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CHARACTERIZATIONS FOR THE FRACTIONAL INTEGRAL OPERATOR AND ITS COMMUTATORS IN GENERALIZED WEIGHTED MORREY SPACES ON CARNOT GROUPS

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Abstract. In this paper, we shall give a characterization for the strong and weak type Spanne type boundedness of the fractional integral operator I_α, $0<\alpha<Q$ on Carnot group $\mathbb{G}$ on generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{G},w)$, respectively, where Q is the homogeneous dimension of $\mathbb{G}$. Also we give a characterization for the Spanne type boundedness of the commutator operator $[b,I_\alpha]$ on generalized weighted Morrey spaces.

As applications of the properties of the fundamental solution of sub-Laplacian $\mathscr{L}$ on $\mathbb{G}$, we prove two Sobolev-Stein embedding theorems on generalized weighted Morrey spaces in the Carnot group setting.

1. Introduction

The classical Morrey spaces were introduced by Morrey [32] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [12, 31, 33] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [13, 14, 40]). Komori and Shirai [28] defined weighted Morrey spaces $L_{p,\kappa}(w)$. Guliyev [18] gave a concept of the generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{R}^n,w)$ which could be viewed as extension of both $M_{p,\varphi}(\mathbb{R}^n)$ and $L_{p,\kappa}(w)$. In [18], the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}(\mathbb{R}^n,w)$ was also studied, see also [7, 19, 20, 21, 22, 24, 34].

The spaces $M_{p,\varphi}(\mathbb{R}^n,w)$ defined by the norm

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n,w)} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-1/p} \|f\|_{L_p(B(x,r),w)},$$

where the function φ is a positive measurable function on $\mathbb{R}^n\times(0,\infty)$ and w is a non-negative measurable function on $\mathbb{R}^n$.

Carnot groups appear in quantum physics and many parts of mathematics, including Fourier analysis, several complex variables, geometry and topology. Analysis on

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the groups is also motivated by their role as the simplest and the most important model in the general theory of vector fields satisfying Hörmander's condition. The simplest examples of the Carnot groups are Euclidean space $\mathbb{R}^n$, Heisenberg group $\mathbb{H}_n$ and (Heisenberg)-type groups introduced by Kaplan [27].

For $x \in \mathbb{G}$ and $r > 0$, let $D(x,r)$ denote the $\mathbb{G}$- ball centered at x of radius r and ${}^{\complement}D(x,r)$ denote its complement.

Let $f \in L_1^{\mathrm{loc}}(\mathbb{G})$. The fractional integral operator I_α is defined by

$$I_\alpha f(x) = \int_{\mathbb{G}} \frac{f(y)dy}{\rho(x^{-1}y)^{Q-\alpha}}, \qquad 0 < \alpha < Q,$$

where $|D(x,t)|$ is the Haar measure of the $\mathbb{G}$-ball $D(x,t)$.

The operator I_α play an important role in real and harmonic analysis and applications (see, for example [1, 26, 37, 42]).

In the present work, we shall give a characterization for the Spanne type boundedness of the operator I_α on generalized weighted Morrey spaces, including weak versions. Also we give a characterization for the Spanne type boundedness of the commutator operator $[b, I_\alpha]$ on generalized weighted Morrey spaces.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Notation and preliminary results

We first recall some preliminaries concerning stratified Lie groups (or so-called Carnot groups). We refer the reader to the books [1, 5, 42] for analysis on stratified groups.

Let $\mathscr{G}$ be a finite-dimensional, stratified, nilpotent Lie algebra. Assume that there is a direct sum vector space decomposition

$$\mathscr{G} = V_1 \oplus \cdots \oplus V_m \tag{2.1}$$

so that each element of V_j, $2 \leqslant j \leqslant m$, is a linear combination of $(j-1)$th order commutator of elements of V_1. Equivalently, (2.1) is a stratification provided $[V_i, V_j] = V_{i+j}$ whenever $i + j \leqslant m$ and $[V_i, V_j] = 0$ otherwise. Let $X = X_1, \ldots, X_n$ be a basis for V_1 and X_{ij}, $1 \leqslant i \leqslant k_j$, for V_j consisting of commutators of length j. We set $X_{i1} = X_i$, $i = 1, \ldots, n$ and $k_1 = n$, and we call X_{i1} a commutator of length 1.

If $\mathbb{G}$ is the simply connected Lie group associated with $\mathscr{G}$, then the exponential mapping is a global diffeomorphism from $\mathscr{G}$ to $\mathbb{G}$. Thus, for each $g \in \mathbb{G}$, there is $x = (x_{ij}) \in \mathbb{R}^N$, $1 \leqslant i \leqslant k_j$, $1 \leqslant j \leqslant m$, $N = \sum\limits_{j=1}^{m} k_j$, such that $g = \exp(\sum x_{ij} X_{ij})$. A homogeneous norm function $|\cdot|$ on $\mathbb{G}$ is defined by $|g| = \left(\sum |x_{ij}|^{2m!/j}\right)^{1/(2m!)}$, and $Q = \sum\limits_{j=1}^{m} jk_j$ is said to be the *homogeneous dimension* of $\mathbb{G}$, since $d(\delta_r x) = r^Q dx$ for

$r > 0$. The dilation δ_r on $\mathbb{G}$ is defined by

$$\delta_r(g) = \exp\left(\sum r^j x_{ij} X_{ij}\right) \quad \text{if} \quad g = \exp\left(\sum x_{ij} X_{ij}\right).$$

The convolution operation on $\mathbb{G}$ is defined by

$$f * h(x) = \int_{\mathbb{G}} f(xy^{-1})h(y)dy = \int_{\mathbb{G}} f(y)h(y^{-1}x)dy,$$

where y^{-1} is the inverse of y and xy^{-1} denotes the group multiplication of x by y^{-1}. It is known that for any left invariant vector field X on $\mathbb{G}$, $X(f*h) = f*(Xh)$.

Since $\mathbb{G}$ is nilpotent, the exponential map is diffeomorphism from $\mathbb{G}$ onto $\mathbb{G}$ which takes Lebesgue measure on $\mathbb{G}$ to a biinvariant Haar measure dx on $\mathbb{G}$. The group identity of $\mathbb{G}$ will be referred to as the origin and denoted by e.

