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GENERALIZATIONS AND SHARPENINGS OF CERTAIN BERNSTEIN AND TURÁN TYPES OF INEQUALITIES FOR THE POLAR DERIVATIVE OF A POLYNOMIAL

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Abstract. Let $p(z)$ be a polynomial of degree n. The polar derivative of $p(z)$ with respect to a complex number α is defined by

$$D_\alpha p(z) = np(z) + (\alpha - z)p'(z).$$

If $p(z)$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, then for $|\alpha| \geqslant k$, Aziz and Rather [Math. Inequal. Appl., 1, (1998), 231-238] proved

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant n\left(\frac{|\alpha| - k}{1 + k^n}\right) \max_{|z|=1} |p(z)|.$$

In this paper, we first improve as well as generalize the above inequality. Besides, we are able to prove an improvement of a result due to Govil and Mctume [Acta Math. Hungar., 104, (2004), 115–126] and also prove an inequality for a subclass of polynomials having all its zeros in $|z| \geqslant k$, $k \leqslant 1$.

1. Introduction

Let $p(z) = \sum_{j=0}^{n} c_j z^j$ be a polynomial of degree n over the set of complex numbers. We will use $Q(z)$ to represent the polynomial $z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

According to the famous Bernstein's inequality [6],

$$\max_{|z|=1} |p'(z)| \leqslant n \max_{|z|=1} |p(z)|. \tag{1}$$

Inequality (1) is sharp and equality holds for $p(z) = \alpha z^n, \alpha \neq 0$.

The above inequality can be sharpened, if the zeros of $p(z)$ are restricted. In this direction, Erdös conjectured and later Lax [18] proved that if $p(z)$ has all its zeros in $|z| \geqslant 1$, then

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{2} \max_{|z|=1} |p(z)|. \tag{2}$$

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Inequality (2) is best possible for $p(z)=\alpha+\beta z^n$, where $|\alpha|=|\beta|$.

On the other hand, in 1939, Turán [19] provided a lower bound estimate of the derivative to the size of the polynomial by restricting its zeros, and proved that if $p(z)$ has all its zeros in $|z|\leqslant 1$, then

$$\max_{|z|=1}|p'(z)|\geqslant\frac{n}{2}\max_{|z|=1}|p(z)|. \tag{3}$$

Aziz and Dawood [2] further refined inequality (3) by involving $\min\limits_{|z|=1}|p(z)|$

$$\max_{|z|=1}|p'(z)|\geqslant\frac{n}{2}\left\{\max_{|z|=1}|p(z)|+\min_{|z|=1}|p(z)|\right\}. \tag{4}$$

Both these inequalities (3) and (4) are best possible and equality holds if $p(z)$ has all its zeros on $|z|=1$.

Inequalities (3) and (4) have been extended and generalized in different directions (see [4], [5], [8], [10], [16]). For polynomial $p(z)$ having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, Govil [8] proved that

$$\max_{|z|=1}|p'(z)|\geqslant\frac{n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{5}$$

Govil [10] further improved inequality (5) for the same class of polynomials which also is a generalization of (4) by proving

$$\max_{|z|=1}|p'(z)|\geqslant\frac{n}{1+k^n}\max_{|z|=1}|p(z)|+\frac{n}{1+k^n}\min_{|z|=k}|p(z)|. \tag{6}$$

Inequalities (5) and (6) are sharp and equality holds for $p(z)=z^n+k^n$.

Govil [9] proved a generalization of (2) to a subclass of polynomials having all its zeros in $|z|\geqslant k$, $k\leqslant 1$, by proving that if $|p'(z)|$ and $|Q'(z)|$ attain maximum at the same point on $|z|=1$, then

$$\max_{|z|=1}|p'(z)|\leqslant\frac{n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{7}$$

It is easy to see that $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}\left[\frac{D_\alpha p(z)}{\alpha}\right]=p'(z).$$

In 1998, Aziz and Rather [3] extended inequality (5) to polar derivative by proving that if $p(z)$ is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then for every real or complex number α with $|\alpha|\geqslant k$,

$$\max_{|z|=1}|D_\alpha p(z)|\geqslant n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)|. \tag{8}$$

Govil and Mctume [12] established the polar derivative extension of inequality (6) and proved

$$\max_{|z|=1}|D_\alpha p(z)| \geqslant n\left(\frac{|\alpha|-k}{1+k^n}\right)\max_{|z|=1}|p(z)|+n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)\min_{|z|=k}|p(z)|, \tag{9}$$

for any complex number α with $|\alpha| \geqslant 1+k+k^n$.

