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A NOTE ON TWO WEIGHTED DISCRETE CARLEMAN INEQUALITIES

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Abstract. In this paper, we re-examine two weighted discrete Carleman inequalities, discuss their correctness and optimal constants in detail, and get some correct and relatively complete conclusions.

1. Introduction

As we know, Carleman inequality was firstly established by T. Carleman in 1923 [1]. In fact, Carleman inequality is a derived result of Hardy inequality when the index tends to infinity. The basic form of Carleman inequality is [2]:

$$\sum_{n=1}^{\infty}(a_1a_2\cdots a_n)^{\frac{1}{n}} \leqslant e\sum_{n=1}^{\infty}a_n,$$

where $\{a_n\}$ is a non-negative sequence, $\sum\limits_{n=1}^{\infty} a_n$ is convergent. The equality holds if and only if $a_n = 0$, $n = 1,2,\cdots$, and e is the best possible.

The corresponding integral form is [2]:

$$\int_0^{\infty}\exp\left\{\frac{1}{x}\int_0^x \ln f(t)dt\right\}dx < e\int_0^{\infty} f(x)dx,$$

where $f(x)$ is positive and measurable in $(0,\infty)$, $\int_0^{\infty} f(x)dx$ is convergent, and e is the best possible.

Carleman inequality has found much attention among several mathematicians, and in many papers different proofs have been provided, such as [3–5].

There are a large number of results on the refinement of Carleman inequality, for example, some results can be seen in [6–12]. On the other hand, some other authors focused on the weighted forms of Carleman inequality, such as [13–16].

In the paper [17], G. Sunouchi and N. Takagi presented two weighted discrete Carleman inequalities:

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LEMMA 1.1. [17] *Let*

$$a_n \geqslant 0,\ \ \lambda_n > 0,\ \ (n = 1,2,3,\cdots)$$

and

$$\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n,$$

then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum_{n=1}^{\infty} \lambda_n a_n,$$

where the constant e is the best possible, provided that the right-hand side series is convergent.

LEMMA 1.2. [17] *There exists the inequality*

$$\sum_{n=1}^{\infty} \frac{\Lambda_n}{\lambda_n} (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum_{n=1}^{\infty} \lambda_n a_n, \tag{1.1}$$

under the same condition as in Lemma 1.1, where the constant e is the best possible.

However, we have some different opinions about the above two conclusions.

Lemma 1.1 also can be seen in [2, 18], but there were no records on these two monographs about the optimal constant of this inequality.

If Lemma 1.2 is correct, let $a_n = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$, we have $\lambda_1 \geqslant \frac{e-1}{e}$. However, there is no such a restriction on the sequence $\{\lambda_n\}$.

There is another version of (1.1) in [18]:

$$\sum_{n=1}^{\infty} \frac{n}{\lambda_n} (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.2}$$

In [18], the authors also pointed out that e is the best possible. But if the inequality (1.2) is correct, we also set $a_n = \begin{cases} 1, & n = 1 \\ 0, & n > 1 \end{cases}$, then we have $\lambda_1 \geqslant (\frac{e-1}{e})^{\frac{1}{2}}$. However, there is no such a restriction on the sequence $\{\lambda_n\}$.

It is worth mentioning that in the paper [19], J. Pečarić and K. B. Stolarsky also pointed out that there were some flaws on Lemma 1.2.

In this paper, we re-examine Lemma 1.1 and 1.2, discuss their correctness and optimal constants in detail, and get some correct and relatively complete conclusions. Our conclusions are as follow:

THEOREM 1.3. *Suppose that $\{\lambda_n\}$ is a positive sequence, $\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, $n = 1,2,3,\cdots$.*

Let $\{a_n\}$ be a non-negative sequence, and $\sum_{n=1}^{\infty} \lambda_n a_n$ be convergent. Then, we have

(1) There exists the following inequality:

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.3}$$

(2) The equality holds if and only if $a_n = 0,\ n = 1,2,3,\cdots$.

