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ON A MORE ACCURATE HILBERT–TYPE INEQUALITY INVOLVING THE PARTIAL SUMS

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Abstract. By means of the weight coefficients, Hermite-Hadamard's inequality, the Euler-Maclaurin summation formula and Abel's summation by parts formula, a more accurate Hilbert-type inequality with the partial sums is given. The equivalent conditions of the best possible constant factor related to several parameters and some particular inequalities are also obtained.

1. Introduction

Assuming that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_m, b_n \geqslant 0$, $0<\sum_{m=1}^{\infty} a_m^p<\infty$, and $0<\sum_{n=1}^{\infty} b_n^q<\infty$, we have the following well known Hardy-Hilbert's inequality with the best possible constant factor (cf. [1], Theorem 315):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{1}$$

The more accurate extension of (1) was provided as follows (cf. [1], Theorem 323):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n-1}<\frac{\pi}{\sin(\pi/p)}\left(\sum_{m=1}^{\infty}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\right)^{\frac{1}{q}}. \tag{2}$$

Since $\frac{1}{m+n}<\frac{1}{m+n-1}$, inequality (2) deduces to (1).

In 2006, by introducing parameters $\lambda_i \in (0,2]$, $\lambda_1+\lambda_2=\lambda\in(0,4]$, an extension of (1) was provided by Krnic et al. [2] as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B(\lambda_1,\lambda_2)\left[\sum_{m=1}^{\infty}m^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}n^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{3}$$

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where, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible, and

$$B(u,v) := \int_0^\infty \frac{t^{u-1}}{(1+t)^{u+v}}dt \ (u,v>0)$$

is the beta function. For $\lambda = 1$, $\lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, inequality (3) reduces to (1); for $p=q=2$, $\lambda_1=\lambda_2=\frac{\lambda}{2}$, (3) reduces to Yang's inequality in [3].

Recently, applying inequality (3), Adiyasuren et. al. [4] gave a Hilbert-type inequality involving partial sums as follows: For $\lambda_i \in (0.1]\cap(0,\lambda)$ $(\lambda\in(0,2];\ i=1,2)$, $\lambda_1+\lambda_2=\lambda$,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^\lambda} < \lambda_1\lambda_2 B(\lambda_1,\lambda_2)\left(\sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q\right)^{\frac{1}{q}}, \tag{4}$$

where the constant factor $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible, and the partial sums $A_m := \sum_{i=1}^{m} a_i$ and $B_n := \sum_{k=1}^{n} b_k$ $(m,n\in\mathbf{N}=\{1,2,\cdots\})$, satisfying

$$0<\sum_{m=1}^{\infty} m^{-p\lambda_1-1}A_m^p<\infty$$

and

$$0<\sum_{n=1}^{\infty} n^{-q\lambda_2-1}B_n^q<\infty.$$

Inequalities (1), (2) and the integral analogues play an important role in analysis and its applications (cf. [5]–[15]).

In 1934, a half-discrete Hilbert-type inequality was given as follows (cf. [1], Theorem 351): If $K(t)$ $(t>0)$ is a decreasing function, $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $0<\phi(s):=\int_0^\infty K(t)t^{s-1}dt<\infty$, $a_n\geqslant 0$, $0<\sum_{n=1}^{\infty} a_n^p<\infty$, then

$$\int_0^\infty x^{p-2}\left(\sum_{n=1}^{\infty} K(nx)a_n\right)^p dx < \phi^p\left(\frac{1}{q}\right)\sum_{n=1}^{\infty} a_n^p. \tag{5}$$

Some new extensions of (5) were provided by [16]–[20].

In 2016, by means of the techniques of real analysis, Hong et al. [21] considered some equivalent statements of the extensions of (1) with the best possible constant factor related to a few parameters. The other similar works were provided by [22]–[32].

In this paper, following the way of [4, 21], by means of the weight coefficients, the idea of introduced parameters, Hermite-Hadamard's inequality, the Euler-Maclaurin summation formula and Abel's summation by parts formula, a more accurate extension of (4) is given. The equivalent conditions of the best possible constant factor related to several parameters and some particular inequalities are considered.

2. Some lemmas

In what follows, we suppose that $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $\lambda\in(0,1]$, $\lambda_i\in(0,\frac{1}{2}]\cap(0,\lambda)$, $\eta_i\in[0,\frac{1}{4}]$, $k_\lambda(\lambda_i):=B(\lambda_i,\lambda-\lambda_i)$ $(i=1,2)$, $\widehat{\lambda}_1:=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\widehat{\lambda}_2:=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, $\eta=\eta_1+\eta_2$. For $a_m,b_n\geqslant 0$ $(m,n\in\mathbf{N})$, the partial sums $A_m:=\sum_{i=1}^m a_i$ and $B_n:=\sum_{k=1}^n b_k$ satisfy $A_m=o(e^{t(m-\eta_1)})$, $B_n=o(e^{t(n-\eta_2)})$ $(t>0;\ m,n\to\infty)$,

$$0<\sum_{m=1}^{\infty} m^{-p\widehat{\lambda}_1-1}A_m^p<\infty \text{ and } 0<\sum_{n=1}^{\infty} n^{-q\widehat{\lambda}_2-1}B_n^q<\infty.$$