A homogenous norm on $\mathbb{G}$ is a continuous function $x \to \rho(x)$ from $\mathbb{G}$ to $[0,\infty)$, which is C^∞ on $\mathbb{G}\setminus\{0\}$ and satisfies $\rho(x^{-1}) = \rho(x)$, $\rho(\delta_t x) = t\rho(x)$ for all $x \in \mathbb{G}$, $t > 0$; $\rho(e) = 0$ (the group identity). Moreover, there exists a constant $c_0 \geqslant 1$ such that $\rho(xy) \leqslant c_0(\rho(x)+\rho(y))$ for all $x,\ y \in G$.

We call a curve $\gamma : [a,b] \to \mathbb{G}$ a horizontal curve connecting two points $x,y \in \mathbb{G}$ if $\gamma(a) = x$, $\gamma(b) = y$ and $\gamma'(t) \in V_1$ for all t. Then the Carnot-Caratheodory distance between x, y is defined as

$$d_{cc}(x,y) = \inf_{\gamma} \int_a^b \langle \gamma'(t), \gamma'(t)\rangle^{\frac{1}{2}} dt,$$

where the infimum is taken over all horizontal curves γ connecting x and y. It is known that any two points x, y on $\mathbb{G}$ can be joined by a horizontal curve of finite length and then d_{cc} is a left invariant metric on $\mathbb{G}$. We can define the metric ball centered at x and with radius r associated with this metric by

$$B_{cc}(x,r) = \{y \in \mathbb{G} : d_{cc}(x,y) < r\}.$$

We must notice that this metric d_{cc} is equivalent to the pseudo-metric $\rho(x,y) = \rho(x^{-1}y)$ defined by the homogeneous norm $|\cdot|$ in the following sense (see [1]):

$$C^{-1}\rho(x,y) \leqslant d_{cc}(x,y) \leqslant C\rho(x,y).$$

We denote the metric ball associated with ρ as $D(x,r) = \{y \in \mathbb{G} : \rho(x,y) < r\}$. An important feature of both of these distance functions is that these distances and thus the associated metric balls are left invariant, namely,

$$d_{cc}(zx,zy) = d_{cc}(x,y), \quad B_{cc}(x,r) = xB_{cc}(e,r)$$

and

$$\rho(zx,zy) = \rho(x,y), \quad D(x,r) = xD(e,r).$$

From now on, we will always use the metric d_{cc} and drop the subscript from d_{cc}. Similarly, we will use $B(x,r)$ to denote $B_{cc}(x,r)$.

With this norm, we define the $\mathbb{G}$ - ball centered at x with radius r by $D(x,r) = \{y \in \mathbb{G} : \rho(y^{-1}x) < r\}$, and we denote by $D_r = D(e,r) = \{y \in \mathbb{G} \,:\, \rho(y) < r\}$ the open ball centered at e, the identity element of $\mathbb{G}$, with radius r. By ${}^{\complement}D(x,r) = \mathbb{G} \setminus D(x,r)$ we denote the complement of $D(x,r)$.

One easily recognizes that there exist $c_1 = c_1(\mathbb{G})$, and $c_2 = c_2(\mathbb{G})$ such that

$$|B(x,r)| = c_1\, r^Q, \;\; |D(x,r)| = c_2\, r^Q, \;\; x \in \mathbb{G}, \; r > 0.$$

The most basic partial differential operator in a Carnot group is the sub- Laplacian associated with X is the second-order partial differential operator on G given by $\mathscr{L} = \sum_{i=1}^{n} X_i^2$.

3. Generalized weighted Morrey spaces

By a weight function, briefly weight, we mean a locally integrable function on $\mathbb{G}$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$, and denote the Lebesgue measure of E by $|E|$ and the characteristic function of E by χ_E.

If w is a weight function, we denote by $L_{p,w}(\mathbb{G})$ the weighted Lebesgue space defined by finiteness of the norm

$$\|f\|_{L_{p,w}(\mathbb{G})} = \Big(\int_{\mathbb{G}} |f(x)|^p w(x) dx \Big)^{\frac{1}{p}} < \infty, \quad if \quad 1 \leqslant p < \infty$$

and

$$\|f\|_{L_{\infty,w}(\mathbb{G})} = \operatorname*{ess\,sup}_{x \in \mathbb{G}} |f(x)| w(x), \quad if \quad p = \infty.$$

We define the generalized weighed Morrey spaces as follows.

DEFINITION 1. Let $1 \leqslant p < \infty$, φ be a positive measurable function on $\mathbb{G} \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{G}$. We denote by $M_{p,\varphi}(\mathbb{G},w) \equiv M_{p,\varphi}(w)$ the generalized weighted Morrey space, the space of all functions $f \in L^{loc}_{p,w}(\mathbb{G})$ with finite norm

$$\|f\|_{M_{p,\varphi}(w)} = \sup_{x \in \mathbb{G}, r > 0} \varphi(x,r)^{-1} w(D(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w}(D(x,r))},$$

where $L_{p,w}(D(x,r))$ denotes the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,w}(D(x,r))} \equiv \|f \chi_{D(x,r)}\|_{L_{p,w}(\mathbb{G})} = \Big(\int_{D(x,r)} |f(y)|^p w(y) dy \Big)^{\frac{1}{p}}.$$

Furthermore, by $WM_{p,\varphi}(w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL^{loc}_{p,w}(\mathbb{G})$ for which

$$\|f\|_{WM_{p,\varphi}(w)} = \sup_{x \in \mathbb{G}, r > 0} \varphi(x,r)^{-1} w(D(x,r))^{-\frac{1}{p}} \|f\|_{WL_{p,w}(D(x,r))} < \infty,$$

where $WL_{p,w}(D(x,r))$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_{p,w}(D(x,r))} \equiv \|f\chi_{D(x,r)}\|_{WL_{p,w}(\mathbb{G})} = \sup_{t>0} t\Big(\int_{\{y\in D(x,r):|f(y)|>t\}} w(y)dy\Big)^{\frac{1}{p}}.$$