(3) If $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ *is convergent, then* e *is not the optimal constant.*

(4) If $\lim_{n\to\infty} \frac{\lambda_n}{\Lambda_n} = 0$*, but* $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ *is divergent, then* e *is the optimal constant.*

(5) If e *is the optimal constant, then* $\inf_{n\in N^+} \frac{\lambda_n}{\Lambda_n} = 0$.

THEOREM 1.4. *Suppose that* $\{\lambda_n\}$ *is a positive sequence,* $\Lambda_n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, $n = 1,2,3,\cdots$.

Let $\{a_n\}$ *be a non-negative sequence, and* $\sum_{n=1}^{\infty} \lambda_n a_n$ *be convergent. Then, we have*

(1) There exists the following inequality:

$$\sum_{n=1}^{\infty} \Lambda_n (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum_{n=1}^{\infty} \lambda_n a_n. \tag{1.4}$$

(2) The equality holds if and only if $a_n = 0,\ n = 1,2,3,\cdots$.

(3) If $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ *is convergent, then* e *is not the optimal constant.*

(4) If $\lim_{n\to\infty} \frac{\lambda_n}{\Lambda_n} = 0$*, but* $\sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ *is divergent, then* e *is the optimal constant.*

2. The proofs of main conclusions

Now we start to prove Theorem 1.1.

Proof. (1) Let $\{c_n\}$ be an arbitrary positive sequence. Based on weighted mean inequality, for any $n \in N^+$, we have

$$(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} = \left[\frac{(c_1 a_1)^{\lambda_1} (c_2 a_2)^{\lambda_2} \cdots (c_n a_n)^{\lambda_n}}{c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n}} \right]^{\frac{1}{\Lambda_n}} \leqslant \frac{\sum_{j=1}^{n} \lambda_j c_j a_j}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}},$$

and the equality holds if and only if $c_1 a_1 = c_2 a_2 = \cdots = c_n a_n$. Therefore,

$$\begin{aligned} \sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} &\leqslant \sum_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} \sum_{j=1}^{n} \lambda_j c_j a_j \\ &= \sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} \right] a_j. \end{aligned} \tag{2.1}$$

Let's select a proper sequence $\{c_n\}$, such that

$$\sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < \frac{e}{c_j}, \quad j = 1,2,3,\cdots.$$

Moreover, we demand that $\{c_n\}$ must be non-decreasing. Thus, $\{c_n\}$ has a generalized limit, which is denoted as $\lim\limits_{n\to\infty} c_n = c$, then $c \in (0,\infty]$.

If $c = \infty$, then $\sum\limits_{n=j}^{\infty} (\frac{1}{c_n} - \frac{1}{c_{n+1}}) = \frac{1}{c_j}$, $j = 1,2,3,\cdots$.

If $c < \infty$, then $\sum\limits_{n=j}^{\infty} (\frac{1}{c_n} - \frac{1}{c_{n+1}}) = \frac{1}{c_j} - \frac{1}{c}$, $j = 1,2,3,\cdots$.

Therefore, $\sum\limits_{n=j}^{\infty} (\frac{1}{c_n} - \frac{1}{c_{n+1}}) \leqslant \frac{1}{c_j}$, $j = 1,2,3\cdots$. Base on the last inequality, we can futher demand that $\{c_n\}$ satisfy the following inequality:

$$\sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < e \sum_{n=j}^{\infty} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\right), \quad j = 1,2,3,\cdots.$$

Moreover, we can let $\{c_n\}$ satisfy the following inequality:

$$\frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < e\left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\right), \quad n = 1,2,3,\cdots. \tag{2.2}$$

Now, let's construct a positive and non-decreasing sequence $\{c_n\}$, such that the inequality (2.2) holds.