LEMMA 1. (cf. [5], (2.2.3)) *(i) If* $(-1)^i\frac{d^i}{dt^i}g(t)>0,\ t\in[m,\infty)$ $(m\in\mathbf{N})$ *with* $g^{(i)}(\infty)=0$ $(i=0,1,2,3)$, $P_i(t)$, B_i $(i\in\mathbf{N})$ *are the Bernoulli functions and the Bernoulli numbers of* i*-order, then*

$$\int_0^\infty P_{2q-1}(t)g(t)dt=-\varepsilon_q\frac{B_{2q}}{2q}g(m)\ (0<\varepsilon_q<1;\ q=1,2,\cdots). \tag{6}$$

In particular, for $q=1$, *in view of* $B_2=\frac{1}{6}$, *we have*

$$-\frac{1}{12}g(m)<\int_0^\infty P_1(t)g(t)dt<0; \tag{7}$$

for $q=2$, *in view of* $B_4=-\frac{1}{30}$, *we have*

$$0<\int_0^\infty P_3(t)g(t)dt<\frac{1}{120}g(m). \tag{8}$$

(ii) (cf. [5], (2.3.2)) *If* $f(t)(>0)\in C^3[m,\infty)$, $f^{(i)}(\infty)=0$ $(i=0,1,2,3)$, *then we have the following Euler-Maclaurin summation formula:*

$$\sum_{k=m}^{\infty} f(k)=\int_m^\infty f(t)dt+\frac{1}{2}f(m)+\int_m^\infty P_1(t)f'(t)dt, \tag{9}$$

$$\int_m^\infty P_1(t)f'(t)dt=-\frac{1}{12}f'(m)+\frac{1}{6}\int_m^\infty P_3(t)f'''(t)dt. \tag{10}$$

LEMMA 2. *For* $s\in(0,3]$, $s_i\in(0,\frac{3}{2}]\cap(0,s)$, $k_s(s_i)=B(s_i,s-s_i)$ $(i=1,2)$, *define the following weight coefficient:*

$$\varpi(s_2,m):=(m-\eta_1)^{s-s_2}\sum_{n=1}^{\infty}\frac{(n-\eta_2)^{s_2-1}}{(m+n-\eta)^s}\ (m\in\mathbf{N}). \tag{11}$$

We have the following inequalities:

$$0<k_s(s_2)\Big(1-O\Big(\frac{1}{(m-\eta_1)^{s_2}}\Big)\Big)<\varpi(s_2,m)<k_s(s_2)\ (m\in\mathbf{N}), \tag{12}$$

where, we indicate that

$$O\Big(\frac{1}{(m-\eta_1)^{s_2}}\Big):=\frac{1}{k_s(s_2)}\int_0^{\frac{1-\eta_2}{m-\eta_1}}\frac{u^{s_2-1}}{(1+u)^s}du>0.$$

Proof. For fixed $m \in \mathbf{N}$, we set the following real function:

$$g(m,t) := \frac{(t-\eta_2)^{s_2-1}}{(m-\eta+t)^s} \ (t > \eta_2).$$

In the following we divide two cases of $s_2 \in (0,1) \cap (0,s)$ and $s_2 \in [1, \frac{3}{2}] \cap (0,s)$ to prove inequalities (12).

(i) $s_2 \in (0,1) \cap (0,s)$. Since

$$(-1)^i g^{(i)}(m,t) > 0 \ (t > \eta_2;\ i = 0,1,2),$$

by Hermite-Hadamard's inequality, setting $u = \frac{t-\eta_2}{m-\eta_1}$, we have

$$\begin{aligned}\varpi(s_2,m) &= (m-\eta_1)^{s-s_2} \sum_{n=1}^{\infty} g(m,n) < (m-\eta_1)^{s-s_2} \int_{\frac{1}{2}}^{\infty} g(m,t)dt \\ &= \int_{\frac{\frac{1}{2}-\eta_2}{m-\eta_1}}^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} \leqslant \int_0^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = k_s(s_2).\end{aligned}$$

On the other hand, in view of the decreasingness property of series, setting $u = \frac{t-\eta_2}{m-\eta_1}$, we obtain

$$\begin{aligned}\varpi(s_2,m) &= (m-\eta_1)^{s-s_2} \sum_{n=1}^{\infty} g(m,n) > (m-\eta_1)^{s-s_2} \int_1^{\infty} g(m,t)dt \\ &= \int_{\frac{1-\eta_2}{m-\eta_1}}^{\infty} \frac{u^{s_2-1}du}{(1+u)^s} = k_s(s_2) - \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}du}{(1+u)^s} \\ &= k_s(s_2)\Big(1 - O\Big(\frac{1}{(m-\eta_1)^{s_2}}\Big)\Big) > 0,\end{aligned}$$

where, $O(\frac{1}{(m-\eta_1)^{s_2}}) = \frac{1}{k_s(s_2)} \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}}{(1+u)^s} du > 0$ satisfying

$$0 < \int_0^{\frac{1-\eta_2}{m-\eta_1}} \frac{u^{s_2-1}}{(1+u)^s} du < \int_0^{\frac{1-\eta_2}{m-\eta_1}} u^{s_2-1} du = \frac{1}{s_2}\left(\frac{1-\eta_2}{m-\eta_1}\right)^{s_2} \ (m \in \mathbf{N}).$$

Hence, we obtain (12) for the case (i).

