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DETERMINANT FOR POSITIVE OPERATORS AND OPPENHEIM'S INEQUALITY

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Abstract. In this paper, by virtue of the Specht ratio, we show Oppenheim type inequalities for the normalized determinant of positive invertible operators on a Hilbert space, and we moreover discuss Hadamard type inequalities for positive definite matrices.

1. Introduction

There are some attempts to extend the notion of the determinant for matrices. In 1950s, Fuglede and Kadison [2, 3] and Arveson [1] introduced the normalized determinant for invertible operators A in a II_1-factor with the canonical normalized trace τ as

$$\Delta(A) = \exp\tau(\log|A|)$$

and discussed properties of the determinant. Inspired by the notion of the determinant due to Fuglede-Kadison and Arveson, Fujii et al. [6, 5] discussed the normalized determinant $\Delta_x(A)$ for positive invertible operators A on a Hilbert space H and a fixed unit vector $x \in H$ defined by

$$\Delta_x(A) = \exp\langle \log A x, x\rangle, \tag{1.1}$$

and discussed it as a continuous geometric mean and observed some inequalities around the determinant from this point of view. In the matrix case, the definition of (1.1) is a generalization of the usual determinant det for positive definite matrices: In fact, for any positive definite $n \times n$ matrix A with the spectrum $\sigma(A) = \{l_1, \dots, l_n\}$

$$\Delta_x(A) = \prod_{i=1}^{n} l_i^{1/n} = (\det A)^{1/n}$$

for some unit vector $x \in \mathbb{C}^n$.

We want to consider Oppenheim type inequality for the normalized determinant $\Delta_x(A)$. For this, we recall the Hadamard product of operators [8, 4]. The Hadamard product is expressed as the deformation of the tensor product, which is one of the most powerful tools for the study of the Hadamard product of operators on a separable Hilbert

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space: Let $\{e_j\}$ be an orthogonal basis of a Hilbert space H and $A \otimes B$ be tensor product of operators A and B on H regarding to $\{e_j\}$. Let $U : H \mapsto H \otimes H$ be the isometry such that $Ue_j = e_j \otimes e_j$. The Hadamard product $A \circ B$ regarding to $\{e_j\}$ is expressed as

$$A \circ B = U^*(A \otimes B)U. \tag{1.2}$$

In the finite dimensional case, if the matrices $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbb{M}_n(\mathbb{C})$, then the Hadamard product $A \circ B$ has an associated matrix $A \circ B = (a_{ij}b_{ij})$ in $\mathbb{M}_n(\mathbb{C})$. We recall the Hadamard inequality and Oppenheim's inequality for the Hadamard product [11, p. 218, p. 242]: Let A and B be $n \times n$ positive semidefinite matrices with diagonal entries a_{ii} and b_{ii}, respectively. Then the Hadamard determinant inequality says that

$$\det A \leqslant \prod_{i=1}^{n} a_{ii} = \det A \circ I, \tag{1.3}$$

and Oppenheim's inequality says that

$$\det A \det B \leqslant \det(A \circ B) \leqslant \prod_{i=1}^{n} a_{ii}b_{ii}. \tag{1.4}$$

In this paper, by virtue of the Specht ratio, we show Oppenheim type inequalities for the normalized determinant of positive invertible operators on a Hilbert space and we moreover discuss Hadamard type determinant inequalities for positive definite matrices.

2. Oppenheim type inequality

Let $B(H)$ be the space of all bounded linear operators on a Hilbert space H, and I stands for the identity operator on H. An operator A in $B(H)$ is said to be positive (in symbol: $A \geqslant 0$) if $\langle Ax,x\rangle \geqslant 0$ for all $x \in H$. In particular, $A > 0$ means that A is positive and invertible. For a pair A,B of selfadjoint operators the order relation $A \geqslant B$ means as usual that $A - B$ is positive.