Let $d_1 = 0$, $d_n \geqslant 0$, $n = 2,3,4,\cdots$, $c_n = \exp\left\{\sum\limits_{i=1}^{n} d_i\right\}$, $n = 1,2,3,\cdots$, then

$$c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n} = \prod_{i=1}^{n} c_i^{\lambda_i} = \prod_{i=1}^{n} \exp\left\{\lambda_i \sum_{j=1}^{i} d_j\right\} = \exp\left\{\sum_{i=1}^{n} \lambda_i \sum_{j=1}^{i} d_j\right\}.$$

For any $n \in N^+$, we have

$$\begin{aligned}\sum_{i=1}^{n} \lambda_i \sum_{j=1}^{i} d_j &= \sum_{j=1}^{n} d_j \sum_{i=j}^{n} \lambda_i = \sum_{j=1}^{n} (\Lambda_n - \Lambda_{j-1}) d_j = \Lambda_n \sum_{j=1}^{n} d_j - \sum_{j=1}^{n} \Lambda_{j-1} d_j \\ &= \Lambda_n \sum_{j=2}^{n} d_j - \sum_{j=2}^{n} \Lambda_{j-1} d_j,\end{aligned}$$

where we supplementary agree that $\Lambda_0 = 0$. Set $d_j = \frac{\lambda_{j-1}}{\Lambda_{j-1}}$, $j = 2,3,4,\cdots$, then

$$c_n = \exp\left\{\sum_{j=2}^{n} \frac{\lambda_{j-1}}{\Lambda_{j-1}}\right\} = \exp\left\{\sum_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j}\right\},$$

$$\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{i}d_j=\Lambda_n\sum_{j=2}^{n}\frac{\lambda_{j-1}}{\Lambda_{j-1}}-\sum_{j=2}^{n}\lambda_{j-1}=\Lambda_n\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\sum_{j=1}^{n-1}\lambda_j=\Lambda_n\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\Lambda_{n-1}.$$

Then

$$c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n}=\exp\left\{\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{i}d_j\right\}=\exp\left\{\Lambda_n\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\Lambda_{n-1}\right\},$$

$$\begin{aligned}(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}&=\exp\left\{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\frac{\Lambda_{n-1}}{\Lambda_n}\right\}=\exp\left\{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}-\frac{\Lambda_n-\lambda_n}{\Lambda_n}\right\}\\&=e^{-1}\exp\left\{\sum_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}\right\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}&=\frac{\frac{\lambda_n}{\Lambda_n}}{e^{-1}\exp\left\{\sum\limits_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}\right\}}<e\frac{\exp\left\{\frac{\lambda_n}{\Lambda_n}\right\}-1}{\exp\left\{\sum\limits_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}\right\}}\\&=e\left[\frac{1}{\exp\left\{\sum\limits_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}\right\}}-\frac{1}{\exp\left\{\sum\limits_{j=1}^{n}\frac{\lambda_j}{\Lambda_j}\right\}}\right]=e\left(\frac{1}{c_n}-\frac{1}{c_{n+1}}\right).\end{aligned}$$

Till now, we have obtained a sequence $\{c_n\}$, which is positive and increasing and satisfies the inequality (2.2).

Now, we set $c_n=\exp\left\{\sum\limits_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}\right\}$, $n=1,2,3,\cdots$, then

$$c=\lim_{n\to\infty}c_n=\lim_{n\to\infty}\exp\left\{\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}\right\}=\exp\left\{\lim_{n\to\infty}\sum_{j=1}^{n-1}\frac{\lambda_j}{\Lambda_j}\right\}=\exp\left\{\sum_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}\right\}.$$

Therefore, if $\sum\limits_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ is convergent, then $c<\infty$, else if $\sum\limits_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ is divergent, then $c=\infty$.