(ii) $s_2 \in [1, \frac{3}{2}] \cap (0,s)$. By (9), we have

$$\begin{aligned}\sum_{n=\infty}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \frac{1}{2} g(m,1) + \int_1^{\infty} P_1(t) g'(m,t)dt, \\ &= \int_{\eta_2}^{\infty} g(m,t)dt - h(m),.\end{aligned}$$

where, $h(m)$ is indicated as

$$h(m) := \int_{\eta_2}^{1} g(m,t)dt - \frac{1}{2} g(m,1) - \int_1^{\infty} P_1(t) g'(m,t)dt.$$

We obtain $-\frac{1}{2}g(m,1)=\frac{-(1-\eta_2)^{s_2-1}}{2(m-\eta+1)^s}$, and integrating by parts, it follows that

$$\begin{aligned}
\int_{\eta_2}^1 g(m,t)dt &= \int_{\eta_2}^1 \frac{(t-\eta_2)^{s_2-1}}{(m-\eta+t)^s}dt \\
&= \frac{1}{s_2}\int_{\eta_2}^1 \frac{d(t-\eta_2)^{s_2}}{(m-\eta+t)^s} \\
&= \frac{1}{s_2}\frac{(t-\eta_2)^{s_2}}{(m-\eta+t)^s}\Big|_{\eta_2}^1+\frac{s}{s_2}\int_{\eta_2}^1 \frac{(t-\eta_2)^{s_2}dt}{(m-\eta+t)^{s+1}} \\
&= \frac{1}{s_2}\frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s}+\frac{s}{s_2(s_2+1)}\int_{\eta_2}^1 \frac{d(t-\eta_2)^{s_2+1}}{(m-\eta+t)^{s+1}} \\
&> \frac{1}{s_2}\frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s}+\frac{s}{s_2(s_2+1)}\frac{(t-\eta_2)^{s_2+1}}{(m-\eta+t)^{s+1}}\Big|_{\eta_2}^1 \\
&\quad +\frac{s(s+1)}{s_2(s_2+1)(m-\eta+1)^{s+1}}\int_{\eta_2}^1 (t-\eta_2)^{s_2+1}dt \\
&= \frac{1}{s_2}\frac{(1-\eta_2)^{s_2}}{(m-\eta+1)^s}+\frac{s}{s_2(s_2+1)}\frac{(1-\eta_2)^{s_2+1}}{(m-\eta+1)^{s+1}} \\
&\quad +\frac{s(s+1)(1-\eta_2)^{s_2+2}}{s_2(s_2+1)(s_2+2)(m-\eta+1)^{s+2}}.
\end{aligned}$$

We find

$$\begin{aligned}
-g'(m,t) &= -\frac{(s_2-1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}-\frac{s(t-\eta_2)^{s_2-1}}{(m-\eta+t)^{s+1}} \\
&= \frac{(1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}+\frac{s(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}-\frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}} \\
&= \frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}-\frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}},
\end{aligned}$$

and for $s_2\in[1,\frac{3}{2}]\cap(0,s)$, it follows that

$$\begin{aligned}
(-1)^i\frac{d^i}{dt^i}\Big[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}\Big] &> 0, \\
(-1)^i\frac{d^i}{dt^i}\Big[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\Big] &> 0\ (t>\eta_2;\ i=0,1,2,3).
\end{aligned}$$

By (8), (9) and (10), setting $a:=1-\eta_2\ \ (\in[\frac{3}{4},1])$, we obtain

$$(s+1-s_2)\int_1^\infty P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}dt>-\frac{s+1-s_2}{12(m-\eta+1)^s}a^{s_2-2},$$

$$
\begin{aligned}
&-(m-\eta_1)s\int_1^{\infty}P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}dt\\
&> \frac{(m-\eta_1)s}{12(m-\eta+1)^{s+1}}a^{s_2-2}-\frac{(m-\eta_1)s}{720}\left[\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}\right]''_{t=1}\\
&> \frac{(m-\eta+1)s-as}{12(m-\eta+1)^{s+1}}a^{s_2-2}-\frac{(m-\eta+1)s}{720}\\
&\quad\times\left[\frac{(s+1)(s+2)a^{s_2-2}}{(m-\eta+t)^{s+3}}+\frac{2(s+1)(2-s_2)a^{s_2-3}}{(m-\eta+t)^{s+2}}+\frac{(2-s_2)(3-s_2)a^{s_2-4}}{(m-\eta+t)^{s+1}}\right]\\
&= \frac{sa^{s_2-2}}{12(m-\eta+1)^{s}}-\frac{sa^{s_2-1}}{12(m-\eta+1)^{s+1}}-\frac{s}{720}\\
&\quad\times\left[\frac{(s+1)(s+2)a^{s_2-2}}{(m-\eta+t)^{s+2}}+\frac{2(s+1)(2-s_2)a^{s_2-3}}{(m-\eta+t)^{s+1}}+\frac{(2-s_2)(3-s_2)a^{s_2-4}}{(m-\eta+t)^{s}}\right],
\end{aligned}
$$