We list important properties of the normalized determinant, also see [6]: For each unit vector $x \in H$

(i) *continuity*: The map $A \mapsto \Delta_x(A)$ is norm continuous;

(ii) *bounds*: $\langle A^{-1}x,x\rangle^{-1} \leqslant \Delta_x(A) \leqslant \langle Ax,x\rangle$;

(iii) *continuous mean*: $\langle A^p x,x\rangle^{1/p} \downarrow \Delta_x(A)$ for $p \downarrow 0$ and $\langle A^p x,x\rangle^{1/p} \uparrow \Delta_x(A)$ for $p \uparrow 0$;

(iv) *power equality*: $\Delta_x(A^t) = \Delta_x(A)^t$ for all $t > 0$;

(v) *homogeneity*: $\Delta_x(tA) = t\Delta_x(A)$ and $\Delta_x(tI) = t$ for all $t > 0$;

(vi) *monotonicity*: $0 \leqslant A \leqslant B$ implies $\Delta_x(A) \leqslant \Delta_x(B)$;

(vii) *multiplicativity*: $\Delta_x(AB) = \Delta_x(A)\Delta_x(B)$ for commuting A and B;

(viii) *Ky Fan type inequality*: $\Delta_x((1-\alpha)A+\alpha B) \geqslant \Delta_x(A)^{1-\alpha}\Delta_x(B)^{\alpha}$ for $0<\alpha<1$.

First of all, we consider the matrix case. By (1.4),

$$\det(A\circ B) \leqslant \prod_{i=1}^{n} a_{ii} \prod_{i=1}^{n} b_{ii} = \det(A\circ I)\det(B\circ I).$$

The diagonal operator formed from an operator A can be obtained by the Hadamard multiplication with the identity I. From this view point, we would expect the following Oppenheim's inequality for operators: For any positive invertible operators A and B

$$\Delta_x(A\circ B) \leqslant \Delta_x(A\circ I)\Delta_x(B\circ I)$$

for every unit vector $x\in H$. Unfortunately, we have the following counterexamples: Let

$$A = B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\Delta_x(A\circ B) = 5 > \Delta_x(A\circ I)\Delta_x(B\circ I) = 4 \qquad \text{for } x = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\Delta_x(A\circ B) = 3 < \Delta_x(A\circ I)\Delta_x(B\circ I) = 4 \qquad \text{for } x = \tfrac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Thus, we investigate the upper and lower boundary of the ratio

$$\Delta_x(A\circ B)/\Delta_x(A\circ I)\Delta_x(B\circ I) \qquad \text{for every unit vector } x\in H$$

in terms of the spectra of A and B.

Next, we recall the Specht ratio. Specht [10] estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1,\ldots,x_n \in [m,M]$

$$\sqrt[n]{x_1x_2\cdots x_n} \leqslant \frac{x_1+x_2+\cdots+x_n}{n} \leqslant S(h)\sqrt[n]{x_1x_2\cdots x_n} \tag{2.1}$$

where $h=\frac{M}{m}$ and the Specht ratio is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e\log h}\ (h\neq 1) \quad \text{and} \quad S(1)=1.$$

In [5], we showed an operator version of (2.1): Let A be a positive invertible operator such that $mI \leqslant A \leqslant MI$ for some scalars $0<m<M$. Then

$$\Delta_x(A) \leqslant \langle Ax,x\rangle \leqslant S(h)\Delta_x(A)$$

for every unit vector $x\in H$, where $h=\frac{M}{m}$.

The following Lemma is well-known, but we give a proof for readers' convenience.

LEMMA 2.1. *Let A be a positive invertible operator such that $mI \leqslant A \leqslant MI$ for some scalars $0 < m < M$, and Φ a unital positive linear map from $B(H)$ into $B(K)$. Then*

$$\Phi(\log A) \leqslant \log \Phi(A) \leqslant \Phi(\log A) + \log S(h)I \tag{2.2}$$

where $h = \frac{M}{m}$.

Proof. Since $\log t$ is operator concave, the first inequality of (2.2) follows from Jensen operator inequality.