Base on (2.1) and (2.2), we have

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}&\leqslant\sum_{j=1}^{\infty}\lambda_j\left[c_j\sum_{n=j}^{\infty}\frac{\lambda_n}{\Lambda_n(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}\right]a_j\\&\leqslant e\sum_{j=1}^{\infty}\lambda_j\left[c_j\sum_{n=j}^{\infty}\left(\frac{1}{c_n}-\frac{1}{c_{n+1}}\right)\right]a_j.\end{aligned}\tag{2.3}$$

If $c<\infty$, then

$$\begin{aligned}\sum_{n=1}^{\infty}\lambda_n(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}&\leqslant e\sum_{j=1}^{\infty}\lambda_j\left[c_j\sum_{n=j}^{\infty}\left(\frac{1}{c_n}-\frac{1}{c_{n+1}}\right)\right]a_j\\&=e\sum_{j=1}^{\infty}\lambda_jc_j\left(\frac{1}{c_j}-\frac{1}{c}\right)a_j=e\sum_{j=1}^{\infty}\lambda_j\left(1-\frac{c_j}{c}\right)a_j\\&\leqslant e\sum_{j=1}^{\infty}\lambda_j\left(1-\frac{c_1}{c}\right)a_j=e\left(1-\frac{1}{c}\right)\sum_{j=1}^{\infty}\lambda_ja_j,\end{aligned}\tag{2.4}$$

then $\sum\limits_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum\limits_{n=1}^{\infty} \lambda_n a_n$.

If $c = \infty$, then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}} \right) \right] a_j = e \sum_{j=1}^{\infty} \lambda_j a_j.$$

In summary, we have $\sum\limits_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \sum\limits_{n=1}^{\infty} \lambda_n a_n$.

(2) The sufficiency is obvious, we only need to prove the necessity.

The equality holds, so

$$\sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} \right] a_j = e \sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}} \right) \right] a_j.$$

But for any $n \in N^+$, $\lambda_n > 0$, $c_n > 0$, $a_n \geqslant 0$ and $\frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} < e(\frac{1}{c_n} - \frac{1}{c_{n+1}})$ due to the inequality (2.1), therefore, $a_n = 0$, $n = 1, 2, 3, \cdots$.

(3) If $\sum\limits_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ is convergent, then $c < \infty$. According to (2.4), we have

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant e \Big(1 - \frac{1}{c} \Big) \sum_{j=1}^{\infty} \lambda_j a_j.$$

Therefore, e is not the optimal constant.

(4) $c_n = \exp \left\{ \sum\limits_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j} \right\}$, $\frac{\lambda_n}{\Lambda_n} \in (0, 1]$, $n = 1, 2, 3, \cdots$, thus, the convergence of $\sum\limits_{n=1}^{\infty} \frac{\lambda_n}{c_n}$ is the same as $\sum\limits_{n=1}^{\infty} \lambda_n \exp \left\{ -\sum\limits_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \right\}$. For any $n > 1$, we have

$$\begin{aligned} \sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} &= 1 + \sum_{j=2}^{n} \frac{\lambda_j}{\Lambda_j} = 1 + \sum_{j=2}^{n} \frac{\Lambda_j - \Lambda_{j-1}}{\Lambda_j} = 1 + \sum_{j=2}^{n} \int_{\Lambda_{j-1}}^{\Lambda_j} \frac{dx}{\Lambda_j} \\ &< 1 + \sum_{j=2}^{n} \int_{\Lambda_{j-1}}^{\Lambda_j} \frac{dx}{x} = 1 + \int_{\Lambda_1}^{\Lambda_n} \frac{dx}{x} = 1 + \ln \frac{\Lambda_n}{\Lambda_1}. \end{aligned}$$

So $\sum\limits_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} < 1 + \ln \frac{\Lambda_n}{\Lambda_1}$, then

$$\lambda_n \exp \left\{ -\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \right\} > \lambda_n \exp \left\{ -1 - \ln \frac{\Lambda_n}{\Lambda_1} \right\} = e^{-1} \Lambda_1 \frac{\lambda_n}{\Lambda_n}.$$

$\sum\limits_{n=1}^{\infty} \frac{\lambda_n}{\Lambda_n}$ is divergent, so $\sum\limits_{n=1}^{\infty} \lambda_n \exp \left\{ -\sum\limits_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \right\}$ is also divergent, namely $\sum\limits_{n=1}^{\infty} \frac{\lambda_n}{c_n}$ is divergent.