In literature there exist some recent results which improve inequality (8) by involving certain coefficients, for example: Govil and Kumar [11], Kumar [14], Kumar and Dhankhar [15] and Rather et al. [17]. We can improve inequality (8) by a method different from those adopted by these authors.

2. Main results

In this paper, we get some results concerning polar derivative of a polynomial by using a lemma of Dubinin [7]. We begin, by presenting the following generalization and refinement of inequality (8) due to Aziz and Rather [3].

THEOREM 1. *If $p(z)=z^s\sum_{j=0}^{n-s}c_jz^j$, $0\leqslant s\leqslant n$, is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then for any complex number α with $|\alpha|\geqslant k$,*

$$\max_{|z|=1}|D_\alpha p(z)| \geqslant (|\alpha|-k)\left(\frac{n+s}{1+k^n}+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{(1+k^n)k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\max_{|z|=1}|p(z)|. \tag{10}$$

REMARK 1. Since the polynomial $h(z)=\frac{p(z)}{z^s}=\sum_{j=0}^{n-s}c_jz^j$ has all its zeros in $|z|\leqslant k$, $k\geqslant 1$ and

$$\left|\frac{c_0}{c_{n-s}}\right| \leqslant k^{n-s},$$

which is equivalent to

$$k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|} \geqslant \sqrt{|c_0|}$$

Dividing both sides of (10) by $|\alpha|$ and taking limit as $|\alpha|\to\infty$, we get the following generalization and refinement of inequality (5) due to Govil [8].

COROLLARY 1. *If $p(z)=z^s\sum_{j=0}^{n-s}c_jz^j$, $0\leqslant s\leqslant n$, is a polynomial of degree n having all its zeros in $|z|\leqslant k$, $k\geqslant 1$, then*

$$\max_{|z|=1}|p'(z)| \geqslant \left(\frac{n+s}{1+k^n}+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{(1+k^n)k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\max_{|z|=1}|p(z)|. \tag{11}$$

When $s = 0$, Theorem 1 in particular gives an improvement of inequality (8) proved by Aziz and Rather [3].

COROLLARY 2. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then for any complex number* $|\alpha|$ *with* $|\alpha| \geqslant k$

$$\max_{|z|=1} |D_\alpha p(z)| \geqslant (|\alpha| - k)\left(\frac{n}{1+k^n} + \frac{k^{\frac{n}{2}}\sqrt{|c_n|} - \sqrt{|c_0|}}{(1+k^n)k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\max_{|z|=1}|p(z)|. \tag{12}$$

Dividing both sides of (12) by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we get

COROLLARY 3. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \geqslant \left(\frac{n}{1+k^n} + \frac{k^{\frac{n}{2}}\sqrt{|c_n|} - \sqrt{|c_0|}}{(1+k^n)k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\max_{|z|=1}|p(z)|. \tag{13}$$

Inequality (13) is best possible for $p(z) = z^n + k^n$.

REMARK 2. Taking $k = 1$ in Corollary 3, inequality (13) provides a refinement of inequality (3) due to Turán.

EXAMPLE 1. Consider the polynomial $p(z) = z^2(z^2 - 9)$. Then $p(z)$ is a polynomial of degree 4 having all its zeros in $|z| \leqslant 3$. For this polynomial, we have $\max_{|z|=1}|p(z)| = 10$ and $\min_{|z|=3}|p(z)| = 0$. Then it can be easily seen that by inequalities (5) and (6), we have $\max_{|z|=1}|p'(z)| \geqslant \frac{40}{82}$, while our inequality (11) gives $\max_{|z|=1}|p'(z)| \geqslant \frac{60}{82}$, an improvement of 50% over the bounds obtained from (5) and (6). Also, the inequality (13) gives $\max_{|z|=1}|p'(z)| \geqslant \frac{50}{82}$, an improvement of 25% over the bounds obtained from (5) and (6).