We recall a weight function w is in the Muckenhoupt's class $A_p(\mathbb{G})$, $1<p<\infty$ [29], if

$$[w]_{A_p} := \sup_D [w]_{A_p(D)} = \sup_D \Big(\frac{1}{|D|}\int_D w(x)dx\Big)\Big(\frac{1}{|D|}\int_D w(x)^{1-p'}dx\Big)^{p-1} < \infty, \tag{3.1}$$

where the supremum is taken with respect to all the balls D and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls D Hölder's inequality is

$$[w]_{A_p(D)}^{\frac{1}{p}} = |D|^{-1}\|w\|_{L_1(D)}^{\frac{1}{p}}\|w^{-\frac{1}{p}}\|_{L_{p'}(D)} \geqslant 1. \tag{3.2}$$

For $p=1$, $w\in A_1(\mathbb{G})$ is defined by the condition $Mw(x)\leqslant Cw(x)$ with $[w]_{A_1} = \sup_{x\in\mathbb{G}}\frac{Mw(x)}{w(x)}$, and for $p=\infty$ $A_\infty(\mathbb{G}) = \cup_{1\leqslant p<\infty}A_p(\mathbb{G})$ and $[w]_\infty = \inf_{1\leqslant p<\infty}[w]_{A_p}$.

A weight function w is in the Muckenhoupt-Wheeden class $A_{p,q}(\mathbb{G})$, $1<p<\infty$ [30], if

$$\begin{aligned}[w]_{A_{p,q}} &:= \sup_D [w]_{A_{p,q}(D)} \\ &= \sup_D \Big(\frac{1}{|D|}\int_B w(x)^q dx\Big)^{1/q}\Big(\frac{1}{|D|}\int_D w(x)^{-p'}dx\Big)^{1/p'} < \infty,\end{aligned}$$

where the supremum is taken with respect to all the balls D and $\frac{1}{p}+\frac{1}{p'}=1$. Note that, for all balls D Hölder's inequality is

$$[w]_{A_{p,q}(D)} = |D|^{\frac{1}{p}-\frac{1}{q}-1}\|w\|_{L_q(D)}\|w^{-1}\|_{L_{p'}(D)} \geqslant 1 \tag{3.3}$$

While $p=1$, $w\in A_{1,q}(\mathbb{G})$ with $1<q<\infty$ if

$$\begin{aligned}[w]_{A_{1,q}} &:= \sup_D [w]_{A_{1,q}(D)} \\ &= \sup_D \Big(\frac{1}{|D|}\int_D w(x)^q dx\Big)^{\frac{1}{q}}\Big(\operatorname*{ess\,sup}_{x\in D}\frac{1}{w(x)}\Big) < \infty.\end{aligned} \tag{3.4}$$

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)w(s)ds, \qquad H_w^* g(t) := \int_t^\infty \Big(1+\frac{s}{t}\Big)g(s)w(s)ds, \qquad 0<t<\infty.$$

where w is a weight. The following theorem was proved in [15].

THEOREM 3.1. [15] *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w g(t) \leqslant C \sup_{t>0} v_1(t) g(t)$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\,ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty$$

THEOREM 3.2. [18] *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t) H_w^* g(t) \leqslant C \sup_{t>0} v_1(t) g(t)$$

holds for some $C>0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \left(1+\frac{s}{t}\right) \frac{w(s)\,ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty.$$

Let $\mathscr{D} = \{D(x,r) : x \in \mathbb{G}, r>0\}$. For a non-negative measurable function w, we denote by $\mathscr{G}_w^p$ the set of all almost decreasing functions $\varphi : \mathbb{G}\times(0,\infty) \to (0,\infty)$ such that

$$\inf_{D\in\mathscr{D}: r_D \leqslant r_{D_0}} \varphi(D) \gtrsim \varphi(D_0) \quad for\ all \quad D_0 \in \mathscr{D}$$

and

$$\inf_{D\in\mathscr{D}: r_D \geqslant r_{D_0}} \varphi(D)\, w^p(D)^{\frac{1}{p}} \gtrsim \varphi(D_0)\, w^p(D_0)^{\frac{1}{p}},$$

where r_D and r_{D_0} denote the radius of the $\mathbb{G}$-balls D and D_0, respectively.

For proving our main results, we need the following estimate.

LEMMA 3.1. *Let $D_0 := D(x_0,r_0)$. If $\varphi \in \mathscr{G}_w^p$, then there exist $C>0$ such that*

$$\frac{1}{\varphi(D_0)} \leqslant \|\chi_{D_0}\|_{M_{p,\varphi}(w^p)} \leqslant \frac{C}{\varphi(D_0)}.$$

Proof.

$$\begin{aligned} \|\chi_{D_0}\|_{M_{p,\varphi}(w^p)} &= \sup_D \varphi(D)^{-1} w^p(D)^{-\frac{1}{p}} (w(D\cap D_0))^{\frac{1}{p}} \\ &\geqslant \varphi(D_0)^{-1} w^p(D_0)^{-\frac{1}{p}} w^p(D_0)^{\frac{1}{p}} = \varphi(D_0)^{-1}. \end{aligned}$$

Now if $r \leqslant r_0$, then $\varphi(D_0) = \varphi(x_0, r_0) \leqslant C\varphi(x,r) = C\varphi(D)$ and

$$\varphi(D)^{-1} w^p(D)^{-\frac{1}{p}} \|\chi_{D_0}\|_{L_{p,w^p}(D)} = \varphi(D)^{-1} w^p(D)^{-\frac{1}{p}} (w(D \cap D_0))^{\frac{1}{p}} \leqslant \varphi(D)^{-1} \leqslant C\varphi(D_0)^{-1}.$$

Therefore, $\|\chi_{D_0}\|_{M_{p,\varphi}(w^p)} \leqslant C\varphi(D_0)^{-1}$.