For any positive integer N, take $a_n = \begin{cases} \frac{1}{c_n}, \; n = 1,2,\cdots,N \\ 0, \; n > N \end{cases}$, then

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} = \sum_{n=1}^{N} \frac{\lambda_n}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} = \sum_{n=1}^{N} \frac{\lambda_n}{e^{-1} \exp\left\{ \sum\limits_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \right\}},$$

$$\sum_{n=1}^{\infty} \lambda_n a_n = \sum_{n=1}^{N} \frac{\lambda_n}{c_n}.$$

$\left\{ \sum\limits_{n=1}^{N} \frac{\lambda_n}{c_n} \right\}$ is strictly increasing. $\sum\limits_{n=1}^{\infty} \frac{\lambda_n}{c_n}$ is divergent, so $\lim\limits_{N\to\infty} \sum\limits_{n=1}^{N} \frac{\lambda_n}{c_n} = \infty$. Based on Stolz theorem, we have

$$\begin{aligned}
\lim_{N\to\infty} \frac{\sum\limits_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum\limits_{n=1}^{\infty} \lambda_n a_n} &= \lim_{N\to\infty} \frac{\sum\limits_{n=1}^{N} \frac{\lambda_n}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}}{\sum\limits_{n=1}^{N} \frac{\lambda_n}{c_n}} \\
&= \lim_{N\to\infty} \frac{\frac{\lambda_N}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_N^{\lambda_N})^{\frac{1}{\Lambda_N}}}}{\frac{\lambda_N}{c_N}} = \lim_{N\to\infty} \frac{\exp\left\{ \sum\limits_{j=1}^{N-1} \frac{\lambda_j}{\Lambda_j} \right\}}{e^{-1} \exp\left\{ \sum\limits_{j=1}^{N} \frac{\lambda_j}{\Lambda_j} \right\}} = e \lim_{N\to\infty} \exp\left\{ -\frac{\lambda_N}{\Lambda_N} \right\} \\
&= e.
\end{aligned}$$

Therefore, e is the optimal constant.

(5) Let $r = \inf\limits_{n\in N^+} \frac{\lambda_n}{\Lambda_n}$, then $r \in [0,1)$. If the conclusion is wrong, then $r > 0$.

Take an arbitrary non-negative sequence $\{a_n\}$, based on (2.1), we have

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant \sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} \right] a_j.$$

For any $j \in N^+$,

$$\begin{aligned}
c_j \sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} &= c_j \sum_{n=j}^{\infty} \frac{\frac{\lambda_n}{\Lambda_n}}{e^{-1} \exp\left\{ \sum\limits_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \right\}} = e c_j \sum_{n=j}^{\infty} \frac{\frac{\lambda_n}{\Lambda_n}}{\exp\left\{ \frac{\lambda_n}{\Lambda_n} \right\} - 1} \frac{\exp\left\{ \frac{\lambda_n}{\Lambda_n} \right\} - 1}{\exp\left\{ \sum\limits_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \right\}} \\
&= e c_j \sum_{n=j}^{\infty} \frac{\frac{\lambda_n}{\Lambda_n}}{\exp\left\{ \frac{\lambda_n}{\Lambda_n} \right\} - 1} \left[\frac{1}{\exp\left\{ \sum\limits_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j} \right\}} - \frac{1}{\exp\left\{ \sum\limits_{j=1}^{n} \frac{\lambda_j}{\Lambda_j} \right\}} \right] = e c_j \sum_{n=j}^{\infty} \frac{\frac{\lambda_n}{\Lambda_n}}{\exp\left\{ \frac{\lambda_n}{\Lambda_n} \right\} - 1} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}} \right).
\end{aligned}$$

$\frac{x}{e^x - 1}$ is strictly decreasing in $(0,\infty)$, $\frac{\lambda_n}{\Lambda_n} \geqslant r > 0$, $n = 1,2,3,\cdots$, then

$$\frac{\frac{\lambda_n}{\Lambda_n}}{\exp\left\{ \frac{\lambda_n}{\Lambda_n} \right\} - 1} \leqslant \frac{r}{e^r - 1}, \quad n = 1,2,3,\cdots.$$