As an application of Theorem 1, we get the following refinement of inequality (9) due to Govil and Mctume [12].

THEOREM 2. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros* *in* $|z| \leqslant k,\ k \geqslant 1$, *then for any complex number* α *with* $|\alpha| \geqslant 1+k+k^n$

$$\begin{aligned}
&\max_{|z|=1} |D_\alpha p(z)| \\
\geqslant\ & \frac{(|\alpha|-k)}{1+k^n}\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\max_{|z|=1}|p(z)| \\
&+\left[n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{|\alpha|-k}{1+k^n}\left(\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\right]m, \quad (14)
\end{aligned}$$

where $m = \min_{|z|=k} |p(z)|$ *and* $\theta_0 = \arg\{p(e^{i\phi_0})\}$ *such that* $|p(e^{i\phi_0})| = \max_{|z|=1} |p(z)|$.

REMARK 3. If $p(z) = \sum_{j=0}^{n} c_j z^j$ is a polynomial of degree n having all its zeros in $|z| \leqslant k,\ k \geqslant 1$, then for any complex number $|\lambda| e^{i\theta_0}$ with $|\lambda| < 1$, by Rouche's theorem it follows that the polynomial $p(z)+|\lambda|e^{i\theta_0}m = (c_0+|\lambda|e^{i\theta_0}m)+c_1 z+\cdots+c_n z^n$ has all its zeros in $|z| \leqslant k$, where $m = \min_{|z|=k}|p(z)|$, then

$$k^n \geqslant \left|\frac{c_0+|\lambda|e^{i\theta_0}m}{c_n}\right|,$$

which implies that

$$k^{\frac{n}{2}}\sqrt{|c_n|} \geqslant \sqrt{|c_0+|\lambda|e^{i\theta_0}m|}.$$

Taking $|\lambda| \to 1$, we get

$$k^{\frac{n}{2}}\sqrt{|c_n|} \geqslant \sqrt{|c_0+e^{i\theta_0}m|}.$$

REMARK 4. Diving both sides of (14) by $|\alpha|$ and taking limit as $|\alpha| \to \infty$, we have the following refinement of inequality (6) due to Govil [10].

COROLLARY 4. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant k,\ k \geqslant 1$, *then*

$$\begin{aligned}
\max_{|z|=1}|p'(z)| \geqslant\ & \frac{1}{1+k^n}\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\max_{|z|=1}|p(z)| \\
&+\left[\frac{n}{1+k^n}+\frac{1}{1+k^n}\left(\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\right]m, \qquad (15)
\end{aligned}$$

where $m = \min_{|z|=k} |p(z)|$ *and* $\theta_0 = \arg\{p(e^{i\phi_0})\}$ *such that* $|p(e^{i\phi_0})| = \max_{|z|=1} |p(z)|$.

Inequality (15) is best possible for $p(z) = z^n + k^n$.

REMARK 5. Taking $k = 1$ in Corollary 4, inequality (15) reduces to a refinement of inequality (4) due to Aziz and Dawood [2].

COROLLARY 5. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \geqslant \frac{1}{2}\left(n + \frac{\sqrt{|c_n|} - \sqrt{|c_0 + e^{i\theta_0} m|}}{\sqrt{|c_n|}}\right) \max_{|z|=1} |p(z)| + \frac{1}{2}\left[n + \left(\frac{\sqrt{|c_n|} - \sqrt{|c_0 + e^{i\theta_0} m|}}{\sqrt{|c_n|}}\right)\right] m, \tag{16}$$

where $m = \min_{|z|=1} |p(z)|$ *and* $\theta_0 = \arg\left\{p(e^{i\phi_0})\right\}$ *such that* $|p(e^{i\phi_0})| = \max_{|z|=1} |p(z)|$.

Further, we are able to prove an improvement of inequality (7) due to Govil [9].

THEOREM 3. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \geqslant k$, $k \leqslant 1$. *If* $|p'(z)|$ *and* $|Q'(z)|$ *attain their maxima at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \left[\frac{n}{1+k^n} - \frac{\left(\sqrt{|c_0|} - k^{\frac{n}{2}} \sqrt{|c_n|}\right) k^n}{(1+k^n)\sqrt{|c_0|}}\right] \max_{|z|=1} |p(z)|. \tag{17}$$

The result is sharp and equality in (17) *holds for* $p(z) = z^n + k^n$.

