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APPROXIMATION OF BÖGEL CONTINUOUS FUNCTIONS AND DEFERRED WEIGHTED A–STATISTICAL CONVERGENCE BY BERNSTEIN–KANTOROVICH TYPE OPERATORS ON A TRIANGLE

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Abstract. The present article is in continuation of the work done by Kajla (Math. Methods Appl. Sci., 42(12), (2019), 4365–4377) on Bernstein-Kantorovich type operators on a triangle. We discuss the deferred weighted A-statistical approximation and τ-th order generalization of these operators by means of a Taylor polynomial. We also investigate the convergence estimates for the functions in a Bögel space by these operators.

1. Introduction

Stancu [28] introduced a new kind of Bernstein operators involving two parameters $r,s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ as

$$\mathscr{L}_{\nu,r,s}(g;x) = \sum_{\mu_1=0}^{\nu-sr} b_{\nu-sr,\mu_1}(x) \sum_{\mu_2=0}^{s} b_{s,\mu_2}(x) g\left(\frac{\mu_1+\mu_2 r}{\nu}\right), \tag{1.1}$$

where $b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $0 \leqslant x \leqslant 1$. Clearly, for any $r \in \mathbb{N}_0$ and $s = 0$, the operators (1.1) reduce to the classical Bernstein polynomials. Abel et al. [1] proposed a Durrmeyer version of the sequence of operators (1.1) and derived some approximation properties and a complete asymptotic expansion for these operators. Kajla [17] considered the Kantorovich variant of the operators (1.1) as

$$\mathscr{K}_{\nu,r,s}(g;x) = \sum_{\mu_1=0}^{\nu-sr} b_{\nu-sr,\mu_1}(x) \sum_{\mu_2=0}^{s} b_{s,\mu_2}(x) \int_0^1 g\left(\frac{\mu_1+\mu_2 r+t}{\nu}\right) dt, \tag{1.2}$$

and studied some direct approximation theorems and the A-statistical convergence by these operators. Later, Kajla [18] proposed a bivariate generalized Bernstein-Kantorovich type operator on a triangle, associated with the operators (1.2) as follows:

$$\mathscr{U}_{\nu,r,s}(g;\mathbf{x}) = \sum_{\mu_1=0}^{\nu-sr} \sum_{\mu_2=0}^{\nu-sr-\mu_1} w_{\nu-sr,\mu}(\mathbf{x}) \sum_{\nu_1=0}^{s} \sum_{\nu_2=0}^{s-\nu_1} w_{s,\nu}(\mathbf{x}) \\ \times \int_0^1 \int_0^1 g\left(\frac{\mu_1+r\nu_1+t_1}{\nu}, \frac{\mu_2+r\nu_2+t_2}{\nu}\right) dt_1 dt_2 \tag{1.3}$$

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for all $r,s \in \mathbb{N}_0$, $g \in C(T)$, $C(T)$ being the space of all continuous functions on T, $\nu \in \mathbb{N}$, $x \in T$, where $T := \left\{\mathbf{x} = (x_1,x_2) \in \mathbb{R}^2 : x_1,x_2 \geqslant 0, x_1+x_2 \leqslant 1\right\}$ and $w_{\nu,\mu}(x) = \dfrac{\nu!}{\mu!(\nu-|\mu|)!}\, x^{\mu}(1-|x|)^{\nu-|\mu|}$, $\mu = (\mu_1,\mu_2) \in \mathbb{N}_0^2$, $|x| = x_1+x_2$, $x^{\mu} = x_1^{\mu_1}x_2^{\mu_2}$, $|\mu| = \mu_1+\mu_2$, $\mu! = \mu_1!\mu_2!$, and studied the approximation degree with the aid of the Peetre's K-functional and the modulus of continuity. Korovkin and Voronovskaja type theorems were also established in [18] by the author using the weighted A-statistical convergence. Note that for any $r \in \mathbb{N}_0$ and $s = 0$, (1.3) includes the classical bivariate Bernstein-Kantorovich operator on a triangle having a slightly different form than the operators introduced by Pop and Fărcas [25]. Deshwal et al. [9] studied the rate of convergence in terms of the moduli of continuity for the bivariate operators defined in [25] and also examined the approximation degree with the aid of the Peetre's K-functional for the associated GBS (Generalized Boolean Sum) operator. For other significant contributions in this direction, we refer the reader to the book [14] and the references therein.

In the present paper, we study the deferred weighted A-statistical convergence properties of the bivariate operator $\mathscr{U}_{\nu,r,s}$. Also, a τ^{th}-generalization of $\mathscr{U}_{\nu,r,s}$ by means of a Taylor polynomial is considered to approximate the functions in $C^{\tau}(T)$, the space of τ times continuously differentiable functions in T. The GBS case of the operator $\mathscr{U}_{\nu,r,s}$ is introduced and the approximation degree for the GBS operators is obtained with the help of the Lipschitz class of Bögel continuous functions and the mixed modulus of smoothness.

2. Preliminaries

We give some basic results for (1.3), using the test functions $e_{i,j}(\mathbf{x}) = x_1^i x_2^j$, $(i,j = 0,1,2)$ as follows:

LEMMA 1. [18] *For the operators $\mathscr{U}_{\nu,r,s}$ given by (1.3), we have*

1. $\mathscr{U}_{\nu,r,s}(e_{0,0};\mathbf{x}) = 1;$
2. $\mathscr{U}_{\nu,r,s}(e_{1,0};\mathbf{x}) = x_1 + \dfrac{1}{2\nu};$
3. $\mathscr{U}_{\nu,r,s}(e_{0,1};\mathbf{x}) = x_2 + \dfrac{1}{2\nu};$
4. $\mathscr{U}_{\nu,r,s}(e_{2,0};\mathbf{x}) = x_1^2 + \dfrac{x_1(1-x_1)}{\nu}\left(1+\dfrac{sr(r-1)}{\nu}\right) + \dfrac{x_1}{\nu} + \dfrac{1}{3\nu^2};$
5. $\mathscr{U}_{\nu,r,s}(e_{0,2};\mathbf{x}) = x_2^2 + \dfrac{x_2(1-x_2)}{\nu}\left(1+\dfrac{sr(r-1)}{\nu}\right) + \dfrac{x_2}{\nu} + \dfrac{1}{3\nu^2};$
6. $$\mathscr{U}_{\nu,r,s}(e_{4,0};\mathbf{x}) = x_1^4 + \frac{x_1^4}{5\nu^4}\Big[55\nu^2 - 30\nu^3 + 30\nu^2 rs - 30(-1+r)rs - 30\nu^2 r^2 s + 15(r-1)r^2(s-2)s - 15(r-1)r^2 s(s+2) + 10\nu(-3+(r-1)r(7+4r)s)\Big]$$

$$+\frac{x_1^3}{5\nu^4}\Big[40\nu^3-80\nu-120\nu^2-30\nu^2rs+80(r-1)rs+60\nu^2r^2s$$
$$+50(r-1)r^2s-30\nu(r-1)rs(2r+5)-30r^3s(r-1)(s-2)$$
$$+15r^2s^2(r-1)+15r^2s^2(r-1)(s+2)\Big]$$
$$+\frac{x_1^2}{5\nu^4}\Big[75\nu^2-75\nu-75rs(r-1)-65r^2s(r-1)-5r^3s(r-1)$$
$$+20\nu rs(r-1)+15r^3s(s-2)-15r^2s^2(r-1)\Big]$$
$$+\frac{x_1}{5\nu^4}\Big[30\nu+25rs(r-1)+15r^2s(r-1)+5r^3s(r-1)\Big]+\frac{1}{5\nu^4};$$