and then we have

$$
h(m) > \frac{a^{s_2-4}h_1}{(m-\eta+1)^s}+\frac{sa^{s_2-3}h_2}{(m-\eta+1)^{s+1}}\frac{s(s+1)a^{s_2-2}h_3}{(m-\eta+1)^{s+2}},
$$

where, $h_i\ (i=1,2,3)$ are indicated as

$$
\begin{aligned}
h_1 &:= \frac{a^4}{s_2}-\frac{a^3}{2}-\frac{(1-s_2)a^2}{12}-\frac{s(2-s_2)(3-s_2)}{720},\\
h_2 &:= \frac{a^4}{s_2(s_2+1)}-\frac{a^2}{12}-\frac{(s+1)(2-s_2)}{360}, \text{ and}\\
h_3 &:= \frac{a^4}{s_2(s_2+1)(s_2+2)}-\frac{s+2}{720}.
\end{aligned}
$$

For $s_2\in[1,\ \frac{3}{2}]\cap(0,s)\ \ (s\in(0,3])$, $a\in[\frac{3}{4},1]$, we find

$$
h_1 > \frac{a^2}{12s_2}[s_2^2-(6a+1)s_2+12a^2]-\frac{1}{120}.
$$

In view of

$$
\frac{d}{da}[s_2^2-(6a+1)s_2+12a^2]=6(4a-s_2)\geqslant 6\left(4\times\frac{3}{4}-\frac{3}{2}\right)>0,
$$

and

$$
\begin{aligned}
&\frac{d}{ds_2}[s_2^2-(6a+1)s_2+12a^2]\\
&=2s_2-(6a+1)\leqslant 2\times\frac{3}{2}-\left(6\times\frac{3}{4}+1\right)=3-\frac{11}{2}<0,
\end{aligned}
$$

we obtain

$$\begin{aligned} h_1 &> \frac{(3/4)^2}{12(3/2)}\left[\left(\frac{3}{2}\right)^2 - \left(6\times\frac{1}{4}+1\right)s_2 + 12\left(\frac{3}{4}\right)^2\right] - \frac{1}{120} \\ &= \frac{3}{128} - \frac{1}{120} > 0, \\ h_2 &> a^2\left(\frac{4a^2}{15} - \frac{1}{12}\right) - \frac{1}{90} \geqslant \left(\frac{3}{4}\right)^2\left[\frac{4(3/4)^2}{15} - \frac{1}{12}\right] - \frac{1}{90} \\ &= \frac{3}{80} - \frac{1}{90} > 0, \\ h_3 &> \frac{8a^4}{105} - \frac{5}{720} \geqslant \frac{8(3/4)^4}{105} - \frac{1}{144} = \frac{27}{1120} - \frac{1}{144} > 0, \end{aligned}$$

and then $h(m) > 0$, which follows that

$$\sum_{n=1}^{\infty} g(m,n) < \int_{\eta_2}^{\infty} g(m,t)dt.$$

On the other hand, we also have

$$\begin{aligned} \sum_{n=1}^{\infty} g(m,n) &= \int_1^{\infty} g(m,t)dt + \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt \\ &= \int_1^{\infty} g(m,t)dt + H(m),. \end{aligned}$$

where, $H(m)$ is indicated as

$$H(m) := \frac{1}{2}g(m,1) + \int_1^{\infty} P_1(t)g'(m,t)dt.$$

We have obtained that $\frac{1}{2}g(m,1) = \frac{a^{s_2-1}}{2(m-\eta+1)^s}$ and

$$g'(m,t) = -\frac{(s+1-s_2)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s} + \frac{s(m-\eta_1)(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}.$$

For $s_2 \in [1,\frac{3}{2}] \cap (0,s)$ $(s \in (0,3])$, by (7), we obtain

$$-(s+1-s_2)\int_1^{\infty} P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^s}dt > 0,$$

$$\begin{aligned} & s(m-\eta_1)\int_1^{\infty} P_1(t)\frac{(t-\eta_2)^{s_2-2}}{(m-\eta+t)^{s+1}}dt \\ &> \frac{-s(m-\eta_1)a^{s_2-2}}{12(m-\eta+1)^{s+1}} = \frac{[-s(m-\eta+1)+as]a^{s_2-2}}{12(m-\eta+1)^{s+1}} \\ &= \frac{-sa^{s_2-2}}{12(m-\eta+1)^s} + \frac{sa^{s_2-1}}{12(m-\eta+1)^{s+1}} \\ &> \frac{-sa^{s_2-2}}{12(m-\eta+1)^s}. \end{aligned}$$

Hence, we have

$$\begin{aligned} H(m) &> \frac{a^{s_2-1}}{2(m-\eta+1)^s} - \frac{sa^{s_2-2}}{12(m-\eta+1)^s} \\ &= \frac{(\frac{a}{2}-\frac{s}{12})a^{s_2-2}}{(m-\eta+1)^s} \\ &> \frac{(\frac{3/4}{2}-\frac{3}{12})a^{s_2-2}}{(m-\eta+1)^s} \\ &= \frac{(\frac{3}{8}-\frac{3}{12})a^{s_2-2}}{(m-\eta+1)^s} > 0. \end{aligned}$$

Therefore, we obtain the following inequalities:

$$\int_1^\infty g(m,t)dt < \sum_{n=1}^\infty g(m,n) < \int_{\eta_2}^\infty g(m,t)dt.$$

In view of the the results in the case (i), we still can obtain (12).