REMARK 6. Taking $k = 1$ in Theorem 3, we get the following improvement of (2) due to Erdös and Lax for a subclass of polynomials having all its zeros in $|z| \geqslant 1$.

COROLLARY 6. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* n *having all its zeros in* $|z| \geqslant 1$. *If* $|p'(z)|$ *and* $|Q'(z)|$ *attain their maxima at the same point on* $|z| = 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{1}{2}\left(n - \frac{\sqrt{|c_0|} - \sqrt{|c_n|}}{\sqrt{|c_0|}}\right) \max_{|z|=1} |p(z)|. \tag{18}$$

3. Lemmas

We need the following lemmas to prove our theorems.

LEMMA 1. *If* $0 \leqslant x \leqslant 1$ *and* $0 \leqslant y \leqslant 1$, *then*

$$\frac{2}{1+x} \geqslant 1 + \sqrt{y} - \sqrt{xy}. \tag{19}$$

Proof of Lemma 1. The inequality is trivially true if $x = 1$. So, let us assume that $x < 1$, then

$$\frac{1+\sqrt{x}}{1+x} > 1 \geqslant \sqrt{y},$$

which implies

$$\frac{1-x}{1+x} > \sqrt{y}\frac{1-x}{1+\sqrt{x}} = \sqrt{y} - \sqrt{xy},$$

and hence it follows that

$$\frac{2}{1+x} > 1 + \sqrt{y} - \sqrt{xy}. \quad \square$$

LEMMA 2. *If* $p(z) = \sum_{j=0}^{n} c_j z^j$ *is a polynomial of degree* $n \geqslant 1$ *having all its zeros in* $|z| \leqslant 1$, *then for all* z *on* $|z| = 1$ *with* $p(z) \neq 0$

$$\Re\left(z\frac{p'(z)}{p(z)}\right) \geqslant \frac{n+1}{2} - \frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_n|}}. \tag{20}$$

The above result was first proved by Dubinin [7]. Here we present an alternative proof which we think is new by the principle of mathematical induction.

Proof of Lemma 2. Without loss of generality, let us assume $c_n = 1$. We use the principle of mathematical induction on the degree of $p(z)$.

If $n = 1$, then $p(z) = z - z_0$ with $|z_0| \leqslant 1$, and for $|z| = 1$ and $z \neq z_0$

$$\Re\left(z\frac{p'(z)}{p(z)}\right) = \Re\left(\frac{z}{z-z_0}\right) \geqslant \frac{1}{1+|z_0|},$$

and with some simple calculations it is easy to obtain that, for $|z_0| \leqslant 1$

$$\frac{1}{1+|z_0|} \geqslant \frac{1}{2}\left(2 - \sqrt{|z_0|}\right).$$

So,

$$\Re\left(z\frac{p'(z)}{p(z)}\right) \geqslant \frac{1}{2}\left(2 - \sqrt{|z_0|}\right), \tag{21}$$

which is nothing but (20) for $n = 1$.

Let us assume that (20) is true for all polynomials with degree $\leqslant N$.

Let $p(z) = (z-w)Q(z)$ with $|w| \leqslant 1$, where $Q(z) = \sum_{j=0}^{N} c_j z^j$ is a polynomial of degree N having all its zeros in $|z| \leqslant 1$, then

$$\begin{aligned}\Re\left(z\frac{p'(z)}{p(z)}\right) &= \Re\left(\frac{z}{z-w}\right) + \Re\left(z\frac{Q'(z)}{Q(z)}\right)\\ &\geqslant \frac{1}{1+|w|} + \frac{1}{2}\left(N + 1 - \sqrt{|c_0|}\right),\end{aligned}$$

for all z on $|z|=1$ with $p(z) \neq 0$.