7. $$\mathscr{U}_{\nu,r,s}(e_{04};\mathbf{x})=x_2^4+\frac{x_2^4}{5\nu^4}\Big[55\nu^2-30\nu^3+30\nu^2rs-30(-1+r)rs-30\nu^2r^2s$$
$$+15(r-1)r^2(s-2)s-15(r-1)r^2s(s+2)+10\nu(-3+(r-1)r(7+4r)s)\Big]$$
$$\frac{x_2^3}{5\nu^4}\Big[40\nu^3-80\nu-120\nu^2-30\nu^2rs+80(r-1)rs+60\nu^2r^2s$$
$$+50(r-1)r^2s-30\nu(r-1)rs(2r+5)-30r^3s(r-1)(s-2)$$
$$+15r^2s^2(r-1)+15r^2s^2(r-1)(s+2)\Big]$$
$$+\frac{x_2^2}{5\nu^4}\Big[75\nu^2-75\nu-75rs(r-1)-65r^2s(r-1)-5r^3s(r-1)$$
$$+20\nu rs(r-1)+15r^3s(s-2)-15r^2s^2(r-1)\Big]$$
$$+\frac{x_2}{5\nu^4}\Big[30\nu+25rs(r-1)+15r^2s(r-1)+5r^3s(r-1)\Big]+\frac{1}{5\nu^4}.$$

Let $p=t_1-x_1,\ q=t_2-x_2$ and $\mathbf{t}=(t_1,t_2)$. As a consequence of the above lemma, we obtain:

LEMMA 2. [18] *The operators $\mathscr{U}_{\nu,r,s}$ defined by* (1.3) *satisfy the following indentities:*

(i) $\mathscr{U}_{\nu,r,s}(p;\mathbf{x})=\dfrac{1}{2\nu}$;

(ii) $\mathscr{U}_{\nu,r,s}(q;\mathbf{x})=\dfrac{1}{2\nu}$;

(iii) $\mathscr{U}_{\nu,r,s}(p^2;\mathbf{x})=\dfrac{x_1(1-x_1)}{\nu}\left(1+\dfrac{sr(r-1)}{\nu}\right)+\dfrac{1}{3\nu^2}$;

(iv) $\mathscr{U}_{\nu,r,s}(q^2;\mathbf{x})=\dfrac{x_2(1-x_2)}{\nu}\left(1+\dfrac{sr(r-1)}{\nu}\right)+\dfrac{1}{3\nu^2}$;

$$(v)\quad \mathscr{U}_{\nu,r,s}(p^4;\mathbf{x}) = \frac{3x_1^4\left(\nu^2+rs(2+r^3(s-2)+rs-2r^2s)+2\nu(r^2s-rs-1)\right)}{\nu^4}$$
$$+\frac{2x_1^3\left(\nu(6rs-6r^2s+8)-30n^2+rs(r-1)(8-3r^2(s-2)+r(8+2s))\right)}{n^4}$$
$$+\frac{x_1^2\left(3\nu^2+3\nu(2r^2s-2rs-5)+rs(r-1)(r^2(3s-7)-r(3s+13)-15)\right)}{\nu^4}$$
$$+\frac{x_1\left(rs(r^3+2r^2+2r-5)+5\nu\right)}{\nu^4}+\frac{1}{5\nu^4};$$

$$(vi)\quad \mathscr{U}_{\nu,r,s}(q^4;\mathbf{x}) = \frac{3x_2^4\left(\nu^2+rs(2+r^3(s-2)+rs-2r^2s)+2\nu(r^2s-rs-1)\right)}{\nu^4}$$
$$+\frac{2x_2^3\left(\nu(6rs-6r^2s+8)-30\nu^2+rs(r-1)(8-3r^2(s-2)+r(8+2s))\right)}{n^4}$$
$$+\frac{x_2^2\left(3n^2+3n(2r^2s-2rs-5)+rs(r-1)(r^2(3s-7)-r(3s+13)-15)\right)}{n^4}$$
$$+\frac{x_2\left(rs(r^3+2r^2+2r-5)+5\nu\right)}{\nu^4}+\frac{1}{5\nu^4}.$$

From Lemma 2, we have the following important basic result:

REMARK 1. [18] For all $\mathbf{x}\in T$ and $\nu\in\mathbb{N}$, there holds

$$\lim_{\nu}\nu\mathscr{U}_{\nu,r,s}(p^2;\mathbf{x}) = x_1(1-x_1);$$
$$\lim_{\nu}\nu\mathscr{U}_{\nu,r,s}(q^2;\mathbf{x}) = x_2(1-x_2);$$
$$\lim_{\nu}\nu^2\mathscr{U}_{\nu,r,s}(p^4;\mathbf{x}) = 3x_1^2(x_1^2-20x_1+1);$$
$$\lim_{\nu}\nu^2\mathscr{U}_{\nu,r,s}(q^4;\mathbf{x}) = 3x_2^2(x_2^2-20x_2+1).$$

Clearly, $\|\mathscr{U}_{\nu,r,s}(p^2)\| = \|\mathscr{U}_{\nu,r,s}(q^2)\| = \upsilon_{\nu,r,s}$, where $\|.\|$ denotes the sup-norm on T.

3. Deferred A-statistical convergence

Firstly, Zygmund [31], introduced the notion of the statistical convergence. Karakaya and Chishti [19] gave the concept of weighted statistical convergence. Mohiuddin (see [16], [23], [24]) established the relation of statistical weighted A-summability of a sequence with weighted A-statistical convergence. Srivastava et al. [27] defined the concept of deferred weighted A-statistical convergence.

Let $A=(a_{\nu k})$ be a non-negative infinite summability matrix. For any sequence (x_ν) of real or complex numbers, the A-transform $(Ax)_\nu$ is defined as

$$(Ax)_\nu = \sum_{k=1}^{\infty} a_{\nu k}x_k$$

such that the series converges for each ν. A is called regular if $\lim_\nu (Ax)_\nu = \alpha$ whenever $\lim_\nu x_\nu = \alpha$. Then, $x = (x_\nu)$ is said to be A-statistically convergent to α, i.e. $st_A - \lim_\nu x_\nu = \alpha$ if for every $\varepsilon > 0$, $\lim_\nu \sum_{k:|x_k-\alpha|\geqslant\varepsilon} a_{\nu k} = 0$.