The lemma is proved. □

LEMMA 3. *For* $s \in (0,3]$, $s_i \in (0, \frac{3}{2}] \cap (0,s)$ $(i = 1,2)$, *we have the following more accurate Hardy-Hilbert's inequality:*

$$\begin{aligned} I &:= \sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{(m+n-\eta)^s} \\ &\leqslant (k_s(s_2))^{\frac{1}{p}} (k_s(s_1))^{\frac{1}{q}} \left\{ \sum_{m=1}^\infty (m-\eta_1)^{p[1-(\frac{s-s_2}{p}+\frac{s_1}{q})]-1} a_m^p \right\}^{\frac{1}{p}} \\ &\quad \times \left\{ \sum_{n=1}^\infty (n-\eta_2)^{q[1-(\frac{s-s_1}{q}+\frac{s_2}{p})]-1} b_n^q \right\}^{\frac{1}{q}}. \end{aligned} \tag{13}$$

Proof. In the same way, by the symmetry, we obtain the following inequalities for the next weight coefficient:

$$\begin{aligned} 0 &< k_s(s_1)\Big(1-O\Big(\frac{1}{(n-\eta_2)^{s_1}}\Big)\Big) \\ &< \omega(s_1,n) := (n-\eta_2)^{s-s_1} \sum_{m=1}^\infty \frac{(m-\eta_1)^{s_1-1}}{(m+n-\eta)^s} \\ &< k_s(s_1) \ (n \in \mathbf{N}), \end{aligned} \tag{14}$$

By Hölder's inequality (cf. [33]), we obtain

$$\begin{aligned} I &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n-\eta)^s}\left[\frac{(m-\eta_1)^{(1-s_1)/q}a_m}{(n-\eta_2)^{(1-s_2)/p}}\right]\cdot\left[\frac{(n-\eta_2)^{(1-s_2)/p}b_n}{(m-\eta_1)^{(1-s_1)/q}}\right] \\ &\leqslant \left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(m+n-\eta)^s}\frac{(m-\eta_1)^{(1-s_1)p/q}}{(n-\eta_2)^{1-s_2}}a_m^p\right]^{\frac{1}{p}} \\ &\quad\times\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n-\eta)^s}\frac{(n-\eta_2)^{(1-s_2)q/p}}{(m-\eta_1)^{1-s_1}}b_n^q\right]^{\frac{1}{q}} \\ &= \left\{\sum_{m=1}^{\infty}\varpi(s_2,m)(m-\eta_1)^{p[1-(\frac{s-s_2}{p}+\frac{s_1}{q})-1]}a_m^p\right\}^{\frac{1}{p}} \\ &\quad\times\left\{\sum_{n=1}^{\infty}\omega(s_1,n)(n-\eta_2)^{q[1-(\frac{s-s_1}{q}+\frac{s_2}{p})-1]}b_n^q\right\}^{\frac{1}{q}}. \end{aligned}$$

Then by (12) and (14), we obtain (13).

The lemma is proved. □

REMARK 1. In particular, for $s=\lambda+2\in(2,3]$, $s_i=\lambda_i+1\in(1,\frac{3}{2}]\cap(1,\lambda+1)$ in (13), replacing a_m (resp. b_n) by A_m (resp. B_n), in view of the assumptions of A_m and B_n, we have

$$\begin{aligned} I_0 &:= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{A_mB_n}{(m+n-\eta)^{\lambda+2}} \\ &< (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \\ &\quad\times\left[\sum_{m=1}^{\infty}(m-\eta_1)^{-p\widehat{\lambda}_1-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta_2)^{-q\widehat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}}. \end{aligned} \tag{15}$$

where, $\lambda\in(0,1]$, $\lambda_i\in(0,\frac{1}{2}]\cap(0,\lambda)$ $(i=1,2)$, satisfying the assumption.

LEMMA 4. *For $t>0$, we have the following inequalities:*

$$\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}a_m \leqslant t\sum_{m=1}^{\infty}e^{-t(m-\eta_1)}A_m, \tag{16}$$

$$\sum_{n=1}^{\infty}e^{-t(n-\eta_2)}b_n \leqslant t\sum_{n=1}^{\infty}e^{-t(n-\eta_2)}B_n. \tag{17}$$

Proof. In view of $A_m e^{-t(m-\eta_1)} = o(1)\ \ (m \to \infty)$, by Abel's summation by parts formula, we find

$$\begin{aligned}
&\sum_{m=1}^{\infty} e^{-t(m-\eta_1)} a_m \\
&= \lim_{m\to\infty} A_m e^{-t(m-\eta_1)} + \sum_{m=1}^{\infty} [e^{-t(m-\eta_1)} - e^{-t(m-\eta_1+1)}] A_m \\
&= \sum_{m=1}^{\infty} [e^{-t(m-\eta_1)} - e^{-t(m-\eta_1+1)}] A_m = (1-e^{-t}) \sum_{m=1}^{\infty} e^{-t(m-\eta_1)} A_m.
\end{aligned}$$

Since $1-e^{-t} < t\ \ (t>0)$, we have inequality

$$\sum_{m=1}^{\infty} e^{-t(m-\eta_1)} a_m \leqslant t \sum_{m=1}^{\infty} e^{-t(m-\eta_1)} A_m,$$

namely, (16) follows. In the same way, we obtain (17).