On the other hand if $r \geqslant r_0$, then $\varphi(D)\, w^p(D)^{\frac{1}{p}} \geqslant C\varphi(D_0)\, w^p(D_0)^{\frac{1}{p}}$ and

$$\varphi(D_0)^{-1} \geqslant C\,\varphi(D)^{-1} w^p(D)^{-\frac{1}{p}}\, w^p(D_0)^{\frac{1}{p}} \geqslant C\,\varphi(D)^{-1}\, w^p(D)^{-\frac{1}{p}}\, w^p(D \cap D_0)^{\frac{1}{p}}.$$

Then

$$\|\chi_{D_0}\|_{M_{p,\varphi}(w^p)} = \sup_{D \in \mathscr{D}} \varphi(D)^{-1}\, w^p(D)^{-\frac{1}{p}}\, w^p(D \cap D_0)^{\frac{1}{p}} \leqslant C\varphi(D_0)^{-1}.$$

Because of this, $\|\chi_{D_0}\|_{M_{p,\varphi}(w^p)} \leqslant C\varphi(D_0)^{-1}$. □

REMARK 3.1. Lemma 3.1 was proved in [8] for the case of $\mathbb{G} = \mathbb{R}^n$.

4. Fractional integral operator in the spaces $M_{p,\varphi}(\mathbb{G}, w)$

In this section, we shall give a characterization for the Spanne type boundedness of the operator I_α on generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{G}, w)$, including weak versions. In the case of $\mathbb{G} = \mathbb{R}^n$ Spanne type result for the operator I_α in the space $M_{p,\varphi}(\mathbb{R}^n, w)$ was proved in [18], see also [19, 20, 22].

The following weighted local estimates are valid (see [18]).

THEOREM 4.3. *Let* $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, *and* $\omega \in A_{p,q}(\mathbb{G})$. *Then, for* $1 \leqslant p < q < \frac{Q}{\alpha}$, *the inequality*

$$\|I_\alpha f\|_{L_{q,w^q}(D(x,r))} \lesssim w^q(D(x,r))^{\frac{1}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}$$

holds for any boll $D(x,r)$ *and for all* $f \in L^{loc}_{p,w}(\mathbb{G})$.

Moreover, for $p = 1$ *the inequality*

$$\|I_\alpha f\|_{WL_{q,w^q}(D(x,r))} \lesssim w^q(D(x,r))^{\frac{1}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_{1,w}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t} \qquad (4.5)$$

holds for any boll $D(x,r)$ *and for all* $f \in L^{loc}_{1,w}(\mathbb{G})$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and $w \in A_{p,q}(\mathbb{G})$. For arbitrary $x \in \mathbb{G}$, set $D = D(x,r)$, $2c_0 D = D(x, 2c_0 r)$.

We present f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2c_0D}(y), \quad f_2(y) = f(y)\chi_{(2c_0D)^{\complement}}(y), \quad r > 0, \tag{4.6}$$

and have

$$\|I_\alpha f\|_{L_{q,w^q}(D)} \leqslant \|I_\alpha f_1\|_{L_{q,w^q}(D)} + \|I_\alpha f_2\|_{L_{q,w^q}(D)}.$$

Since $f_1 \in L_{p,w^p(\mathbb{G})}$, $I_\alpha f_1 \in L_{q,w^q}(\mathbb{G})$ and from the boundedness of I_α from $L_{p,w^p}(\mathbb{G})$ to $L_{q,w^q}(\mathbb{G})$ (see [2, 35]) it follows that:

$$\|I_\alpha f_1\|_{L_{q,w^q}(D)} \leqslant \|I_\alpha f_1\|_{L_{q,w^q}} \leqslant C\|f_1\|_{L_{p,w^p}} = C\|f\|_{L_{p,w^p}(2c_0D)},$$

where the constant $C > 0$ does not depend on f.

It is clear that $z \in D$, $y \in (2c_0D)^{\complement}$ implies $\frac{1}{2c_0}\rho(x^{-1}y) \leqslant \rho(z^{-1}y) \leqslant \frac{3c_0}{2}\rho(x^{-1}y)$. We get

$$|I_\alpha f_2(z)| \leqslant (2c_0)^{Q-\alpha} \int_{(2c_0D)^{\complement}} \frac{|f(y)|}{\rho(x^{-1}y)^{Q-\alpha}}dy.$$

By the Fubini's theorem we have

$$\begin{aligned}\int_{(2c_0D)^{\complement}} \frac{|f(y)|}{\rho(x^{-1}y)^{Q-\alpha}}dy &\approx \int_{(2c_0D)^{\complement}} |f(y)|\Big(\int_{\rho(x^{-1}y)}^{\infty} \frac{dt}{t^{Q+1-\alpha}}\Big)dy \\ &= \int_{2c_0r}^{\infty}\Big(\int_{2c_0r\leqslant\rho(x^{-1}y)<t} |f(y)|dy\Big)\frac{dt}{t^{Q+1-\alpha}} \\ &\leqslant \int_{2c_0r}^{\infty}\Big(\int_{D(x,t)} |f(y)|dy\Big)\frac{dt}{t^{Q+1-\alpha}}.\end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned}\int_{(2c_0D)^{\complement}} \frac{|f(y)|}{\rho(x^{-1}y)^{Q-\alpha}}dy &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_{p,\omega^p}(D(x,t))}\|w^{-1}\|_{L_{p'}(D(x,t))}\frac{dt}{t^{Q+1-\alpha}} \\ &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_{p,\omega^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}}\frac{dt}{t}.\end{aligned}$$

Moreover, for all $p \in [1,\infty)$,

$$\|I_\alpha f_2\|_{L_{q,w^q}(D)} \lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,\omega^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}}\frac{dt}{t}. \tag{4.7}$$

Thus,

$$\|I_\alpha f\|_{L_{q,w^q}(D)} \lesssim \|f\|_{p,w^p}(2c_0D) + w^q(D(x,t))^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,\omega^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}}\frac{dt}{t}$$

On the other hand, since, $w \in A_{p,q}(\mathbb{G})$, by the Hölder's inequality

$$[w]_{A_{p,q}} \geqslant |D|^{\frac{1}{p}-\frac{1}{q}-1} w^q(D)^{\frac{1}{q}}\|w^{-1}\|_{L_{p'}(D)} = |D|^{\frac{\alpha}{Q}-1} w^q(D)^{\frac{1}{q}}\|w^{-1}\|_{L_{p'}(D)} \geqslant 1.$$