Let (s_ν) be a sequence of non-negative real numbers such that $s_1 > 0$. Further, let $R_\nu = \sum_{i=1}^{\nu} s_i$ and (x_ν) be a sequence of real or complex numbers. A matrix $A = (c_{ij})$ is called a weighted regular matrix if

$$\lim_\nu \frac{1}{R_\nu} \sum_{i=1}^{\nu} \sum_{j=1}^{\infty} s_i a_{ij} x_j = L,$$

whenever

$$\lim_\nu x_\nu = L.$$

Let (a_ν), (b_ν) be the sequences of non-negative integers satisfying the regularity conditions

(i) $a_\nu < b_\nu$, $\nu \in \mathbb{N}$ and

(ii) $\lim_{\nu\to\infty} b_\nu = \infty$.

Further, let $S_\nu = \sum_{m=a_\nu+1}^{b_\nu} s_m$.

A matrix $A = (a_{\nu k})$ is called deferred weighted regular matrix if

$$\lim_\nu \frac{1}{S_\nu} \sum_{m=a_\nu+1}^{b_\nu} \sum_{k=1}^{\infty} s_m a_{mk} x_k = L$$

whenever

$$\lim_\nu x_\nu = L.$$

If $A = (a_{\nu k})$ be a non-negative deferred weighted regular matrix then the sequence $x = (x_\nu)$ of real or complex numbers is said to be deferred weighted A-statistically convergent to a number α, if for every $\varepsilon > 0$,

$$\lim_\nu \frac{1}{S_\nu} \sum_{m=a_\nu+1}^{b_\nu} \sum_{k\in E_\varepsilon} s_m a_{mk} = 0,$$

where $E_\varepsilon = \{k \in \mathbb{N} : |x_\nu - \alpha| \geqslant \varepsilon\}$. Let $A = (a_{\nu k})$ be a non negative deferred weighted regular matrix and (d_ν) be a positive non increasing sequence. Then the sequence (x_ν) is said to converge deferred weighted A-statistically to the number α with the rate $o(d_\nu)$ provided for every $\varepsilon > 0$,

$$\lim_\nu \frac{1}{d_\nu} \left\{ \frac{1}{S_\nu} \sum_{m=a_\nu+1}^{b_\nu} \sum_{k\in E_\varepsilon} s_m a_{mk} \right\} = 0.$$

We denote it as $x_\nu - \alpha = \tilde{st}_A^{D(W)} - o(d_\nu)$. If $a_\nu = 0$, $b_\nu = \nu$, for all $\nu \in \mathbb{N}$, then deferred weighted A-statistical convergence coincides with weighted A-statistical convergence

[24]. Furthermore, if $a_\nu = 0$, $b_\nu = \nu$, $A = C_1$ and $s_\nu = 1$, then deferred weighted A-statistical convergence reduces to statistical convergence [12].

Throughout this section, let us assume that $A = (a_{\nu k})$ is a non-negative deferred weighted regular matrix and (a_ν), (b_ν) are the sequences of non-negative integers. First, we prove the Korovkin type theorem for the operators (1.3) using deferred weighted A-statistical convergence.

THEOREM 1. *For $g \in C(T)$, we have*

$$\tilde{st}_A^{D(W)} - \lim_\nu \| \mathscr{U}_{\nu,r,s}(g) - g \| = 0.$$

Proof. Following ([13], Theorem 1), it is adequate to show that

$$\tilde{st}_A^{D(W)} - \lim_\nu \| \mathscr{U}_{\nu,r,s}(e_{ij}) - e_{ij} \| = 0,$$

where $e_{ij}(\mathbf{t}) = t_1^i t_2^j$, $0 \leqslant i+j \leqslant 1$ and $\tilde{st}_A^{D(W)} - \lim_\nu \| \mathscr{U}_{\nu,r,s}(e_{20}+e_{02}) - (e_{20}+e_{02}) \| = 0$. Enforcing Lemma 1, we have

$$\| \mathscr{U}_{\nu,r,s}(e_{00}) - e_{00} \| = 0.$$

Hence,

$$\tilde{st}_A^{D(W)} - \lim_\nu \| \mathscr{U}_{\nu,r,s}(e_{00}) - e_{00} \| = 0.$$

Again applying Lemma 1,

$$\begin{aligned} \| \mathscr{U}_{\nu,r,s}(e_{10}) - e_{10} \| &= \sup_{\mathbf{x} \in T} \left| x_1 + \frac{1}{2\nu} - x_1 \right| \\ &= \frac{1}{2\nu}. \end{aligned}$$

For every $\varepsilon > 0$, let us define the sets:

$$\Pi = \{\nu \in \mathbb{N} : \| \mathscr{U}_{\nu,r,s}(e_{10}) - e_{10} \| \geqslant \varepsilon\}$$

and

$$\Pi_1 = \{\nu \in \mathbb{N} : \frac{1}{2\nu} \geqslant \varepsilon\}.$$

Then, $\Pi \subseteq \Pi_1$, which implies that

$$\frac{1}{S_\nu} \sum_{m=a_\nu+1}^{b_\nu} s_m \sum_{k \in \Pi} a_{mk} \leqslant \frac{1}{S_\nu} \sum_{m=a_\nu+1}^{b_\nu} s_m \sum_{k \in \Pi_1} a_{mk}.$$

Since

$$\tilde{st}_A^{D(W)} - \lim_\nu \frac{1}{2\nu} = 0,$$

we have

$$\tilde{st}_A^{D(W)} - \lim_{\nu} \| \mathscr{U}_{\nu,r,s}(e_{10}) - e_{10} \| = 0.$$

Similarly,

$$\tilde{st}_A^{D(W)} - \lim_{\nu} \| \mathscr{U}_{\nu,r,s}(e_{01}) - e_{01} \| = 0.$$

Now, let us consider

$$\begin{aligned} \| \mathscr{U}_{\nu,r,s}(e_{20}+e_{02}) - (e_{20}+e_{02}) \| &= \sup_{\mathbf{x}\in T} \left| \frac{x_1(1-x_1)}{\nu}\left(1+\frac{sr(r-1)}{\nu}\right) + \frac{x_1}{\nu} + \frac{1}{3\nu^2} \right| \\ &\quad + \left| \frac{x_2(1-x_2)}{\nu}\left(1+\frac{sr(r-1)}{\nu}\right) + \frac{x_2}{\nu} + \frac{1}{3\nu^2} \right| \\ &\leqslant \frac{5}{2\nu} + \frac{3sr|r-1|+4}{6\nu^2}. \end{aligned}$$

We define the following sets:

$$\begin{aligned} \Pi_2 &= \{\nu \in \mathbb{N} : \| \mathscr{U}_{\nu,r,s}(e_{20}+e_{02}) - (e_{20}+e_{02}) \| \geqslant \varepsilon\} \\ \Pi_3 &= \left\{\nu \in \mathbb{N} : \frac{5}{2\nu} \geqslant \frac{\varepsilon}{2}\right\} \\ \Pi_4 &= \left\{\nu \in \mathbb{N} : \frac{3sr(r-1)+4}{6\nu^2} \geqslant \frac{\varepsilon}{2}\right\}. \end{aligned}$$