Then,

$$c_j \sum_{n=j}^{\infty} \frac{\lambda_n}{\Lambda_n (c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} = ec_j \sum_{n=j}^{\infty} \frac{\frac{\lambda_n}{\Lambda_n}}{\exp\left\{\frac{\lambda_n}{\Lambda_n}\right\} - 1} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\right)$$
$$\leqslant \frac{r}{e^r - 1} ec_j \sum_{n=j}^{\infty} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\right) \leqslant \frac{r}{e^r - 1} e, \quad j = 1, 2, 3, \cdots.$$

Therefore,

$$\sum_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant \frac{r}{e^r - 1} e \sum_{j=1}^{\infty} \lambda_j a_j.$$

It follow from $\frac{r}{e^r-1} e < e$, that e is not the optimal constant, which is a contradiction. Therefore, $\inf_{n \in N^+} \frac{\lambda_n}{\Lambda_n} = 0$. $\square$

Let's start the proof of Theorem 1.2.

(1) We still set $c_n = \exp\left\{\sum_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j}\right\}$, $n = 1, 2, 3, \cdots$, $c = \lim_{n \to \infty} c_n$. Still based on weighted mean inequality, we have

$$\sum_{n=1}^{\infty} \Lambda_n (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}} \leqslant \sum_{n=1}^{\infty} \Lambda_n (e^{\frac{\lambda_n}{\Lambda_n}} - 1) \frac{\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_n} c_j a_j}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}$$
$$= \sum_{n=1}^{\infty} (e^{\frac{\lambda_n}{\Lambda_n}} - 1) \frac{\sum_{j=1}^{n} \lambda_j c_j a_j}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} = \sum_{j=1}^{\infty} \lambda_j \left[c_j \sum_{n=j}^{\infty} \frac{e^{\frac{\lambda_n}{\Lambda_n}} - 1}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} \right] a_j.$$

For any $j \in N^+$,

$$c_j \sum_{n=j}^{\infty} \frac{e^{\frac{\lambda_n}{\Lambda_n}} - 1}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} = c_j \sum_{n=j}^{\infty} \frac{e^{\frac{\lambda_n}{\Lambda_n}} - 1}{e^{-1} \exp\left\{\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j}\right\}}$$
$$= ec_j \sum_{n=j}^{\infty} \left[\frac{1}{\exp\left\{\sum_{j=1}^{n-1} \frac{\lambda_j}{\Lambda_j}\right\}} - \frac{1}{\exp\left\{\sum_{j=1}^{n} \frac{\lambda_j}{\Lambda_j}\right\}} \right] = ec_j \sum_{n=j}^{\infty} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\right).$$

If $c < \infty$, then

$$c_j \sum_{n=j}^{\infty} \frac{e^{\frac{\lambda_n}{\Lambda_n}} - 1}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} = ec_j \left(\frac{1}{c_j} - \frac{1}{c}\right) = e\left(1 - \frac{c_j}{c}\right) \leqslant e\left(1 - \frac{c_1}{c}\right) = e\left(1 - \frac{1}{c}\right) < e. \tag{2.5}$$

If $c = \infty$, then

$$c_j \sum_{n=j}^{\infty} \frac{e^{\frac{\lambda_n}{\Lambda_n}} - 1}{(c_1^{\lambda_1} c_2^{\lambda_2} \cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}} = ec_j \sum_{n=j}^{\infty} \left(\frac{1}{c_n} - \frac{1}{c_{n+1}}\right) = e.$$

Therefore,

$$\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\leqslant e\sum_{n=1}^{\infty}\lambda_n a_n.$$

(2) The sufficiency is obvious, we only need to prove the necessity. Let's discuss the condition of equality of inequality (1.4) by case.

(i) The case $c<\infty$.