We need to show that

$$\Re\left(z\frac{p'(z)}{p(z)}\right) \geqslant \frac{1}{2}\left(N+2-\sqrt{|w||c_0|}\right) \quad \text{on} \quad |z|=1. \tag{22}$$

Clearly, inequality (22) holds if

$$\frac{1}{1+|w|}+\frac{1}{2}(N+1-\sqrt{c_0}) \geqslant \frac{1}{2}\left(N+2-\sqrt{|w||c_0|}\right),$$

which is equivalent to

$$\frac{2}{1+|w|} \geqslant 1+\sqrt{|c_0|}-\sqrt{|w||c_0|}. \tag{23}$$

Since all the zeros of $p(z)$ lies on $|z| \leqslant 1$, therefore $0 \leqslant |c_0| \leqslant 1$ and $0 \leqslant |w| \leqslant 1$, the inequality (23) follows from Lemma 1.

This completes the proof of Lemma 2 by using induction principle. □

LEMMA 3. *If* $p(z)=z^s\sum_{j=0}^{n-s} c_j z^j$, $0 \leqslant s \leqslant n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant 1$, *with* s*-fold zeros at the origin, then on* $|z|=1$ *with* $p(z) \neq 0$

$$\Re\left(z\frac{p'(z)}{p(z)}\right) \geqslant \frac{n+s+1}{2}-\frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_{n-s}|}}. \tag{24}$$

Proof of Lemma 3. Let $p(z)=z^sQ(z)$ where $Q(z)=\sum_{j=0}^{n-s} c_j z^j$ is a polynomial of degree $n-s$ having all its zeros in $|z| \leqslant 1$. Then for any complex number z with $p(z) \neq 0$

$$\Re\left(z\frac{p'(z)}{p(z)}\right) = s+\Re\left(z\frac{Q'(z)}{Q(z)}\right) \tag{25}$$

Applying Lemma 2 to $Q(z)$, it follows from (25) that

$$\Re\left(z\frac{p'(z)}{p(z)}\right) \geqslant s+\frac{n-s+1}{2}-\frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_{n-s}|}} = \frac{n+s+1}{2}-\frac{1}{2}\frac{\sqrt{|c_0|}}{\sqrt{|c_{n-s}|}}. \quad \square$$

LEMMA 4. *Let* $p(z)$ *be a polynomial of degree* n *having all its zeros in* $|z| \leqslant k$, $k \geqslant 1$, *then*

$$\max_{|z|=k}|p(z)| \geqslant \frac{2k^n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{26}$$

The above result appears in Aziz [1].

LEMMA 5. *If* $p(z) = z^s \sum_{j=0}^{n-s} c_j z^j$, $0 \leqslant s \leqslant n$ *is a polynomial of degree* n *having all its zeros in* $|z| \leqslant 1$, *with* s*-fold zeros at the origin, then for any complex number* α *with* $|\alpha| \geqslant 1$ *and on* $|z| = 1$

$$|D_\alpha p(z)| \geqslant (|\alpha| - 1)\left(\frac{n+s}{2} + \frac{\sqrt{|c_{n-s}|} - \sqrt{|c_0|}}{2\sqrt{|c_{n-s}|}}\right)|p(z)| \tag{27}$$

Proof of Lemma 5. If $q(z) = z^n \overline{p(\frac{1}{\bar{z}})}$, then it is easy to verify that for $|z| = 1$

$$\left|q'(z)\right| = \left|np(z) - zp'(z)\right|.$$

Also, $p(z)$ has all its zeros in $|z| \leqslant 1$, and it is a well-known fact that on $|z| = 1$

$$\left|p'(z)\right| \geqslant \left|q'(z)\right|. \tag{28}$$

We have for $|\alpha| \geqslant 1$ and $|z| = 1$

$$\begin{aligned} |D_\alpha p(z)| &= \left|np(z) + (\alpha - z)p'(z)\right| \\ &\geqslant |\alpha|\left|p'(z)\right| - \left|np(z) - zp'(z)\right|. \end{aligned}$$

This gives with (28)

$$|D_\alpha p(z)| \geqslant (|\alpha| - 1)\left|p'(z)\right| \tag{29}$$

on $|z| = 1$, and using Lemma 3, we get on $|z| = 1$

$$\left|p'(z)\right| \geqslant \left(\frac{n+s}{2} + \frac{\sqrt{|c_{n-s}|} - \sqrt{|c_0|}}{2\sqrt{|c_{n-s}|}}\right)|p(z)|. \tag{30}$$

Combining (29) and (30) gives inequality (27). □

LEMMA 6. *If* $p(z)$ *is a polynomial of degree* n, *then on* $|z| = 1$

$$\left|p'(z)\right| + \left|Q'(z)\right| \leqslant n \max_{|z|=1} |p(z)|. \tag{31}$$

The above result is due to Govil and Rahman [13].