Then, we can write $\Pi_2 \subseteq \Pi_3 \cup \Pi_4$, which leads us to

$$\frac{1}{S_\nu}\sum_{m=a_\nu+1}^{b_\nu} s_m \sum_{k\in\Pi_2} a_{mk} \leqslant \frac{1}{S_\nu}\sum_{m=a_\nu+1}^{b_\nu} s_m \sum_{k\in\Pi_3} a_{mk} + \frac{1}{S_\nu}\sum_{m=a_\nu+1}^{b_\nu} s_m \sum_{k\in\Pi_4} a_{mk}.$$

Now, since

$$\tilde{st}_A^{D(W)} - \lim_{\nu} \frac{5}{2\nu} = 0,$$

and

$$\tilde{st}_A^{D(W)} - \lim_{\nu} \frac{3sr(r-1)+4}{6\nu^2} = 0,$$

it follows that

$$\tilde{st}_A^{D(W)} - \lim_{\nu} \| \mathscr{U}_{\nu,r,s}(e_{20}+e_{02}) - (e_{20}+e_{02}) \| = 0.$$

This completes the proof of the theorem. □

Next, we establish the Voronovskaja type theorem for the operators $\mathscr{U}_{\nu,r,s}$ in deferred weighted A-statistical approximation.

THEOREM 2. *For every $g'' \in C(T)$, we have*

$$\begin{aligned} \tilde{st}_A^{D(W)} - \lim_{\nu}(\mathscr{U}_{\nu,r,s}(g;\mathbf{x}) - g(\mathbf{x})) &= \frac{1}{2}g_{x_1}(\mathbf{x}) + \frac{1}{2}g_{x_2}(\mathbf{x}) \\ &\quad + \frac{1}{2}\{x_1(1-x_1)g_{x_1x_1}(\mathbf{x}) + x_2(1-x_2)g_{x_2x_2}(\mathbf{x})\}, \end{aligned}$$

uniformly in $\mathbf{x} \in T$.

Proof. For $g'' \in C(T)$, we may write

$$g(\mathbf{t}) = g(\mathbf{x}) + g_{x_1}(\mathbf{x})p + g_{x_2}(\mathbf{x})q + \frac{1}{2}\{g_{x_1x_1}(\mathbf{x})p^2 + 2g_{x_1x_2}(\mathbf{x})pq + g_{x_2x_2}(\mathbf{x})q^2\} + \theta(\mathbf{t};\mathbf{x})\sqrt{(p^4+q^4)}, \tag{3.1}$$

where $\theta(\mathbf{t};\mathbf{x}) \in C(T)$ and $\theta(\mathbf{t},\mathbf{x}) \to 0$, as $\mathbf{t} \to \mathbf{x}$. Applying $\mathscr{U}_{\nu,r,s}(;\mathbf{x})$ on (3.1), we get

$$\begin{aligned}\mathscr{U}_{\nu,r,s}(g;\mathbf{x}) = {} & g(\mathbf{x}) + g_{x_1}(\mathbf{x})\mathscr{U}_{\nu,r,s}(p;\mathbf{x}) + g_{x_2}(\mathbf{x})\mathscr{U}_{\nu,r,s}(q;\mathbf{x}) \\ & + \frac{1}{2}\{g_{x_1x_1}(\mathbf{x})\mathscr{U}_{\nu,r,s}(p^2;\mathbf{x}) \\ & + 2g_{x_1x_2}(\mathbf{x})\mathscr{U}_{\nu,r,s}(pq;\mathbf{x}) + g_{x_2x_2}(\mathbf{x})\mathscr{U}_{\nu,r,s}(q^2;\mathbf{x})\} \\ & + \mathscr{U}_{\nu,r,s}(\theta(\mathbf{t};\mathbf{x})\sqrt{(p^4+q^4)};\mathbf{x}).\end{aligned}$$

In view of Lemma 2, we get

$$\begin{aligned}\tilde{st}_A^{D(W)} - \lim_{\nu} \nu(\mathscr{U}_{\nu,r,s}(g;\mathbf{x}) - g(\mathbf{x})) = {} & \frac{1}{2}g_{x_1}(\mathbf{x}) + \frac{1}{2}g_{x_2}(\mathbf{x}) \\ & + \frac{1}{2}\{x_1(1-x_1)g_{x_1x_1}(\mathbf{x}) + x_2(1-x_2)g_{x_2x_2}(\mathbf{x})\} \\ & + \tilde{st}_A^{D(W)} - \lim_{\nu} \nu\mathscr{U}_{\nu,r,s}(\theta(\mathbf{t};\mathbf{x})\sqrt{(p^4+q^4)};\mathbf{x}).\end{aligned}$$

Using Cauchy-Schwarz inequality

$$|\mathscr{U}_{\nu,r,s}(\theta(\mathbf{t};\mathbf{x})\sqrt{p^4+q^4};\mathbf{x})| \leqslant \left(\mathscr{U}_{\nu,r,s}(\theta^2(\mathbf{t};\mathbf{x});\mathbf{x})\right)^{\frac{1}{2}} \{\sqrt{\mathscr{U}_{\nu,r,s}(p^4;\mathbf{x}) + \mathscr{U}_{\nu,r,s}(q^4;\mathbf{x})}\}. \tag{3.2}$$

Applying Theorem 1, we have

$$\tilde{st}_A^{D(W)} - \lim_{\nu} \mathscr{U}_{\nu,r,s}(\theta^2(\mathbf{t};\mathbf{x}) = \theta^2(\mathbf{x}) = 0,$$

uniformly in $\mathbf{x} \in T$, as $\theta^2(\mathbf{t};\mathbf{x}) \in C(T)$. Further by Lemma 2, $\tilde{st}_A^{D(W)} - \lim_{\nu} \nu^2\mathscr{U}_{\nu,r,s}(p^4;\mathbf{x}) = 3x_1^2(x_1^2 - 20x_1 + 1)$, $\tilde{st}_A^{D(W)} - \lim_{\nu} \nu^2\mathscr{U}_{\nu,r,s}(q^4;\mathbf{x}) = 3x_2^2(x_2^2 - 20x_2 + 1)$, uniformly in $\mathbf{x} \in T$, hence from (3.2) we obtain

$$\tilde{st}_A^{D(W)} - \lim_{\nu} \nu\mathscr{U}_{\nu,r,s}(\theta(\mathbf{t};\mathbf{x})\sqrt{p^4+q^4};\mathbf{x}) = 0,$$

uniformly in $\mathbf{x} \in T$. □

The following theorem yields us the rate of the deferred weighted A-statistical convergence of the operators $\mathscr{U}_{\nu,r,s}(g)$ for $g \in C(T)$.