Based on (2.5), we have

$$\sum_{n=1}^{\infty}\Lambda_n\Big(e^{\frac{\lambda_n}{\Lambda_n}}-1\Big)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\leqslant e(1-\frac{1}{c})\sum_{j=1}^{\infty}\lambda_j a_j.$$

The equality of (1.4) holds, so

$$e\Big(1-\frac{1}{c}\Big)\sum_{n=1}^{\infty}\lambda_n a_n=e\sum_{n=1}^{\infty}\lambda_n a_n,$$

then $\sum\limits_{n=1}^{\infty}\lambda_n a_n=0$. But $\lambda_n>0$, $a_n\geqslant 0$, $n=1,2,3,\cdots$, so $a_n=0$, $n=1,2,3,\cdots$.

(ii) The case $c=\infty$.

The equality of (1.4) holds, so

$$\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}=\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)\frac{\sum\limits_{j=1}^{n}\frac{\lambda_j}{\Lambda_n}c_j a_j}{(c_1^{\lambda_1}c_2^{\lambda_2}\cdots c_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}.$$

Based on the condition of equality of weighted mean inequality, we have the following equality

$$c_1a_1=c_2a_2=\cdots=c_na_n=\cdots.$$

Let $a=c_na_n$, $n=1,2,3,\cdots$, then $a_n=\frac{a}{c_n}$, $\lambda_n a_n=a\frac{\lambda_n}{c_n}$, $n=1,2,3,\cdots$. $\sum\limits_{n=1}^{\infty}\lambda_n a_n$ is convergent, but as we know in the proof of Theorem 1.1, $\sum\limits_{n=1}^{\infty}\frac{\lambda_n}{c_n}$ is divergent, so $a=0$, namely $a_n=0$, $n=1,2,3,\cdots$.

(3) If $\sum\limits_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ is convergent, then $c<\infty$. Based on (2.5), we have

$$\sum_{n=1}^{\infty}\Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}}-1)(a_1^{\lambda_1}a_2^{\lambda_2}\cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}\leqslant e\Big(1-\frac{1}{c}\Big)\sum_{j=1}^{\infty}\lambda_j a_j.$$

It follow from $(1-\frac{1}{c})e<e$, that e is not the optimal constant.

(4) $\lim\limits_{n\to\infty}\frac{\lambda_n}{\Lambda_n}=0$, but $\sum\limits_{n=1}^{\infty}\frac{\lambda_n}{\Lambda_n}$ is divergent. Based on Theorem 1.1, e is the optimal constant of inequality (1.3). Set

$$A=\left\{\{a_n\}\,\middle|\,a_n\geqslant 0,\ \ n=1,2,3,\cdots,\ \ 0<\sum_{n=1}^{\infty}\lambda_n a_n<\infty\right\},$$

then

$$e = \sup_{\{a_n\}\in A} \frac{\sum\limits_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum\limits_{n=1}^{\infty} \lambda_n a_n},$$

$$\sup_{\{a_n\}\in A} \frac{\sum\limits_{n=1}^{\infty} \Lambda_n (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum\limits_{n=1}^{\infty} \lambda_n a_n} \leqslant e.$$

On the other hand, for any $n \in N^+$, $0 < \lambda_n < \Lambda_n(e^{\frac{\lambda_n}{\Lambda_n}} - 1)$, then

$$e = \sup_{\{a_n\}\in A} \frac{\sum\limits_{n=1}^{\infty} \lambda_n (a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum\limits_{n=1}^{\infty} \lambda_n a_n} \leqslant \sup_{\{a_n\}\in A} \frac{\sum\limits_{n=1}^{\infty} \Lambda_n (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum\limits_{n=1}^{\infty} \lambda_n a_n} \leqslant e.$$

Therefore,

$$\sup_{\{a_n\}\in A} \frac{\sum\limits_{n=1}^{\infty} \Lambda_n (e^{\frac{\lambda_n}{\Lambda_n}} - 1)(a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_n^{\lambda_n})^{\frac{1}{\Lambda_n}}}{\sum\limits_{n=1}^{\infty} \lambda_n a_n} = e,$$

namely e is the optimal constant. □

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