For the second inequality of (2.2), put $f(t) = \alpha t + \beta - \log t$ on $[m,M]$, where $\alpha = \frac{\log M - \log m}{M-m}$ and $\beta = \frac{M \log m - m \log M}{M-m}$. Since $f'(t) = \alpha - \frac{1}{t}$, putting $t_0 = \frac{1}{\alpha}$, we then have

$$\min_{m \leqslant t \leqslant M} f(t) = f(t_0) = 1 + \beta + \log \alpha.$$

Since $\alpha = \frac{\log h}{m(h-1)}$ and $\beta = \log M - \frac{h \log h}{h-1}$, we have

$$\begin{aligned} 1 + \beta + \log \alpha &= 1 + \log M - \frac{h \log h}{h-1} + \log \left(\frac{\log h}{m(h-1)} \right) \\ &= -\log(h-1) + 1 - \frac{h \log h}{h-1} + \log(\log h) + \log h \\ &= -\log S(h). \end{aligned}$$

Hence it follows that

$$\begin{aligned} \Phi(\log A) - \log \Phi(A) &\geqslant \Phi(\alpha A + \beta I) - \log \Phi(A) = \alpha \Phi(A) + \beta I - \log \Phi(A) \\ &\geqslant (1 + \beta + \log \alpha) I = -\log S(h) I, \end{aligned}$$

which implies $\log \Phi(A) \leqslant \Phi(\log A) + \log S(h)I$. $\square$

We show the following Oppenheim type inequality in terms of the Specht ratio:

THEOREM 2.2. *Let A and B be positive invertible operators on H such that $m_1 I \leqslant A \leqslant M_1 I$ and $m_2 I \leqslant B \leqslant M_2 I$ for some scalars $0 < m_1 < M_1$ and $0 < m_2 < M_2$. Put $h_1 = \frac{M_1}{m_1}$ and $h_2 = \frac{M_2}{m_2}$. Then*

$$\frac{1}{S(h_1)S(h_2)} \Delta_x(A \circ I) \Delta_x(B \circ I) \leqslant \Delta_x(A \circ B) \leqslant S(h_1 h_2) \Delta_x(A \circ I) \Delta_x(B \circ I) \tag{2.3}$$

for every unit vector $x \in H$.

Proof. Since $\log t$ is operator concave, it follows from Jensen operator inequality that

$$U^* [\log(A \otimes I)] U \leqslant \log U^* (A \otimes I) U$$

where the isometry U is defined as (1.2) in the section 1. Since $m_1 m_2 I \leqslant A_1 \otimes A_2 \leqslant M_1 M_2 I$ and $h_1 h_2 = \frac{M_1 M_2}{m_1 m_2}$, it follows from Lemma 2.1 that

$$\log(A \circ B) = \log U^* (A \otimes B) U \leqslant U^* [\log(A \otimes B)] U + \log S(h_1 h_2) I.$$

Moreover, since $A\otimes I$ and $I\otimes B$ commute,

$$U^*\left[\log(A\otimes B)\right]U=U^*\left[\log(A\otimes I)(I\otimes B)\right]U=U^*\left[\log(A\otimes I)+\log(I\otimes B)\right]U.$$

Hence it follows from Lemma 2.1 that for every unit vector $x\in H$

$$\begin{aligned}\langle\log(A\circ B)x,x\rangle&=\langle\log U^*(A\otimes B)Ux,x\rangle\\&\leqslant\langle U^*\log(A\otimes B)Ux,x\rangle+\log S(h_1h_2)\\&=\langle U^*\log(A\otimes I)(I\otimes B)Ux,x\rangle+\log S(h_1h_2)\\&=\langle U^*\log(A\otimes I)Ux,x\rangle+\langle U^*\log(I\otimes B)Ux,x\rangle+\log S(h_1h_2)\\&\leqslant\langle\log U^*(A\otimes I)Ux,x\rangle+\langle\log U^*(I\otimes B)Ux,x\rangle+\log S(h_1h_2)\\&=\langle\log(A\circ I)x,x\rangle+\langle\log(I\circ B)x,x\rangle+\log S(h_1h_2).\end{aligned}$$

By taking the exponential of both sides, we get the desired right hand side of (2.3).