THEOREM 3. *If,* $\omega(g;\sqrt{2\upsilon_{\nu,r,s}}) = \tilde{st}_A^{D(W)} - o(d_\nu)$, *as* $\nu \to \infty$, *where* $g \in C(T)$, *then we have*

$$\|\mathscr{U}_{\nu,r,s}(g) - g\| = \tilde{st}_A^{D(W)} - o(d_\nu), \text{ as } \nu \to \infty,$$

where $\upsilon_{\nu,r,s}$ *is as defined in Remark 1.*

Proof. Since $g \in C(T)$, for any $\upsilon > 0$,

$$\begin{aligned}|\mathscr{U}_{\nu,r,s}(g(\mathbf{t});\mathbf{x}) - g(\mathbf{x})| &\leqslant \mathscr{U}_{\nu,r,s}\left(|g(\mathbf{t}) - g(\mathbf{x});\mathbf{x})|\right)\\ &\leqslant \mathscr{U}_{\nu,r,s}\left(\left(1+\frac{p^2+q^2}{\upsilon^2}\right)\omega(g;\upsilon);\mathbf{x}\right), \upsilon > 0\\ &= \left[1+\frac{1}{\upsilon^2}\{\mathscr{U}_{\nu,r,s}\left(p^2;\mathbf{x}\right)+\mathscr{U}_{\nu,r,s}\left(q^2;\mathbf{x}\right)\}\right]\omega(g;\upsilon).\end{aligned}$$

Hence,

$$\|\mathscr{U}_{\nu,r,s}(g) - g\| \leqslant 2\omega(g;\sqrt{2\upsilon_{\nu,r,s}}), \tag{3.3}$$

where $\upsilon = \sqrt{2\upsilon_{\nu,r,s}}$.

For $\varepsilon > 0$, let us consider the sets:

$$\begin{aligned}L_1 &= \{\nu : \|\mathscr{U}_{\nu,r,s}(g) - g\| \geqslant \varepsilon\} \text{ and}\\ L_2 &= \{\nu : 2\omega(g;\sqrt{2\upsilon_{\nu,r,s}} \geqslant \varepsilon\}.\end{aligned}$$

Then, from (3.3), we have $L_1 \subseteq L_2$.

Thus,

$$\frac{1}{d_\nu}\left\{\frac{1}{S_\nu}\sum_{m=a_\nu+1}^{b_\nu} s_m \sum_{k\in L_1} a_{mk}\right\} \leqslant \frac{1}{d_\nu}\left\{\frac{1}{S_\nu}\sum_{m=a_\nu+1}^{b_\nu} s_m \sum_{k\in L_2} a_{mk}\right\}.$$

Hence taking limit as $\nu \to \infty$ and using $\omega(g;\sqrt{2\upsilon_{\nu,r,s}}) = \tilde{st}_A^{D(W)} - o(d_\nu)$, we reach the desired result. □

4. τ^{th}-order generalization

The studies of Voronovskaja [30] and Korovkin [21] showed that the order of approximation by linear positive operators is, at best, $O\left(\frac{1}{n^2}\right)$, however smooth the function may be. In order to deal with this problem of operators not responding to the smoothness of the function f, Kirov and Popova [20], introduced τ^{th} order generalization of the operators with the help of Taylor's polynomial of f. Taşdelen et al. [29] extended this study to the case of certain linear positive operators defined for functions of two variables. Subsequently, this idea has been applied by researchers to several sequences of operators (cf. [15], [10], [2], [11], [6] and [26] etc.). Motivated by these studies, we define the τ^{th}-generalization of $\mathscr{U}_{\nu,r,s}$ to approximate smooth functions as:

$$\begin{aligned}\mathscr{U}^{\tau}_{\nu,r,s}(g;\mathbf{x}) = &\sum_{\mu_1=0}^{\nu-sr}\sum_{\mu_2=0}^{\nu-sr-\mu_1} w_{\nu-sr,\mu}(\mathbf{x})\sum_{\nu_1=0}^{s}\sum_{\nu_2=0}^{s-\nu_1} w_{s,\nu}(\mathbf{x})\\ &\times\int_0^1\int_0^1\left\{\sum_{s=0}^{\tau}\frac{d^s g(\mathbf{t})}{s!}\right\}dt_1dt_2,\end{aligned} \tag{4.1}$$

where

$$d^s g(\mathbf{t}) = \sum_{i=0}^{s} \binom{s}{i} \frac{\partial^s g(\mathbf{t})}{\partial x_1^{s-i} \partial x_2^i} (x_1 - t_1)^{s-i} (x_2 - t_2)^i. \tag{4.2}$$

Let (α, β) be a unit vector such that $(x_1 - t_1, x_2 - t_2) = w(\alpha, \beta)$, where $w = |(x_1 - t_1, x_2 - t_2)|$. Then, we may write

$$\begin{aligned} g(\mathbf{x}) &= g(t_1 + (x_1 - t_1), t_2 + (x_2 - t_2)) \\ &= g(t_1 + w\alpha, t_2 + w\beta) = Q(w), say. \end{aligned}$$

Hence, $Q^{(\tau)}(w)$ is given by

$$Q^{(\tau)}(w) = \sum_{i=0}^{\tau} \binom{\tau}{i} \frac{\partial^\tau g(t_1 + w\alpha, t_2 + w\beta)}{\partial x_1^{\tau-i} \partial x_2{}^i} \alpha^{\tau-i} \beta^i. \tag{4.3}$$

Thus from (4.2)and (4.3), it is obvious that

$$\begin{aligned} d^\tau g(\mathbf{t}) &= \sum_{i=0}^{\tau} \binom{\tau}{i} \frac{\partial^\tau g(\mathbf{t})}{\partial x_1^{\tau-i} \partial x_2^i} (x_1 - t_1)^{\tau-i} (x_2 - t_2)^i \\ &= w^\tau Q^{(\tau)}(0). \end{aligned} \tag{4.4}$$

Our following result provides the approximation degree for functions in $C^\tau(T)$ by the operators (4.1).

THEOREM 4. *For all $g \in C^\tau(T)$ such that $Q^{(\tau)}(w) \in Lip_M(\xi)$, $\xi \in (0,1]$, we have*

$$\|\mathscr{U}_{\nu,r,s}^{\tau}(g) - g\| \leqslant \frac{M}{(\tau-1)!} \frac{\xi B(\xi, \tau)}{\xi + \tau} \|\mathscr{U}_{n,r,s}\left(|\mathbf{x} - \mathbf{t}|^{\xi+\tau}\right)\|.$$

Proof. Let $g \in C^\tau(T)$ and $\mathbf{x} \in T$. By the definition (4.1) of $\mathscr{U}_{n,r,s}^{\tau}(g;.)$, and $\forall$ $\tau \in \mathbb{N}$

$$\begin{aligned} g(\mathbf{x}) - \mathscr{U}_{\nu,r,s}^{\tau}(g(\mathbf{t});\mathbf{x}) = & \sum_{\mu_1=0}^{\nu-sr} \sum_{\mu_2=0}^{\nu-sr-\mu_1} w_{\nu-sr,\mu}(\mathbf{x}) \sum_{\nu_1=0}^{s} \sum_{\nu_2=0}^{s-\nu_1} w_{s,\nu}(\mathbf{x}) \\ & \times \int_0^1 \int_0^1 \left\{ g(\mathbf{x}) - \sum_{s=0}^{\tau} \frac{d^s g(\mathbf{t})}{s!} \right\} dt_1 dt_2. \end{aligned} \tag{4.5}$$