The lemma is proved. □

3. Main results and some particular inequalities

THEOREM 1. *We have the following more accurate Hilbert-type inequality:*

$$\begin{aligned}
I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}} &< \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \\
&\times \left[\sum_{m=1}^{\infty} (m-\eta_1)^{-p\widehat{\lambda}_1-1} A_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\eta_2)^{-q\widehat{\lambda}_2-1} B_n^q\right]^{\frac{1}{q}}. \qquad (18)
\end{aligned}$$

In particular, for $\lambda_1+\lambda_2=\lambda$, *we have* $k_\lambda(\lambda_1) = B(\lambda_1,\lambda_2)$,

$$0 < \sum_{m=1}^{\infty} m^{-p\lambda_1-1} A_m^p < \infty,\ 0 < \sum_{n=1}^{\infty} n^{-q\lambda_2-1} B_n^q < \infty$$

and the following inequality:

$$\begin{aligned}
\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{(m+n-\eta)^{\lambda}} &< \lambda_1\lambda_2 B(\lambda_1,\lambda_2) \\
&\times \left[\sum_{m=1}^{\infty} (m-\eta_1)^{-p\lambda_1-1} A_m^p\right]^{\frac{1}{p}} \left[\sum_{n=1}^{\infty} (n-\eta_2)^{-q\lambda_2-1} B_n^q\right]^{\frac{1}{q}}. \qquad (19)
\end{aligned}$$

Proof. In view of the formula that

$$\frac{1}{(m+n-\eta)^{\lambda}} = \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} e^{-(m+n-\eta)t} dt,$$

by (16) and (17), it follows that

$$\begin{aligned} I &= \frac{1}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_m b_n \int_0^{\infty} t^{\lambda-1} e^{-(m+n-\eta)t} dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda-1} \sum_{n=1}^{\infty} e^{-(m-\eta_1)t} a_m \sum_{m=1}^{\infty} e^{-(n-\eta_2)t} b_n dt \\ &\leqslant \frac{1}{\Gamma(\lambda)} \int_0^{\infty} t^{\lambda+1} \sum_{n=1}^{\infty} e^{-(m-\eta_1)t} A_m \sum_{m=1}^{\infty} e^{-(n-\eta_2)t} B_n dt \\ &= \frac{1}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_m B_n \int_0^{\infty} t^{(\lambda+2)-1} e^{-(m+n-\eta)t} dt \\ &= \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{A_m B_n}{(m+n-\eta)^{\lambda+2}} = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} I_0. \end{aligned}$$

Then by (15), we have (18).

For $\lambda_1+\lambda_2=\lambda\ (\in(0,1])\ (\lambda_i \in (0,\frac{1}{2}]\cap(0,\lambda),\ i=1,2)$,

$$\begin{aligned} k_{\lambda+2}(\lambda_2+1) = k_{\lambda+2}(\lambda_1+1) &= B(\lambda_1+1,\lambda_2+1) \\ &= \frac{\Gamma(\lambda_1+1)\Gamma(\lambda_2+1)}{\Gamma(\lambda+2)} = \lambda_1\lambda_2 \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda+2)}, \end{aligned}$$

inequality (18) reduces to (19).

The theorem is proved. □

THEOREM 2. *If $\lambda_1+\lambda_2=\lambda$, then the constant factor*

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)} (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}} (k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}$$

in (18) is the best possible. On the other hand, if the same constant factor in (18) is the best possible and $\lambda-\lambda_i \leqslant \frac{1}{2}\ (i=1,2)$, then we have $\lambda_1+\lambda_2=\lambda$.

Proof. We now prove that the constant factor $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ in (19) is the best possible. For any $0<\varepsilon<\min\{p\lambda_1, q\lambda_2\}$, we set $\widetilde{a}_m = m^{\lambda_1-\frac{\varepsilon}{p}-1}, \widetilde{b}_n = n^{\lambda_2-\frac{\varepsilon}{q}-1}$ $(m,n \in \mathbf{N})$. Since for $\lambda_i \leqslant \frac{1}{2}\ (i=1,2)$, both $f(t) := t^{\lambda_1-\frac{\varepsilon}{p}-1}$ and $g(t) := t^{\lambda_2-\frac{\varepsilon}{q}-1}$ are strictly decreasing with respect to $t>0$, by the decreasingness property of series, we have

$$\begin{aligned} \widetilde{A}_m &:= \sum_{i=1}^{m} \widetilde{a}_i = \sum_{i=1}^{m} i^{\lambda_1-\frac{\varepsilon}{p}-1} < \int_0^m t^{\lambda_1-\frac{\varepsilon}{p}-1} dt = \frac{m^{\lambda_1-\frac{\varepsilon}{p}}}{\lambda_1-\frac{\varepsilon}{p}}, \\ \widetilde{B}_n &:= \sum_{k=1}^{n} \widetilde{b}_k = \sum_{k=1}^{n} k^{\lambda_2-\frac{\varepsilon}{q}-1} < \int_0^n t^{\lambda_2-\frac{\varepsilon}{qp}-1} dt = \frac{n^{\lambda_2-\frac{\varepsilon}{q}}}{\lambda_2-\frac{\varepsilon}{q}}. \end{aligned}$$