Then,

$$\begin{aligned}
\|f\|_{L_{p,w^p}(2c_0D)} &\leqslant |D|^{1-\frac{\alpha}{Q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(D)^{\frac{1}{q}} \|w^{-1}\|_{L_{p'}(D)} \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} \|w^{-1}\|_{L_{p'}(D(x,t))} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}. \qquad (4.8)
\end{aligned}$$

Thus,

$$\|I_\alpha f\|_{L_{q,w^q}(D)} \lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}.$$

Let $p=1$. Since, $w \in A_{1,q}$, by the Hölder's inequality

$$[w]_{A_{p,q}} \geqslant |D|^{\frac{1}{p}-\frac{1}{q}-1} w^q(D)^{\frac{1}{q}} \|w^{-1}\|_{L_\infty(D)} = |D|^{\frac{\alpha}{Q}-1} w^q(D)^{\frac{1}{q}} \|w^{-1}\|_{L_\infty(D)} \geqslant 1.$$

Then from the boundedness of I_α from $L_{1,w^p}(\mathbb{G})$ to $WL_{q,w^q}(\mathbb{G})$ (see [2, 35]) it follows that:

$$\begin{aligned}
\|I_\alpha f_1\|_{WL_{q,w^q}(D)} &\leqslant \|I_\alpha f_1\|_{WL_{q,w^q}} \lesssim \|f_1\|_{L_{1,w}} = \|f\|_{L_{1,w}(2c_0D)} \\
&\leqslant |D|^{1-\frac{\alpha}{Q}} \int_{2c_0r}^{\infty} \|f\|_{L_{1,w}(D(x,t))} \frac{dt}{t^{Q+1-\alpha}} \qquad (4.9) \\
&\lesssim w^q(D)^{\frac{1}{q}} \|w^{-1}\|_{L_\infty(D)} \int_{2c_0r}^{\infty} \|f\|_{L_{1,w}(D(x,t))} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{1,w}(D(x,t))} \|w^{-1}\|_{L_\infty(D(x,t))} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{1,w}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}.
\end{aligned}$$

Since

$$\|I_\alpha f_2\|_{WL_{q,w^q}(D)} \leqslant \|I_\alpha f_2\|_{L_{q,w^q}(D)},$$

then

$$\|I_\alpha f_2\|_{WL_{q,w^q}(B)} \lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{1,\omega}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}. \qquad (4.10)$$

Thus, from (4.9) and (4.10) it follows that

$$\|I_\alpha f\|_{WL_{q,w^q}(D)} \lesssim w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}. \qquad \square$$

The following Spanne type result on the space $M_{p,\varphi}(w)$ is valid.

THEOREM 4.4. *Let* $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$, *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^\infty \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w^p D((x,s))^{1/p}}{w^q(D(x,t))^{1/q}} \frac{dt}{t} \leqslant C\varphi_2(x,r) \tag{4.11}$$

where C *does not depend on* x *and* r. *Then the operator* I_α *is bounded from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(w)$ *to* $WM_{q,\varphi_2}(w^q)$ *for* $p = 1$. *Moreover, for* $p > 1$

$$\|I_\alpha f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{p,\varphi_1}(w^p)},$$

and for $p = 1$

$$\|I_\alpha f\|_{WM_{q,\varphi_2}(w^q)} \lesssim \|f\|_{M_{1,\varphi_1}(w)}.$$

Proof. Using the Theorem 3.1 and the Theorem 4.3 for $p > 1$ we get

$$\begin{aligned}
\|I_\alpha f\|_{M_{q,\varphi_2}(w^q)} &= \sup_{x\in\mathbb{G},r>0} \varphi_2(x,r)^{-1} w^q(D(x,r))^{-\frac{1}{q}} \|I_\alpha f\|_{L_{q,w^q}(D(x,r))} \\
&\lesssim \sup_{x\in\mathbb{G},r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{G},r>0} \varphi_1(x,r)^{-1} w^p(D(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(D(x,r))} = \|f\|_{M_{p,\varphi_1}(w^p)}.
\end{aligned}$$

and for $p = 1$

$$\begin{aligned}
\|I_\alpha f\|_{WM_{q,\varphi_2}(w^q)} &= \sup_{x\in\mathbb{G},r>0} \varphi_2(x,r)^{-1} w^q(D(x,r))^{-\frac{1}{q}} \|I_\alpha f\|_{L_{1,w}(D(x,r))} \\
&\lesssim \sup_{x\in\mathbb{G},r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_{1,w}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{G},r>0} \varphi_1(x,r)^{-1} w(D(x,r))^{-1} \|f\|_{L_{1,w}(D(x,r))} = \|f\|_{M_{1,\varphi_1}(w)}. \quad \square
\end{aligned}$$

REMARK 4.2. Note that, in the case $w \equiv 1$, Theorems 4.3 and 4.4 were proved in [16], see also [10, 17, 36].

For proving our main results, we need the following estimate.

LEMMA 4.2. *If* $D_0 := D(x_0, r_0)$, *then*

$$r_0^\alpha \lesssim I_\alpha \chi_{D_0}(x) \quad \textit{for every } x \in D_0.$$

Proof. If $x, y \in D_0$, then $\rho(x^{-1}y) \leqslant c_0\rho(x^{-1}x_0) + c_0\rho(x_0^{-1}y) \leqslant 2c_0 r_0$.
Since $0 < \alpha < Q$,

$$I_\alpha \chi_{D_0}(x) = \int_{D_0} \rho(x^{-1}y)^{\alpha-Q} dy \gtrsim r_0^{\alpha-Q}|D_0| \approx r_0^\alpha. \quad \square$$

The following theorem is one of our main results.

THEOREM 4.5. *Let* $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$, *and* φ_1, φ_2 *positive measurable functions on* $\mathbb{G} \times (0,\infty)$.

1. *Then the condition* (4.11) *is sufficient for the boundedness of* I_α *from* $M_{p,\varphi_1}(w^p)$ *to* $WM_{q,\varphi_2}(w^q)$ *and for* $p > 1$ *from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$.