From Taylor's formula, we can write

$$\begin{aligned} & g(\mathbf{x}) - \sum_{s=0}^{\tau-1} \frac{d^s g(\mathbf{t})}{s!} = \frac{1}{(\tau-1)!} \\ & \times \int_0^1 (1-z)^{\tau-1} \left(\sum_{i=0}^{\tau} \binom{\tau}{i} \frac{\partial^\tau g(t_1 + z(x_1 - t_1), t_2 + z(x_2 - t_2))}{\partial x_1^{\tau-i} \partial x_2^i} (x_1 - t_1)^{\tau-i} (x_2 - t_2)^i \right) dz. \end{aligned}$$

Hence, in view of (4.3), we have

$$g(\mathbf{x})-\sum_{s=0}^{\tau-1}\frac{d^s g(\mathbf{t})}{s!}$$
$$=\frac{w^\tau}{(\tau-1)!}\int_0^1(1-z)^{\tau-1}\left(\sum_{i=0}^{\tau}\binom{\tau}{i}\frac{\partial^\tau g(t_1+zw\alpha,t_2+zw\beta)}{\partial x_1^{\tau-1}\partial x_2^i}\alpha^{\tau-i}\beta^i\right)dz$$
$$=\frac{w^\tau}{(\tau-1)!}\int_0^1(1-z)^{\tau-1}Q^{(\tau)}(zw)dz.$$

From (4.4), it is obvious that

$$g(\mathbf{x})-\sum_{s=0}^{\tau}\frac{d^s g(\mathbf{t})}{s!}=\frac{w^\tau}{(\tau-1)!}\int_0^1(1-z)^{\tau-1}\{Q^{(\tau)}(zw)-Q^{(\tau)}(0)\}dz.$$

Now, since $Q^{(\tau)}(w)\in Lip_M(\xi)$, from the definition of Beta function, it follows that

$$\begin{aligned}\left|g(\mathbf{x})-\sum_{s=0}^{\tau}\frac{d^s g(\mathbf{t})}{s!}\right| &\leqslant \frac{|w|^\tau}{(\tau-1)!}\int_0^1(1-z)^{\tau-1}|Q^{(\tau)}(zw)-Q^{(\tau)}(0)|dz\\ &\leqslant \frac{M|w|^{\tau+\xi}}{(\tau-1)!}\int_0^1 z^\xi(1-z)^{\tau-1}dz\\ &=\frac{M}{(\tau-1)!}\frac{\xi B(\xi,\tau)}{\xi+\tau}|x_1-t_1,x_2-t_2|^{\tau+\xi},\end{aligned} \tag{4.6}$$

Finally using (4.6) in (4.5), we obtain the desired assertion. □

REMARK 2. Let $f(u,v)=|(x_1-u,x_2-v)|^{\tau+\xi}$ then $f(\mathbf{x})=0$ Clearly, $f\in C(T)$, hence applying Theorem 1

$$\lim_{\nu}\|\mathscr{U}_{\nu,r,s}(f)\|=0,$$

and consequently, from Theorem 4

$$\lim_{\nu}\|\mathscr{U}^{\tau}_{\nu,r,s}(g)-g\|=0.$$

REMARK 3. It is well known that for any $h\in C(T)$ and $\delta>0$, there holds the inequality

$$|h(\mathbf{t})-h(\mathbf{x})|\leqslant\{1+\frac{(t_1-x_1)^2+(t_2-x_2)^2}{\delta^2}\}\omega(h;\delta),\ \forall\,\mathbf{t},\mathbf{x}\in T,$$

hence

$$\begin{aligned}\|\mathscr{U}_{\nu,r,s}(f)\| &\leqslant \omega(f;\delta)\{1+\frac{1}{\delta^2}\|\mathscr{U}_{\nu,r,s}\left((x_1-u)^2+(x_2-v)^2\right)\|\}\\ &\leqslant 2\omega(f;\delta),\end{aligned}$$

where $f(u,v)=|(x_1-u,x_2-v)|^{\xi+\tau}$ and $\delta^2=\|\mathscr{U}_{\nu,r,s}\left((x_1-u)^2+(x_2-v)^2\right)\|$.

Consequently,

$$\|\mathscr{U}_{\nu,r,s}(g)-g\|\leqslant\frac{2M}{(\tau-1)!}\frac{\xi}{\xi+\tau}B(\xi,\tau)\omega(g;\sqrt{\|\mathscr{U}_{\nu,r,s}((x_1-u)^2+(x_2-v)^2)\|}).$$

5. GBS operator related to $\mathscr{U}_{v,r,s}$

The idea of Bögel-continuous and Bögel-differentiable functions was initiated by Bögel in [7] and [8]. Badea et al. [5] gave the proof of the very famous "Test function theorem" for Bögel continuous functions. A quantitative Korovkin-type theorem for these functions was established by Badea et al. in [4].

GBS operators are used in uniform approximation of B-continuous (Bögel continuous) functions. For a detailed account of the research work in this direction, we refer the reader to [22], [14] and the references therein.

A real-valued function g defined on T, is called B-continuous at $\mathbf{x} \in T$ if

$$\lim_{\mathbf{t}\to\mathbf{x}} \Upsilon_{(\mathbf{t})} g(\mathbf{x}) = 0,$$

where $\Upsilon_{(\mathbf{t})} g(\mathbf{x}) = g(\mathbf{t}) - g(x_1,t_2) - g(t_1,x_2) + g(\mathbf{x})$. Let $C_b(T) = \{g : g$ is B-continuous on $T\}$. The function $g : T \to \mathbb{R}$ is B-bounded on T if for all $\mathbf{t},\mathbf{x} \in T$

$$|\Upsilon_{(\mathbf{t})} g(\mathbf{x})| \leqslant M,$$

where M is some positive constant. Let $B_b(T)$ denote the set of all B-bounded functions on T, with the norm $\|g\|_B = \sup_{\mathbf{t},\mathbf{x}\in T} |\Upsilon_{(\mathbf{t})} g(\mathbf{x})|$. Let $B(T)$ be the space of all bounded (in the usual sense) functions on T endowed with the sup-norm $\|.\|_\infty$ and $C(T) = \{f \in B(T) : f$ is continuous$\}$. It is obvious that $C(T) \subset C_b(T)$.

