

Since $\zeta\leqslant\sigma(\theta)$ and V is increasing, we have that

$$V(\zeta)\leqslant V^\sigma(\theta).$$

Using the facts that $0 < s,\ p < 1$ and $p < s < \infty$, where $(s-1)/(1-p) < 0$ and $s-p > 0$, we see that

$$V^{\frac{s-1}{1-p}}(\zeta) \geqslant [V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}. \tag{52}$$

Substituting (52) into (51), we get (note $(s-p)/(1-p) > 0$) that

$$\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta} \geqslant \left(\frac{s-p}{1-p}\right)[V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}},$$

and then

$$\int_a^{\sigma(\zeta)}\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta}\Delta\theta \geqslant \left(\frac{s-p}{1-p}\right)\int_a^{\sigma(\zeta)}[V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta.$$

Thus

$$\begin{aligned}
&\int_a^{\sigma(\zeta)}[V^{\sigma}(\theta)]^{\frac{s-1}{1-p}}[v(\theta)]^{\frac{-1}{1-p}}(\sigma(\theta)-a)^{\frac{1}{1-p}}\Delta\theta \\
&\leqslant \left(\frac{1-p}{s-p}\right)\int_a^{\sigma(\zeta)}\left[V^{\frac{s-p}{1-p}}(\theta)\right]^{\Delta}\Delta\theta \\
&= \left(\frac{1-p}{s-p}\right)[V^{\sigma}(\zeta)]^{\frac{s-p}{1-p}}.
\end{aligned} \tag{53}$$

Substituting (53) into (48), we see that

$$\begin{aligned}
&\int_a^b[\phi(A_k f(\sigma(\zeta),\zeta))]^{\frac{q}{p}}u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)} \\
&\leqslant B^{\frac{q}{p}}\left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}}\int_a^b\left(\int_a^{\sigma(\zeta)}k^{\frac{1}{p}}(\zeta,\theta)\psi(g(\theta))[V^{\sigma}(\theta)]^{\frac{1-s}{p}}\Delta\theta\right)^q \\
&\quad\times(V^{\sigma}(\zeta))^{\frac{q(s-p)}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\frac{1}{K^{\frac{q}{p}}(\sigma(\zeta),\zeta)}\Delta\zeta.
\end{aligned} \tag{54}$$

From (54) and the definition of g in (44), we obtain

$$\begin{aligned}
&\int_a^b[\phi(A_k f(\sigma(\zeta),\zeta))]^{\frac{q}{p}}u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)} \\
&\leqslant B^{\frac{q}{p}}\left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}}\int_a^b\left(\int_a^{\sigma(\zeta)}k^{\frac{1}{p}}(\zeta,\theta)\psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}[V^{\sigma}(\theta)]^{\frac{1-s}{p}}\Delta\theta\right)^q \\
&\quad\times\frac{u(\zeta)}{(\sigma(\zeta)-a)}\left[\frac{(V^{\sigma}(\zeta))^{s-p}}{K(\sigma(\zeta),\zeta)}\right]^{\frac{q}{p}}\Delta\zeta.
\end{aligned} \tag{55}$$

By applying Minkowski's inequality on the term

$$\int_a^b \left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q \times \frac{u(\zeta)}{(\sigma(\zeta)-a)}\left[\frac{(V^\sigma(\zeta))^{s-p}}{K(\sigma(\zeta),\zeta)}\right]^{\frac{q}{p}}\Delta\zeta,$$

with $q>1,$ we observe that

$$\begin{aligned}
&\int_a^b \left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q \\
&\times \left(\frac{[V^\sigma(\zeta)]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta \\
&\leqslant \left[\int_a^b \psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\right. \\
&\left.\times\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\left(\int_\theta^b k^{\frac{q}{p}}(\zeta,\theta)\left(\frac{[V^\sigma(\zeta)]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta\right)^{\frac{1}{q}}\Delta\theta\right]^q. \qquad (56)
\end{aligned}$$

Substituting (41) into (56), we see that

$$\begin{aligned}
&\int_a^b \left(\int_a^{\sigma(\zeta)} k^{\frac{1}{p}}(\zeta,\theta)\psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\left[V^\sigma(\theta)\right]^{\frac{1-s}{p}}\Delta\theta\right)^q \\
&\times \left(\frac{[V^\sigma(\zeta)]^{s-p}}{K(\sigma(\zeta),\zeta)}\right)^{\frac{q}{p}}\frac{u(\zeta)}{(\sigma(\zeta)-a)}\Delta\zeta \\
&\leqslant D^q(s)\left[\int_a^b \psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\Delta\theta\right]^q \qquad (57)
\end{aligned}$$

Substituting (57) into (55), we get

$$\begin{aligned}
&\int_a^b \left[\phi\left(A_k f(\sigma(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)} \\
&\leqslant B^{\frac{q}{p}}\left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}} D^q(s)\left[\int_a^b \psi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\Delta\theta\right]^q,
\end{aligned}$$

From (39), we have that

$$\int_a^b \left[\phi\left(A_k f(\sigma(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}$$
$$\leqslant \left(\frac{B}{A}\right)^{\frac{q}{p}}\left(\frac{1-p}{s-p}\right)^{\frac{q(1-p)}{p}} D^q(s)\left[\int_a^b \phi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\Delta\theta\right]^q,$$

and then

$$\left(\int_a^b \left[\phi\left(A_k f(\sigma(\zeta),\zeta)\right)\right]^{\frac{q}{p}} u(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta)-a)}\right)^{\frac{p}{q}}$$
$$\leqslant \left(\frac{B}{A}\right)\left(\frac{1-p}{s-p}\right)^{1-p} D^p(s)\left[\int_a^b \phi^{\frac{1}{p}}(f(\theta))\frac{v^{\frac{1}{p}}(\theta)}{(\sigma(\theta)-a)^{\frac{1}{p}}}\Delta\theta\right]^p,$$

which is the desired inequality (40) with the constant $C=\left(\frac{B}{A}\right)\left(\frac{1-p}{s-p}\right)^{1-p}D^p(s)$. The proof is complete. □

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Essam Awwad
Department of Mathematics
College of Science and Arts, Jouf University
Jouf, Saudi Arabia
e-mail: emawwad@ju.edu.sa
and
Department of Mathematics
Faculty of Science, Benha University
Benha 13518, Egypt
e-mail: essam.awad@fsc.bu.edu.eg