On the other hand, we have

$$\begin{aligned}\langle\log(A\circ B)x,x\rangle&=\langle\log U^*(A\otimes B)Ux,x\rangle\\&\geqslant\langle U^*\log(A\otimes B)Ux,x\rangle\\&=\langle U^*\log(A\otimes I)(I\otimes B)Ux,x\rangle\\&=\langle U^*\log(A\otimes I)Ux,x\rangle+\langle U^*\log(I\otimes B)Ux,x\rangle\\&\geqslant\langle\log U^*(A\otimes I)Ux,x\rangle-\log S(h_1)+\langle\log U^*(I\otimes B)Ux,x\rangle-\log S(h_2)\end{aligned}$$

for every unit vector $x\in H$. By taking the exponential of the both sides, we get the desired left hand sides of (2.3). $\square$

The coefficients in inequality (2.3) of Theorem 2.2 are not symmetric with resprct to the Specht ratio. For this, we examine some properties of the Specht ratio a little more.

LEMMA 2.3. *Let $h>0$. Then the Specht ratio has the following power monotone increasing property:*

$$S(h)^r\leqslant S(h^r)\qquad \textit{for all } r\geqslant 1,$$

or equivalently

$$S(h^r)\leqslant S(h)^r\qquad \textit{for all } 0<r<1.$$

Proof. Since it follows from [7, Theorem 2.16] that $S(h)=S(h^{-1})$ and $S(1)=\lim_{h\to 1}S(h)=1$, we may assume that $h>1$. Noting that

$$\begin{aligned}(1\leqslant)\frac{S(h^r)}{S(h)^r}&=\frac{(h^r-1)h^{\frac{r}{h^r-1}}}{er\log h}\,\frac{e^r(\log h)^r}{(h-1)^rh^{\frac{r}{h-1}}}\\&=\frac{h^r-1}{(h-1)^r}\,\frac{e^{r-1}}{r}(\log h)^{r-1}/h^{\frac{r}{h-1}-\frac{r}{h^r-1}},\end{aligned}$$

we show the following inequality stronger than the desired one:

$$h^{\frac{r}{h-1}-\frac{r}{h^r-1}} \leqslant e^{r-1} \leqslant \frac{e^{r-1}}{r}\frac{h^r-1}{(h-1)^r}(\log h)^{r-1} \tag{2.4}$$

for all $r \geqslant 1$.

To show the left hand side of (2.4), taking the logarithm of both sides of (2.4), it is equivalent to

$$\left(\frac{1}{h-1}-\frac{1}{h^r-1}\right)\log h \leqslant \frac{r-1}{r} \tag{2.5}$$

for all $r \geqslant 1$. To cancel the denominator of inequality (2.5), we get

$$(\log h)(h^r-h)r \leqslant (h-1)(r-1)(h^r-1).$$

Put $f(r)=(h-1)(r-1)(h^r-1)+(\log h)(h-h^r)r$ and then $f(1)=0$ and

$$f'(r)=(h-1)h^r+1-h+h\log h+(\log h)h^r\left((h-1-\log h)r-h\right)$$

and $f'(1)=(h-1)^2-h(\log h)^2 \geqslant 0$. Differentiating $f'(r)$, we get

$$f''(r)=(\log h)h^r\left((h-1-\log h)(\log h)r+2(h-1)-h\log h-\log h\right).$$

and $f''(r)\geqslant 0$ for $r\geqslant 1$ and $h>1$. Since $f'(r)$ is monotone increasing and $f'(1)\geqslant 0$, we have $f'(r)\geqslant 0$. Since $f(1)=0$, it follows that $f(r)$ is monotone increasing and $f(r)\geqslant 0$. Hence we get (2.5).