If there exists a positive constant $M \leqslant \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$, such that (19) is valid when we replace $\lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ by M, then in particular, for $\eta_1=\eta_2=0$, we have

$$\widetilde{I} := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\widetilde{a}_m \widetilde{b}_n}{(m+n)^{\lambda}} < M \left(\sum_{m=1}^{\infty} m^{-p\lambda_1-1} \widetilde{A}_m^p \right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1} \widetilde{B}_n^q \right)^{\frac{1}{q}}. \qquad (20)$$

In the following, we obtain that $M \geqslant \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$, which follows that $M = \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible constant factor in (19) (cf. [4]).

By (20) and the decreasingness property of series, we obtain

$$\begin{aligned}
\widetilde{I} &< \frac{M}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}\left(\sum_{m=1}^{\infty} m^{-p\lambda_1-1}m^{p\lambda_1-\varepsilon}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{-q\lambda_2-1}n^{q\lambda_2-\varepsilon}\right)^{\frac{1}{q}}. \\
&= \frac{M}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}\left(1+\sum_{m=2}^{\infty} m^{-1-\varepsilon}\right) \\
&< \frac{M}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}\left(1+\int_1^{\infty} x^{-1-\varepsilon}dx\right) = \frac{M(\varepsilon+1)}{\varepsilon(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}.
\end{aligned}$$

By (14), for $\eta_i = \eta = 0$ $(i = 1,2)$, $s = \lambda$, $s_1 = \lambda_1 - \frac{\varepsilon}{p}$ $(\in (0,\frac{1}{2})\cap(0,\lambda))$, we have

$$\begin{aligned}
0 &< k_\lambda\Big(\lambda_1-\frac{\varepsilon}{p}\Big)\Big(1-O\Big(\frac{1}{n^{\lambda_1-\frac{\varepsilon}{p}}}\Big)\Big) \\
&< \omega\Big(\lambda_1-\frac{\varepsilon}{p},n\Big) := n^{\lambda_2+\frac{\varepsilon}{p}}\sum_{m=1}^{\infty}\frac{m^{\lambda_1-\frac{\varepsilon}{p}-1}}{(m+n)^{\lambda}} < k_\lambda\Big(\lambda_1-\frac{\varepsilon}{p}\Big)\ (n\in\mathbf{N}),
\end{aligned}$$

Then we find

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\left[n^{\lambda_2+\frac{\varepsilon}{p}}\sum_{m=1}^{\infty}\frac{m^{\lambda_1-\frac{\varepsilon}{p}-1}}{(m+n)^{\lambda}}\right]n^{-\varepsilon-1} = \sum_{n=1}^{\infty}\omega\Big(\lambda_1-\frac{\varepsilon}{p},n\Big)n^{-\varepsilon-1} \\
&> k_\lambda\Big(\lambda_1-\frac{\varepsilon}{p}\Big)\left[\sum_{n=1}^{\infty}n^{-\varepsilon-1}-\sum_{n=1}^{\infty}O\Big(\frac{1}{n^{\lambda_1-\frac{\varepsilon}{p}}}\Big)n^{-\varepsilon-1}\right] \\
&> k_\lambda\Big(\lambda_1-\frac{\varepsilon}{p}\Big)\left[\int_1^{\infty}y^{-\varepsilon-1}dy-\sum_{n=1}^{\infty}O\Big(\frac{1}{n^{\lambda_1+\frac{\varepsilon}{q}+1}}\Big)\right] \\
&= \frac{1}{\varepsilon}B\Big(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\Big)(1-\varepsilon O(1)).
\end{aligned}$$

By (20) and the above results, we have

$$B\Big(\lambda_1-\frac{\varepsilon}{p},\lambda_2+\frac{\varepsilon}{p}\Big)(1-\varepsilon O(1)) \leqslant \frac{M(\varepsilon+1)}{(\lambda_1-\frac{\varepsilon}{p})(\lambda_2-\frac{\varepsilon}{q})}.$$

For $\varepsilon \to 0^+$, in view of the continuity of the beta function, we find $M \geqslant \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$. Hence, $M = \lambda_1\lambda_2 B(\lambda_1,\lambda_2)$ is the best possible constant factor in (19).