The function $g : T \to \mathbb{R}$ is uniformly B-continuous on T if and only if for any $\varepsilon > 0, \exists\, \upsilon = \upsilon(\varepsilon) > 0$ such that

$$|\Upsilon_{(\mathbf{t})} g(\mathbf{x})| < \varepsilon,$$

whenever $\max\{|p|,|q|\} < \upsilon$, for all $\mathbf{t},\mathbf{x} \in T$. Clearly, every $g \in C_b(T)$ is uniformly B-continuous on T. A function $g : T \to \mathbb{R}$ is called Bögel differentiable at $\mathbf{x} \in T$, if

$$\lim_{\mathbf{t}\to\mathbf{x}} \frac{\Upsilon_{(\mathbf{t})} g(\mathbf{x})}{pq} = D_B g(\mathbf{x}) < \infty.$$

Here, $D_B g$ is called the B-derivative of g and the space of all B-differentiable functions is denoted by $D_b(T)$.

The mixed modulus of smoothness of $g \in C_b(T)$ is defined as

$$\omega_{mixed}(g;\upsilon_1,\upsilon_2) = \sup_{\mathbf{x},\mathbf{x}+\mathbf{h}\in T} \ \sup_{0<|h_1|\leqslant\upsilon_1, 0<|h_2|\leqslant\upsilon_2} \{|\Upsilon_{(x_1+h_1,x_2+h_2)} g(\mathbf{x})|\},$$

for any $\mathbf{h} = (h_1,h_2) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+,$. It is known [3] that $\omega_{mixed}(g;\upsilon_1,\upsilon_2)$ is an increasing function of υ_1 and υ_2, and for all positive numbers λ_1,λ_2, there holds

$$\begin{aligned} \omega_{mixed}(g;\lambda_1\upsilon_1,\lambda_2\upsilon_2) &\leqslant (1+]\lambda_1[)\,(1+]\lambda_2[)\,\omega_{mixed}(g;\upsilon_1,\upsilon_2) \\ &\leqslant (1+\lambda_1)(1+\lambda_2)\omega_{mixed}(g;\upsilon_1,\upsilon_2), \end{aligned} \tag{5.1}$$

where $]\lambda[$ denotes the integral part of λ.

For every $g \in C_b(T)$ and each $\mathbf{x} \in T$, the GBS operator $G^*_{\nu,r,s}$ associated to $\mathscr{U}_{\nu,r,s}$ is defined as

$$\begin{aligned} G^*_{\nu,r,s}(g;\mathbf{x}) &= G^*_{\nu,r,s}(g(\mathbf{t});\mathbf{x}) = \mathscr{U}_{\nu,r,s}(g(t_1,x_2)+g(x_1,t_2)-g(\mathbf{t});\mathbf{x}) \\ &= \sum_{\mu_1=0}^{\nu-sr}\sum_{\mu_2=0}^{\nu-sr-\mu_1} w_{\nu-sr,\mu}(\mathbf{x}) \sum_{\nu_1=0}^{s}\sum_{\nu_2=0}^{s-\nu_1} w_{s,\nu}(\mathbf{x}) \int_0^1\int_0^1 \left[g\left(\frac{\mu_1+r\nu_1+t_1}{\nu},x_2\right) \right. \\ &\quad \left. + g\left(x_1,\frac{\mu_2+r\nu_2+t_2}{\nu}\right) - g\left(\frac{\mu_1+r\nu_1+t_1}{\nu},\frac{\mu_2+r\nu_2+t_2}{\nu}\right)\right] dt_1 dt_2. \end{aligned} \tag{5.2}$$

Clearly, $G^*_{\nu,r,s}$ is a linear operator from $C_b(T)$ into $C(T)$ and $G^*_{\nu,r,s}(1;\mathbf{x}) = 1$, $\forall\, \mathbf{x} \in T$.

Our next theorem deals with the order of approximation by $G^*_{\nu,r,s}$ for functions in $C_b(T)$ in terms of ω_{mixed}.

THEOREM 5. *For every $g \in C_b(T)$, there holds the inequality*

$$\|G^*_{\nu,r,s}(g) - g(\mathbf{x})\| \leqslant 4\omega_{mixed}(g;\frac{1}{\sqrt{\nu}},\frac{1}{\sqrt{\nu}}).$$

Proof. Using the definition of $\omega_{mixed}(g;\upsilon_1,\upsilon_2)$ and (5.1), we have

$$\begin{aligned} |\Upsilon_{(\mathbf{t})} g(\mathbf{x})| &\leqslant \omega_{mixed}(g;|p|,|q|) \\ &\leqslant \left(1+\frac{|p|}{\upsilon_1}\right)\left(1+\frac{|q|}{\upsilon_2}\right)\omega_{mixed}(g;\upsilon_1,\upsilon_2), \\ &\leqslant \left(1+\frac{|p|}{\upsilon_1}+\frac{|q|}{\upsilon_2}+\frac{1}{\upsilon_1\upsilon_2}(|p||q|)\right)\omega_{mixed}(g;\upsilon_1,\upsilon_2), \end{aligned}$$

for every $\mathbf{x},\mathbf{t} \in T$ and for any $\upsilon_1,\upsilon_2 > 0$. Further, by the definition of $\Upsilon_{(\mathbf{t})} g(\mathbf{x})$, we get

$$g(t_1,x_2)+g(x_1,t_2)-g(\mathbf{t}) = g(\mathbf{x}) - \Upsilon_{(\mathbf{t})} g(\mathbf{x}). \tag{5.3}$$

Applying $\mathscr{U}_{\nu,r,s}(;\mathbf{x})$ on (5.3)

$$G^*_{\nu,r,s}(g;\mathbf{x}) = g(\mathbf{x})\mathscr{U}_{\nu,r,s}(1;\mathbf{x}) - \mathscr{U}_{\nu,r,s}(\Upsilon_{(\mathbf{t})} g(\mathbf{x});\mathbf{x}).$$

Now considering Lemma 1 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |G^*_{\nu,r,s}(g;\mathbf{x}) - g(\mathbf{x})| &\leqslant \mathscr{U}_{\nu,r,s}(|\Upsilon_{(\mathbf{t})} g(\mathbf{x})|;\mathbf{x}) \\ &\leqslant \left(\mathscr{U}_{\nu,r,s}(1;\mathbf{x}) + \frac{1}{\upsilon_1}\sqrt{\mathscr{U}_{\nu,r,s}(p^2;\mathbf{x})} + \frac{1}{\upsilon_2}\sqrt{\mathscr{U}_{\nu,r,s}(q^2;\mathbf{x})} \right. \\ &\quad \left. + \frac{1}{\upsilon_1\upsilon_2}\sqrt{\mathscr{U}_{\nu,r,s}(p^2;\mathbf{x})\mathscr{U}_{\nu,r,s}(q^2;\mathbf{x})}\right)\omega_{mixed}(g;\upsilon_1,\upsilon_2). \end{aligned}$$

Again applying Lemma 1, Remark 1 and choosing $\upsilon_1 = \upsilon_2 = \frac{1}{\sqrt{\nu}}$, we get the required result. □

In the following theorem, we present an estimate of error in the approximation of a Lipschitz B-continuous function by $G^*_{\nu,r,s}$.