Next we show the right hand side of (2.4). Taking the logarithm, it is equivalent to

$$\log r + r\log(h-1) \leqslant \log(h^r-1)+(r-1)\log(\log h). \tag{2.6}$$

Put $g(r)=\log(h^r-1)+(r-1)\log(\log h)-\log r - r\log(h-1)$. Then we have $g(1)=0$ and

$$g'(r)=\frac{h^r\log h}{h^r-1}+\log(\log h)-\frac{1}{r}-\log(h-1),$$

so that $g'(1)=\frac{h\log h}{h-1}+\log(\log h)-1-\log(h-1)\geqslant 0$. Moreover, since

$$g''(r)=\frac{(h^r-1)^2-(\log h)^2h^rr^2}{r^2(h^r-1)^2},$$

it follows that $g''(r)\geqslant 0$ for all $r\geqslant 1$. Since $g'(r)$ is monotone increasing and $g'(1)\geqslant 0$, it follows that $g'(r)\geqslant 0$, hence $g(r)$ is monotone increasing and $g(1)=0$. Namely, we have $g(r)\geqslant 0$, which proves (2.6). □

LEMMA 2.4. *Let $h>1$. The Specht ratio $S(h)$ is supermultiplicative for $h>1$, i.e.,*

$$S(h_1)S(h_2)\leqslant S(h_1h_2)$$

for $h_1,h_2>1$.

Proof. We may assume that $1 < h_1 < h_2$. Then $h_1 = h_2^{\alpha}$ for some $0 < \alpha < 1$. By Lemma 2.3, it follows that

$$\begin{aligned} S(h_1)S(h_2) &= S(h_2^{\alpha})S(h_2) \leqslant S(h_2)^{\alpha}S(h_2) \\ &= S(h_2)^{1+\alpha} \leqslant S(h_2^{1+\alpha}) = S(h_1h_2). \quad \square \end{aligned}$$

REMARK 2.5. By Theorem 2.2, we have an Oppenheim type inequality:

$$\frac{1}{S(h_1)S(h_2)}\Delta_x(A\circ I)\Delta_x(B\circ I) \leqslant \Delta_x(A\circ B) \leqslant S(h_1h_2)\Delta_x(A\circ I)\Delta_x(B\circ I) \tag{2.7}$$

for every unit vector $x \in H$. The coefficients of inequality (2.7) are not symmetric. Lemma 2.4 proposes a symmetric version of (2.7) as follows:

$$\frac{1}{S(h_1h_2)}\Delta_x(A\circ I)\Delta_x(B\circ I) \leqslant \Delta_x(A\circ B) \leqslant S(h_1h_2)\Delta_x(A\circ I)\Delta_x(B\circ I).$$

By Theorem 2.2, we have the following Hadamard type determinant inequality:

COROLLARY 2.6. *Let A and B be positive invertible operators on H such that $m_1I \leqslant A \leqslant M_1I$ and $m_2I \leqslant B \leqslant M_2I$ for some scalars $0 < m_1 < M_1$ and $0 < m_2 < M_2$. Put $h_1 = \frac{M_1}{m_1}$ and $h_2 = \frac{M_2}{m_2}$. Then*

$$\frac{1}{S(h_1)S(h_2)}\Delta_x(A\circ B\circ I) \leqslant \Delta_x(A\circ B) \leqslant S(h_1h_2)\Delta_x(A\circ B\circ I) \tag{2.8}$$

for every unit vector $x \in H$.

Proof. Since $A\circ I$ and $B\circ I$ commute and $A\circ B\circ I = (A\circ I)(B\circ I)$, we have

$$\Delta_x(A\circ I)\Delta_x(B\circ I) = \Delta_x((A\circ I)(B\circ I)) = \Delta_x(A\circ B\circ I)$$

for every unit vector $x \in H$ and so (2.8) follows from Theorem 2.2. $\square$

3. The Hadamard type determinant inequality

Finally, we deal with the matrix case. Let $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ be the space of $n\times n$ complex matrices. For $A \in \mathbb{M}_n$, we write $A \geqslant 0$ if A is positive semidefinite and $A > 0$ if A is positive definite, that is, A is positive and invertible. For two Hermitian matrices A and B, we write $A \geqslant B$ if $A - B \geqslant 0$, and it is called the Löwner ordering.