4. Proofs of the theorems

Proof of Theorem 1. Since $p(z) = z^s \sum_{j=0}^{n-s} c_j z^j$ has all its zeros in $|z| \leqslant k$, $k \geqslant 1$, the polynomial $p(kz) = z^s \left(k^s c_0 + k^{s+1} c_1 z + \cdots k^n c_n z^{n-s}\right)$ has all its zeros in $|z| \leqslant 1$. Using Lemma 5 to $p(kz)$, we get for $\left|\frac{\alpha}{k}\right| \geqslant 1$

$$\max_{|z|=1} \left|D_{\frac{\alpha}{k}} p(kz)\right| \geqslant \frac{|\alpha| - k}{k}\left(\frac{n+s}{2} + \frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|} - \sqrt{|c_0|}}{2k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\max_{|z|=1}|p(kz)|,$$

which is

$$\max_{|z|=1}\left|np(kz)+\left(\frac{\alpha}{k}-z\right)kp'(kz)\right|$$
$$\geqslant \frac{(|\alpha|-k)}{k}\left(\frac{n+s}{2}+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{2k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\max_{|z|=k}|p(z)|. \tag{32}$$

Using Lemma 4 and the fact that $\max_{|z|=1}\left|np(kz)+\left(\frac{\alpha}{k}-z\right)kp'(kz)\right|=\max_{|z|=k}|D_\alpha p(z)|$, the inequality (32) gives

$$\max_{|z|=k}|D_\alpha p(z)|\geqslant \frac{(|\alpha|-k)}{k}\left(\frac{n+s}{2}+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{2k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\frac{2k^n}{1+k^n}\max_{|z|=1}|p(z)|. \tag{33}$$

As we can see $D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and $k\geqslant 1$, it is well-known that

$$\max_{|z|=k}|D_\alpha p(z)|\leqslant k^{n-1}\max_{|z|=1}|D_\alpha p(z)|.$$

By using this fact, the inequality (33) gives

$$k^{n-1}\max_{|z|=1}|D_\alpha p(z)|$$
$$\geqslant (|\alpha|-k)\left(n+s+\frac{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}-\sqrt{|c_0|}}{k^{\frac{n-s}{2}}\sqrt{|c_{n-s}|}}\right)\frac{k^{n-1}}{1+k^n}\max_{|z|=1}|p(z)|, \tag{34}$$

this gives the desired inequality (10), and thus the proof of Theorem 1 is complete. □

Proof of Theorem 2. If $p(z)$ has a zero on $|z|=k$, then $m=0$ and the result follows trivially from Theorem 1. So, without loss of generality, let us assume that $p(z)$ has all its zeros in $|z|<k$, $k\geqslant 1$, then it follows by Rouche's theorem that for any complex number λ with $|\lambda|<1$, the polynomial $p(z)+\lambda m=(c_0+\lambda m)+c_1z+\cdots+c_nz^n$ also has all its zeros in $|z|<k$, $k\geqslant 1$. Therefore, applying Theorem 1 to $p(z)+\lambda m$ with $s=0$, we get for $|\alpha|\geqslant 1+k+k^n$

$$\max_{|z|=1}|D_\alpha[p(z)+\lambda m]|$$
$$\geqslant (|\alpha|-k)\left(\frac{n}{1+k^n}+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+\lambda m|}}{(1+k^n)k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\max_{|z|=1}|p(z)+\lambda m|. \tag{35}$$

Let $0\leqslant \phi_0<2\pi$, be such that $\left|p(e^{i\phi_0})\right|=\max_{|z|=1}|p(z)|$. Then, the above inequality (35) gives

$$\max_{|z|=1}|D_\alpha p(z)+n\lambda m|$$
$$\geqslant (|\alpha|-k)\left(\frac{n}{1+k^n}+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+\lambda m|}}{(1+k^n)k^{\frac{n}{2}}\sqrt{|c_n|}}\right)|p(e^{i\phi_0})+\lambda m|. \tag{36}$$