2. *If the function* $\varphi_1 \in \mathscr{G}_w^p$, *then the condition*

$$t^\alpha \varphi_1(t) \leqslant C\varphi_2(t) \quad \text{for all} \ \ t > 0, \tag{4.12}$$

where $C > 0$ *does not depend on* t, *is necessary for the boundedness of* I_α *from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$ *and* $M_{p,\varphi_1}(w^p)$ *to* $WM_{q,\varphi_2}(w^q)$.

3. *If the function* $\varphi_1 \in \mathscr{G}_w^p$ *satisfies the regularity condition*

$$\int_t^\infty \frac{\varphi_1(x,r)w^p(D(x,r))^{\frac{1}{p}}}{w^q(D(x,r))^{\frac{1}{q}}} \frac{dr}{r} \leqslant Ct^\alpha \varphi_1(t)$$

for all $t > 0$, *where* $C > 0$ *does not depend on* t, *then condition* (4.12) *is necessary and sufficient for the boundedness of* I_α *from* $M_{p,\varphi_1}(w^p)$ *to* $WM_{q,\varphi_2}(w^q)$ *and for* $p > 1$ *from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$.

Proof. The first part of the theorem proved in Theorem 4.4

We shall now prove the second part.

Let $D_0 = D(x_0, r_0)$ and $x \in D_0$. By Lemma 4.2 we have $r_0^\alpha \lesssim I_\alpha \chi_{D_0}(x)$. Then, $r_0^\alpha w(x) \lesssim I_\alpha \chi_{D_0}(x) w(x)$. Hence,

$$r_0^\alpha w^q(D_0)^{\frac{1}{q}} \lesssim \|I_\alpha \chi_{D_0}\|_{L_{q,w^q}(D_0)}.$$

Therefore,

$$\begin{aligned} r_0^\alpha &\lesssim \varphi_2(D_0)\, \varphi_2^{-1}(D_0) w^q(D_0)^{-\frac{1}{q}} \|I_\alpha \chi_{D_0}\|_{L_{q,w^q}(D_0)} \\ &\leqslant \varphi_2(D_0) \sup_{x \in \mathbb{G}, r>0} \varphi_2^{-1}(D) w^q(D)^{-\frac{1}{q}} \|I_\alpha \chi_{D_0}\|_{L_{q,w^q}(D)} \\ &= \varphi_2(D_0)\, \|I_\alpha \chi_{D_0}\|_{M_{q,\varphi_2}(w^q)} \lesssim \varphi_2(D_0)\, \|\chi_{D_0}\|_{M_{p,\varphi_1}(w^p)}. \end{aligned}$$

According to Lemma 3.1

$$r_0^\alpha \lesssim \frac{\varphi_2(D_0)}{\varphi_1(D_0)}$$

is true for all D_0.

Since this is true for every D_0, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

REMARK 4.3. Note that, in the case $w \equiv 1$, Theorem 4.5 was proved in [9], see also [10, 23].

5. Commutators of fractional integral operators in the spaces $M_{p,\varphi}(\mathbb{G},w)$

In this section, we shall give a characterization for the Spanne type boundedness of the commutators of fractional integral operator $[b,I_\alpha]$ on generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{G},w)$. In the case of $\mathbb{G}=\mathbb{R}^n$ Spanne type result for the operator $[b,I_\alpha]$ in the space $M_{p,\varphi}(\mathbb{R}^n,w)$ was proved in [18], see also [20, 22].

Given a function b locally integrable on $\mathbb{R}^n$ and the operator I_α, we consider the linear commutator $[b,I_\alpha]$ defined by setting, for smooth, compactly supported functions f,

$$[b,I_\alpha](f)=bI_\alpha(f)-I_\alpha(bf).$$

We recall the definition of the space of $BMO(\mathbb{G})$.

DEFINITION 2. Suppose that $b\in L_1^{\rm loc}(\mathbb{G})$, and let

$$\|b\|_*=\sup_{u\in\mathbb{G},r>0}\frac{1}{|D(x,r)|}\int_{D(x,r)}|b(y)-b_{D(x,r)}|dy<\infty,$$

where

$$b_{D(x,r)}=\frac{1}{|D(x,r)|}\int_{D(x,r)}b(y)dy.$$

Define

$$BMO(\mathbb{G})=\{b\in L_1^{\rm loc}(\mathbb{G})\ :\ \|b\|_*<\infty\}.$$

Modulo constants, the space $BMO(\mathbb{G})$ is a Banach space with respect to the norm $\|\cdot\|_*$.

LEMMA 5.3. [30] *Let $w\in A_\infty$. Then the norm $\|\cdot\|_*$ is equivalent to the norm*

$$\|b\|_{*,w}=\sup_{x\in\mathbb{G},r>0}\frac{1}{w(D(x,r))}\int_{D(x,r)}|b(y)-b_{D(x,r),w}|\,w(y)\,dy,$$

where

$$b_{D(x,r),w}=\frac{1}{w(D(x,r))}\int_{D(x,r)}b(y)\,w(y)\,dy.$$

The following lemma was proved in [18].

LEMMA 5.4. [18]

1. *Let $w\in A_\infty$ and $b\in BMO(\mathbb{G})$. Let also $1\leqslant p<\infty$, $x\in\mathbb{G}$ and $r_1,r_2>0$. Then*

$$\Big(\frac{1}{w(D(x,r_1))}\int_{D(x,r_1)}|b(y)-b_{D(x,r_2),w}|^p w(y)dy\Big)^{\frac{1}{p}}\leqslant C\Big(1+\Big|\ln\frac{r_1}{r_2}\Big|\Big)\,\|b\|_*,$$

where $C>0$ is independent of f, w, x, r_1 and r_2.

2. *Let $w \in A_p$ and $b \in BMO(\mathbb{G})$. Let also $1 < p < \infty$, $x \in \mathbb{G}$ and $r_1, r_2 > 0$. Then*

$$\left(\frac{1}{w^{1-p'}(D(x,r_1))}\int_{D(x,r_1)} |b(y) - b_{D(x,r_2),w}|^{p'} w(y)^{1-p'} dy\right)^{\frac{1}{p'}} \leqslant C\left(1 + \left|\ln \frac{r_1}{r_2}\right|\right) \|b\|_*,$$

where $C > 0$ is independent of b, w, x, r_1 and r_2.