For $g \in C_b(T)$, the Lipschitz class $Lip_{M,b}(\xi)$, $0 < \xi \leqslant 1$, of Bögel continuous functions is defined by

$$Lip_{M,b}(\xi) = \left\{ g \in C_b(T) : \left|\Upsilon_{(\mathbf{t})} g(\mathbf{x})\right| \leqslant M\{p^2+q^2\}^{\frac{\xi}{2}}, \text{ for } \mathbf{t}, \mathbf{x} \in T \right\}.$$

THEOREM 6. *For $g \in Lip_{M,b}(\xi)$, we have*

$$\|G^*_{\nu,r,s}(g) - g(\mathbf{x})\| \leqslant 2^{\frac{\xi}{2}} M \upsilon_{\nu,r,s}^{\frac{\xi}{2}}$$

where M is certain positive constant and $\upsilon_{\nu,r,s}$ is as defined in Remark 1.

Proof. By the definition of $G^*_{\nu,r,s}$ and our hypothesis, we get

$$\begin{aligned} \left|G^*_{\nu,r,s}(g;\mathbf{x}) - g(\mathbf{x})\right| &\leqslant \mathscr{U}_{\nu,r,s}(|\Upsilon_{(\mathbf{t})} g(\mathbf{x})|;\mathbf{x}) \\ &\leqslant M \mathscr{U}_{\nu,r,s}\left(\{p^2+q^2\}^{\frac{\xi}{2}};\mathbf{x}\right). \end{aligned}$$

Now, an application of the Hölder's inequality and Lemma 1, easily leads us to the desired assertion. □

Next, we discuss the degree of approximation by the operators $G^*_{\nu,r,s}$ for functions whose Bögel derivative is bounded.

THEOREM 7. *If $g \in D_b(T)$ and $D_B g \in C_b(T) \cap B(T)$, then there holds the inequality*

$$\|G^*_{\nu,r,s}(g) - g\| \leqslant \frac{M}{\nu}\left[3\|D_B g\|_\infty + \omega_{mixed}(D_B g; \nu^{-1/2}, \nu^{-1/2})\right],$$

where M is a constant.

Proof. Considering the mean value theorem, we have

$$\Upsilon_{(\mathbf{t})} g(\mathbf{x}) = (p)(q) D_B g(\alpha,\beta), \text{ where } x_1 < \alpha < t_1;\ x_2 < \beta < t_2. \tag{5.4}$$

By the definition of $\Upsilon_{(\mathbf{t})} g(\mathbf{x})$, we have

$$\begin{aligned} &\Upsilon_{(\mathbf{t})} D_B g(\alpha,\beta) = D_B g(\alpha,\beta) - D_B g(\alpha,t_2) - D_B g(t_1,\beta) + D_B g(\mathbf{t}) \\ &\Rightarrow\ D_B g(\alpha,\beta) = \Upsilon_{(\mathbf{t})} D_B g(\alpha,\beta) + D_B g(\alpha,t_2) + D_B g(t_1,\beta) - D_B g(\mathbf{t}). \end{aligned} \tag{5.5}$$

Since $D_B g \in C_b(T) \cap B(T)$, $|D_B g(\mathbf{x})| \leqslant \|D_B g\|_\infty$ for every $\mathbf{x} \in T$. Hence in view of (5.4) and (5.5), we obtain

$$\begin{aligned}|G^*_{\nu,r,s}(g;\mathbf{x}) - g(\mathbf{x})| &\leqslant \left|\mathscr{U}_{\nu,r,s}(\Upsilon_{(\mathbf{t})} g(\mathbf{x});\mathbf{x})\right| \\ &\leqslant \left|\mathscr{U}_{\nu,r,s}\left(|p||q|(|\Upsilon_{(\mathbf{t})} D_B g(\alpha,\beta)|\right.\right. \\ &\quad \left.\left. + |D_B g(\alpha,t_2)| + |D_B g(t_1,\beta)| + |D_B g(\mathbf{t})|);\mathbf{x}\right)\right| \\ &\leqslant \mathscr{U}_{\nu,r,s}\left(|p||q|\omega_{mixed}(D_B g;|t_1-\alpha|,|t_2-\beta|);\mathbf{x}\right) \\ &\quad +3\,\|D_B f\|_\infty\, \mathscr{U}_{\nu,r,s}(|p||q|;\mathbf{x}). \end{aligned} \tag{5.6}$$

By the properties of ω_{mixed}, for $\upsilon_1, \upsilon_2 > 0$, we can write

$$\begin{aligned}\omega_{mixed}(D_B g;|t_1-\alpha|,|t_2-\beta|) &\leqslant \omega_{mixed}(D_B g;|p|,|q|) \\ &\leqslant \left(1+\frac{|p|}{\upsilon_1}\right)\left(1+\frac{|q|}{\upsilon_2}\right)\omega_{mixed}(D_B g;\upsilon_1,\upsilon_2). \end{aligned} \tag{5.7}$$

Therefore from (5.6), (5.7) and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|G^*_{\nu,r,s}(g;\mathbf{x}) - g(\mathbf{x})| &\leqslant 3\|D_B g\|_\infty \mathscr{U}_{\nu,r,s}(|p||q|;\mathbf{x}) + \Big(\mathscr{U}_{\nu,r,s}(|p||q|;\mathbf{x}) \\ &\quad + \frac{1}{\upsilon_1}\mathscr{U}_{\nu,r,s}(p^2|q|;\mathbf{x}) + \frac{1}{\upsilon_2}\mathscr{U}_{\nu,r,s}(|p|q^2;\mathbf{x}) \\ &\quad + \frac{1}{\upsilon_1\upsilon_2}\mathscr{U}_{\nu,r,s}(p^2q^2;\mathbf{x})\Big)\omega_{mixed}(D_B g;\upsilon_1,\upsilon_2) \\ &\leqslant 3\|D_B g\|_\infty\sqrt{\mathscr{U}_{\nu,r,s}(p^2q^2;\mathbf{x})} + \Big(\sqrt{\mathscr{U}_{\nu,r,s}(p^2q^2;\mathbf{x})} \\ &\quad + \frac{1}{\upsilon_1}\sqrt{\mathscr{U}_{\nu,r,s}(p^4q^2;\mathbf{x})} + \frac{1}{\upsilon_2}\sqrt{\mathscr{U}_{\nu,r,s}(p^2q^4;\mathbf{x})} \\ &\quad + \frac{1}{\upsilon_1\upsilon_2}\mathscr{U}_{\nu,r,s}(p^2q^2;\mathbf{x})\Big)\omega_{mixed}(D_B g;\upsilon_1,\upsilon_2). \end{aligned}$$

From Remark 1,

$$\mathscr{U}_{\nu,r,s}(p^iq^j;\mathbf{x}) \leqslant \frac{M}{\nu^{\frac{[i+j+1]}{2}}},\ \forall \mathbf{x} \in T \quad \text{and} \quad i,j \in \mathbb{N}_0$$

for some constant $M > 0$. Hence by choosing $\upsilon_1 = \dfrac{1}{\sqrt{\nu}}$ and $\upsilon_2 = \dfrac{1}{\sqrt{\nu}}$, we reach the desired result. □

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