By (1.3) and (1.4), we would expect the following Hadamard inequality for matrices: For a positive definite matrix $A \in \mathbb{M}_n$

$$\Delta_x(A) \leqslant \Delta_x(A\circ I) \qquad \text{for every unit vector } x \in \mathbb{C}^n \tag{3.1}$$

and

$$\Delta_x(A)\Delta_x(B) \leqslant \Delta_x(A\circ B) \qquad \text{for every unit vector } x \in \mathbb{C}^n. \tag{3.2}$$

Unfortunately, these inequalities do not hold in general. In fact, put

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Then

$$\Delta_x(A) = \sqrt{3} < \Delta_x(A \circ I) = 2 \qquad \text{for } x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and

$$\Delta_x(A) = 3 > \Delta_x(A \circ I) = 2 \qquad \text{for } x = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

However, as a complementary result to inequality (3.1), we have the following Hadamard type determinant inequality, also see [9, Lemma 2]:

THEOREM 3.1. *For an $n \times n$ positive definite matrix $A \in \mathbb{M}_n$*

$$\frac{1}{n}\Delta_x(A) \leqslant \Delta_x(A \circ I) \qquad \textit{for every unit vector } x \in \mathbb{C}^n,$$

and the constnat $\frac{1}{n}$ is the best possible.

Proof. Let E be the $n \times n$ matrix all of whose entries are 1. Then it follows that $I - \frac{1}{n}E \geqslant 0$ and thus

$$0 \leqslant A \circ (I - \frac{1}{n}E) = A \circ I - \frac{1}{n}A \circ E = A \circ I - \frac{1}{n}A.$$

Hence we have $\langle \log(A \circ I)x, x\rangle \geqslant \langle \log Ax, x\rangle - \log n$ and this implies the desired inequality

$$\frac{1}{n}\Delta_x(A) \leqslant \Delta_x(A \circ I)$$

for every unit vector $x \in \mathbb{C}^n$.

Considering $A + \varepsilon I$ where A is the $n \times n$ matrix all of whose entries are 1, we see that the coefficient $\frac{1}{n}$ is the best possible. □

REMARK 3.2. The following example shows that there is no upper bound K satisfying $A \circ I \leqslant KA$ for all $A > 0$ in $\mathbb{M}_n$ in Theorem 3.1. Let $A = \begin{pmatrix} 1 & 1-\varepsilon \\ 1-\varepsilon & 1 \end{pmatrix}$ for a fixed $0 < \varepsilon < 1$. Since the spectrum of A is $\{\varepsilon, 2-\varepsilon\}$ and $A \circ I = I$, there is no constant K saatisfying $A \circ I \leqslant KA$ for all $A > 0$ in $\mathbb{M}_n$ by $m_A = \varepsilon$.

As a complementary result to inequality (3.2), we have the following Oppenheim type inequality for matrices:

COROLLARY 3.3. *Let A and B be positive definite matrices in $\mathbb{M}_n$ such that $m_1 I \leqslant A \leqslant M_1 I$ and $m_2 I \leqslant B \leqslant M_2 I$ for some scalars $0 < m_1 < M_1$ and $0 < m_2 < M_2$. Put $h_1 = \frac{M_1}{m_1}$ and $h_2 = \frac{M_2}{m_2}$. Then*

$$\frac{1}{n^2 S(h_1)S(h_2)} \Delta_x(A)\Delta_x(B) \leqslant \Delta_x(A \circ B)$$

for every unit vector $x \in \mathbb{C}^n$.

Proof. By Theorem 3.1 and Theorem 2.2, we have

$$\Delta_x(A \circ B) \geqslant \frac{1}{S(h_1)S(h_2)} \Delta_x(A \circ I)\Delta_x(B \circ I) \geqslant \frac{1}{n^2 S(h_1)S(h_2)} \Delta_x(A)\Delta_x(B). \quad \square$$

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