REMARK 5.4. [5, 42] (1) Let $b \in BMO(\mathbb{G})$. Then

$$\|b\|_* \approx \sup_{x\in\mathbb{G}, r>0} \left(\frac{1}{|D(x,r)|}\int_{D(x,r)} |b(y) - b_{D(x,r)}|^p dy\right)^{\frac{1}{p}} \tag{5.13}$$

for $1 < p < \infty$.

(2) Define $BMO_p(\mathbb{G},w)$ by following:

$$BMO_p(\mathbb{G},w) =: \{b \in L_1^{\mathrm{loc}}(\mathbb{G}) \ : \ \|b\|_{BMO_p(\mathbb{G},w)} < \infty\},$$

where

$$\|b\|_{BMO_p(w)} = \sup_D \left(\frac{1}{w(D)}\int_D |b(x) - b_D|^p w(x) dx\right)^{\frac{1}{p}}.$$

Let $1 \leqslant p < \infty$ and $w \in A_p(\mathbb{G})$. Then

$$BMO(\mathbb{G}) = BMO_p(\mathbb{G},w)$$

and the norms are mutually equivalent, see [25, Theorem 3.1].

(3) Let $b \in BMO(\mathbb{G})$. Then there is a constant $C > 0$ such that

$$\left|b_{D(x,r)} - b_{D(x,\tau)}\right| \leqslant C\|b\|_* \log \frac{\tau}{r} \ \text{ for } \ 0 < 2r < \tau, \tag{5.14}$$

where C is independent of f, x, r and τ.

For the commutator of the fractional integral operator $[b, I_\alpha]$ the following weighted local estimates are valid (see [18]).

THEOREM 5.6. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $b \in BMO(\mathbb{G})$, and $w \in A_{p,q}(\mathbb{G})$. Then the inequality*

$$\|[b,I_\alpha]f\|_{L_{q,w^q}(D(x,r))} \lesssim \|b\|_* w^q(D(x,r))^{\frac{1}{q}} \int_{2c_0 r}^{\infty} \ln\left(e + \frac{t}{r}\right) \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}$$

holds for any ball $D(x,r)$ and for all $f \in L^{loc}_{p,w^p}(\mathbb{G})$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $b \in BMO(\mathbb{G})$ and $w \in A_{p,q}(\mathbb{G})$. For arbitrary $x \in \mathbb{G}$, set $D = D(x,r)$ for the ball centered at x and of radius r. We present f as $f = f_1 + f_2$ with $f = f\chi_{2c_0D}$ and $f_2 = f\chi_{(2c_0D)^{\complement}}$. Hence,

$$\|[b,I_\alpha]f\|_{L_{q,w^q}(D)} \leqslant \|[b,I_\alpha]f_1\|_{L_{q,w^q}(D)} + \|[b,I_\alpha]f_2\|_{L_{q,w^q}(D)}.$$

From the boundedness of $[b,I_\alpha]$ from $L_{p,w^p}(\mathbb{G})$ to $L_{q,w^q}(\mathbb{G})$ (see [2, 35]) it follows that:

$$\|[b,I_\alpha]f_1\|_{L_{q,w^q}(D)} \leqslant \|[b,I_\alpha]f_1\|_{L_{q,w^q}(\mathbb{G})} \lesssim \|b\|_* \|f_1\|_{L_{p,w^p}(\mathbb{G})} = \|b\|_* \|f\|_{L_{p,w^p}(2c_0D)}.$$

For $z \in D$ we have

$$|[b,I_\alpha]f_2(z)| \lesssim \int_{\mathbb{G}} \frac{|b(y)-b(z)|}{\rho(z^{-1}y)^{Q-\alpha}}|f_2(y)|dy \approx \int_{(2c_0D)^{\complement}} \frac{|b(y)-b(z)|}{\rho(x^{-1}y)^{Q-\alpha}}|f(y)|dy.$$

Then,

$$\begin{aligned}
\|[b,I_\alpha]f_2\|_{L_{q,w^q}(D)} &\lesssim \Big(\int_D\Big(\int_{(2c_0D)^{\complement}} \frac{|b(y)-b(z)|}{\rho(x^{-1}y)^{Q-\alpha}}|f(y)|dy\Big)^q w^q(z)dz\Big)^{\frac1q} \\
&\lesssim \Big(\int_D\Big(\int_{(2c_0D)^{\complement}} \frac{|b(y)-b_{D,w}|}{\rho(x^{-1}y)^{Q-\alpha}}|f(y)|dy\Big)^q w^q(z)dz\Big)^{\frac1q} \\
&\quad + \Big(\int_D\Big(\int_{(2c_0D)^{\complement}} \frac{|b(z)-b_{D,w}|}{\rho(x^{-1}y)^{Q-\alpha}}|f(y)|dy\Big)^q w^q(z)dz\Big)^{\frac1q} \\
&= I_1 + I_2.
\end{aligned}$$

Using Fubini's theorem let us estimate I_1 as follows

$$\begin{aligned}
I_1 &= \Big(\int_D\Big(\int_{(2c_0D)^{\complement}} \frac{|b(y)-b_{D,w}|}{\rho(x^{-1}y)^{Q-\alpha}}|f(y)|dy\Big)^q w^q(z)dz\Big)^{\frac1q} \\
&= w^q(D)^{\frac1q} \int_{(2c_0D)^{\complement}} \frac{|b(y)-b_{D,w}|}{\rho(x^{-1}y)^{Q-\alpha}}|f(y)|dy \\
&\approx w^q(D)^{\frac1q} \int_{(2c_0D)^{\complement}} |b(y)-b_{D,w}||f(y)| \int_{\rho(x^{-1}y)}^{\infty} \frac{dt}{t^{Q-\alpha+1}}dy \\
&\approx w^q(D)^{\frac1q} \int_{2c_0r}^{\infty} \int_{2c_0r\leqslant\rho(x^{-1}y)\leqslant t} |b(y)-b_{D,w}||f(y)|dy\frac{dt}{t^{Q-\alpha+1}} \\
&\lesssim w^q(D)^{\frac1q} \int_{2c_0r}^{\infty} \int_{D(x,t)} |b(y)-b_{D,w}||f(y)|dy\frac{dt}{t^{Q-\alpha+1}}.
\end{aligned}$$