On the other hand, since $\widehat{\lambda}_1 = \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}, \widehat{\lambda}_2 = \frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$, and $\lambda_i, \lambda-\lambda_i \in (0,\frac{1}{2}]$ $(i = 1,2)$, we find

$$\begin{aligned}
\widehat{\lambda}_1+\widehat{\lambda}_2 &= \frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p} = \lambda, \\
\widehat{\lambda}_1, \widehat{\lambda}_2 &\leqslant \frac{1/2}{p}+\frac{1/2}{q} = \frac{1}{2},\ \ 0 < \widehat{\lambda}_1, \widehat{\lambda}_2 < \lambda,
\end{aligned}$$

and $\widehat{\lambda}_1\widehat{\lambda}_2B(\widehat{\lambda}_1,\widehat{\lambda}_2) \in \mathbf{R}_+ = (0,\infty)$. By (19), we still have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n-\eta)^{\lambda}} < \widehat{\lambda}_1\widehat{\lambda}_2B(\widehat{\lambda}_1,\widehat{\lambda}_2) \times \left[\sum_{m=1}^{\infty}(m-\eta_1)^{-p\widehat{\lambda}_1-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta_2)^{-q\widehat{\lambda}_2-1}B_n^q\right]^{\frac{1}{q}}. \tag{21}$$

If the constant factor in (18) is the best possible, then by (21), we have the following inequality:

$$\frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}(k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} \leqslant \widehat{\lambda}_1\widehat{\lambda}_2B(\widehat{\lambda}_1,\widehat{\lambda}_2) = \frac{\Gamma(\lambda+2)}{\Gamma(\lambda)}k_{\lambda+2}(\widehat{\lambda}_1+1),$$

which follows that

$$k_{\lambda+2}(\widehat{\lambda}_1+1) \geqslant (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}.$$

By Hölder's inequality, we obtain

$$\begin{aligned} 0 < k_{\lambda+2}(\widehat{\lambda}_1+1) &= k_{\lambda+2}\left(\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}+1\right) \\ &= \int_0^{\infty}\frac{u^{\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}}}{(1+u)^{\lambda+2}}du = \int_0^{\infty}\frac{(u^{\frac{\lambda-\lambda_2}{p}})(u^{\frac{\lambda_1}{q}})}{(1+u)^{\lambda+2}}du \qquad (22) \\ &\leqslant \left[\int_0^{\infty}\frac{u^{\lambda-\lambda_2}}{(1+u)^{\lambda+2}}du\right]^{\frac{1}{p}}\left[\int_0^{\infty}\frac{u^{\lambda_1}}{(1+u)^{\lambda+2}}du\right]^{\frac{1}{q}} \\ &= \left[\int_0^{\infty}\frac{v^{(\lambda_2+1)-1}}{(1+v)^{\lambda+2}}dv\right]^{\frac{1}{p}}\left[\int_0^{\infty}\frac{u^{(\lambda_1+1)-1}}{(1+u)^{\lambda+2}}du\right]^{\frac{1}{q}} \\ &= (k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}}. \qquad (23) \end{aligned}$$

Then we have

$$(k_{\lambda+2}(\lambda_2+1))^{\frac{1}{p}}(k_{\lambda+2}(\lambda_1+1))^{\frac{1}{q}} = k_{\lambda+2}(\widehat{\lambda}_1+1),$$

namely, (23) keeps the form of equality.

We observe that (23) keeps the form of equality if and only if there exist constants A and B, such that they are not both zero satisfying (cf. [33]) $Au^{\lambda-\lambda_2} = Bu^{\lambda_1}$ *a.e.* in. $\mathbf{R}_+$ Assuming that $A \neq 0$, we have $u^{\lambda-\lambda_1-\lambda_2} = B/A$ *a.e.* in. $\mathbf{R}_+$, and then $\lambda-\lambda_1-\lambda_2 = 0$, namely, $\lambda_1+\lambda_2 = \lambda$.

The theorem is proved. □

REMARK 2. (i) For $\lambda = 1$, $\lambda_1 = \lambda_2 = \frac{1}{2}$ in (19), we have the following inequality with the best possible constant factor $\frac{\pi}{4}$:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-\eta} < \frac{\pi}{4}\left[\sum_{m=1}^{\infty}(m-\eta_1)^{-\frac{p}{2}-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}(n-\eta_2)^{-\frac{q}{2}-1}B_n^q\right]^{\frac{1}{q}}. \tag{24}$$

In particular, (a) for $\eta = \eta_1 = \eta_2 = 0$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{4}\left(\sum_{m=1}^{\infty}m^{-\frac{p}{2}-1}A_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-\frac{q}{2}-1}B_n^q\right)^{\frac{1}{q}}; \tag{25}$$

(b) for $\eta = \frac{1}{2}$, $\eta_1 = \eta_2 = \frac{1}{4}$, we have

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n-\frac{1}{2}} < \frac{\pi}{4}\left[\sum_{m=1}^{\infty}\left(m-\frac{1}{4}\right)^{-\frac{p}{2}-1}A_m^p\right]^{\frac{1}{p}}\left[\sum_{n=1}^{\infty}\left(n-\frac{1}{4}\right)^{-\frac{q}{2}-1}B_n^q\right]^{\frac{1}{q}}. \tag{26}$$

(ii) For $\eta = \eta_1 = \eta_2 = 0$, (19) reduces to (4). Hence, (19) (rep. (18)) is a more accurate extension of inequality (4).

4. Conclusions

In this paper, by means of the weight coefficients and the idea of introducing parameters, using Hermite-Hadamard's inequality, the Euler-Maclaurin summation formula and Abel's summation by parts formula, a more accurate extension of (4) involving the partial sums is given in Theorem 1. The equivalent conditions of the best possible constant factor related to several parameters and some particular inequalities are considered in Theorem 2 and Remark 3. The lemmas and theorems provide an extensive account of this type of inequalities.

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