Now,

$$\begin{aligned}\left|p(e^{i\phi_0})+\lambda m\right| &= \left|\left|p(e^{i\phi_0})\right|e^{i\theta_0}+|\lambda|e^{i\phi}m\right| \\ &= \left|\left|p(e^{i\phi_0})\right|+|\lambda|e^{i(\phi-\theta_0)}m\right|.\end{aligned}$$

Setting the argument ϕ such that $\phi=\theta_0$, then $\left|p(e^{i\phi_0})+\lambda m\right|=\left|p(e^{i\phi_0})\right|+|\lambda|m$, and then it follows from inequality (36) that

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)|+n|\lambda|m \\ &\geqslant (|\alpha|-k)\left(\frac{n}{1+k^n}+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+|\lambda|e^{i\theta_0}m|}}{(1+k^n)k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\left(\left|p(e^{i\phi_0})\right|+|\lambda|m\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}&\max_{|z|=1}|D_\alpha p(z)| \\ &\geqslant \frac{(|\alpha|-k)}{1+k^n}\left(n+\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+|\lambda|e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\max_{|z|=1}|p(z)| \\ &+|\lambda|\left[n\left(\frac{|\alpha|-(1+k+k^n)}{1+k^n}\right)+\frac{|\alpha|-k}{1+k^n}\left(\frac{k^{\frac{n}{2}}\sqrt{|c_n|}-\sqrt{|c_0+|\lambda|e^{i\theta_0}m|}}{k^{\frac{n}{2}}\sqrt{|c_n|}}\right)\right]m.\end{aligned}$$

Taking $|\lambda|\to 1$, the above inequality reduces to (14). This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. Since $p(z)$ has all its zeros in $|z|\geqslant k$, $k\leqslant 1$, $Q(z)$ has all its zeros in $|z|\leqslant \frac{1}{k}, \frac{1}{k}\geqslant 1$. Then applying Corollary 3 to $Q(z)$ with $s=0$, we have

$$\begin{aligned}\max_{|z|=1}\left|Q'(z)\right| &\geqslant \left[\frac{n}{1+\frac{1}{k^n}}+\frac{\left(\frac{1}{k}\right)^{\frac{n}{2}}\sqrt{|c_0|}-\sqrt{|c_n|}}{\left(1+\frac{1}{k^n}\right)\left(\frac{1}{k}\right)^{\frac{n}{2}}\sqrt{|c_0|}}\right]\max_{|z|=1}|Q(z)| \\ &= \left[\frac{nk^n}{1+k^n}+\frac{\left(\sqrt{|c_0|}-k^{\frac{n}{2}}\sqrt{|c_n|}\right)k^n}{(1+k^n)\sqrt{|c_0|}}\right]\max_{|z|=1}|p(z)|. \qquad (37)\end{aligned}$$

By Lemma 6 we have on $|z|=1$,

$$\left|p'(z)\right|+\left|Q'(z)\right|\leqslant n\max_{|z|=1}|p(z)|. \qquad (38)$$

Since $|p'(z)|$ and $|Q'(z)|$ attain their maxima at the same point, then

$$\max_{|z|=1}\left\{\left|p'(z)\right|+\left|Q'(z)\right|\right\}=\max_{|z|=1}\left|p'(z)\right|+\max_{|z|=1}\left|Q'(z)\right|. \qquad (39)$$

Combining (37), (38) and (39), we have

$$\max_{|z|=1}|p'(z)|+\left[\frac{nk^n}{1+k^n}+\frac{\left(\sqrt{|c_0|}-k^{\frac{n}{2}}\sqrt{|c_n|}\right)k^n}{(1+k^n)\sqrt{|c_0|}}\right]\max_{|z|=1}|p(z)|\leqslant n\max_{|z|=1}|p(z)|,$$

which is equivalent to

$$\max_{|z|=1}|p'(z)|\leqslant\left[\frac{n}{1+k^n}-\frac{\left(\sqrt{|c_0|}-k^{\frac{n}{2}}\sqrt{|c_n|}\right)k^n}{(1+k^n)\sqrt{|c_0|}}\right]\max_{|z|=1}|p(z)|.$$

This completes the proof of Theorem 3. □

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