Applying Fubini's theorem, Hölder's inequality and the first part of Lemma 5.4 we get

$$\begin{aligned}
I_1 &\lesssim w^q(D)^{\frac1q} \int_{2c_0r}^{\infty} \int_{D(x,t)} |b(y)-b_{D(x,r),w}|w(y)^{-1}w(y)|f(y)|dy\frac{dt}{t^{Q-\alpha+1}} \\
&\leqslant w^q(D)^{\frac1q} \int_{2c_0r}^{\infty} \Big(\int_{D(x,t)} |b(y)-b_{D(x,r),w}|^{p'}w(y)^{-p'}dy\Big)^{-\frac{1}{p'}} \|f\|_{L_{p,w^p}(D(x,t))}\frac{dt}{t^{Q-\alpha+1}} \\
&\lesssim \|b\|_* w^q(D)^{\frac1q} \int_{2c_0r}^{\infty} \Big(1+\ln\frac{t}{r}\Big)\|w^{-1}\|_{L_{p'}(D(x,t))}\|f\|_{L_{p,w^p}(D(x,t))}\frac{dt}{t^{Q-\alpha+1}} \\
&\lesssim [w]_{A_{p,q}}\|b\|_* w^q(D)^{\frac1q} \int_{2c_0r}^{\infty} \ln\Big(e+\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(D(x,t))}w^q(D(x,t))^{-\frac1q}\frac{dt}{t}.
\end{aligned}$$

In order to estimate I_2 we get

$$I_2 = \Big(\int_D |b(z) - b_{D,w}|^q w^q(z) dz\Big)^{1/q} \int_{(2c_0D)^\complement} \frac{|f(y)|}{\rho(x^{-1}y)^{Q-\alpha}} dy.$$

According to the first part of Lemma 5.4, we get

$$I_2 \lesssim \|b\|_* \, w^q(D)^{\frac{1}{q}} \int_{(2c_0D)^\complement} \frac{|f(y)|}{\rho(x^{-1}y)^{Q-\alpha}} dy.$$

Applying Fubini's theorem and Hölder's inequality gives

$$\begin{aligned}\int_{(2c_0D)^\complement} \frac{|f(y)|}{\rho(x^{-1}y)^{Q-\alpha}} dy &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} \|w^{-1}\|_{L_{p'}(D(x,t))} \frac{dt}{t^{Q-\alpha+1}} \\ &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} \, w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}. \end{aligned} \tag{5.15}$$

So, by (5.15)

$$I_2 \lesssim \|b\|_* w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}.$$

Summing I_1 and I_2, for all $p \in (1,\infty)$ we get

$$\begin{aligned}\|[b,I_\alpha]f_2\|_{L_{p,w}(D)} &\lesssim \|b\|_* \, w^q(D)^{\frac{1}{q}} \\ &\times \int_{2c_0r}^{\infty} \ln\Big(e+\frac{t}{r}\Big) \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}. \end{aligned} \tag{5.16}$$

Finally,

$$\begin{aligned}\|[b,I_\alpha]f\|_{L_{p,w}(D)} &= \|b\|_* \|f\|_{L_{p,w^p}(2c_0D)} \\ &+ \|b\|_* w^q(D)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \ln\Big(e+\frac{t}{r}\Big) \|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}} \frac{dt}{t}\end{aligned}$$

and the statement of Theorem 5.6 follows by (4.8). □

The following Spanne type result on the space $M_{p,\varphi}(w)$ is valid.

THEOREM 5.7. *Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$, $b \in BMO(\mathbb{G})$ and (φ_1, φ_2) satisfy the condition*

$$\int_r^{\infty} \ln\Big(e+\frac{t}{r}\Big) \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w^p(D(x,s))^{1/p}}{w^q(D(x,t))^{1/q}} \frac{dt}{t} \leqslant C\varphi_2(x,r) \tag{5.17}$$

where C does not depend on x and r. Then the operator $[b,I_\alpha]$ is bounded from $M_{p,\varphi_1}(w^p)$ to $M_{q,\varphi_2}(w^q)$. Moreover,

$$\|[b,I_\alpha]f\|_{M_{q,\varphi_2}(w^q)} \lesssim \|b\|_* \|f\|_{M_{p,\varphi_1}(w^p)}.$$

Proof. Using the Theorem 3.2 and the Theorem 5.6 we have

$$\begin{aligned}
\|[b,I_\alpha]f\|_{M_{q,\varphi_2}(w^q)} &\lesssim \|b\|_* \sup_{x\in\mathbb{G},r>0} \varphi_2(x,r)^{-1} \\
&\qquad \times \int_r^\infty \ln\Big(e+\frac{t}{r}\Big)\|f\|_{L_{p,w^p}(D(x,t))} w^q(D(x,t))^{-\frac{1}{q}}\frac{dt}{t} \\
&\lesssim \|b\|_* \sup_{x\in\mathbb{G},r>0} \varphi_1(x,r)^{-1} w^p(D(x,r))^{-\frac{1}{p}} \|f\|_{L_{p,w^p}(D(x,r))} \\
&= \|b\|_* \|f\|_{M_{p,\varphi_1}(w^p)}. \qquad \square
\end{aligned}$$

REMARK 5.5. Note that, in the case $w\equiv 1$, Theorems 5.6 and 5.7 were proved in [17]. Note that in [17] a more general case of higher order commutators was studied.

For proving our main results, we need the following estimate.

LEMMA 5.5. *If* $b\in L^1_{\rm loc}(\mathbb{G})$ *and* $D_0 := D(x_0,r_0)$, *then*

$$r_0^\alpha |b(x)-b_{D_0}| \leqslant C|b,I_\alpha|\chi_{D_0}(x)$$

for every $x\in D_0$.

Proof. If $x,y\in D_0$, then $\rho(x^{-1}y)\leqslant c_0\rho(x^{-1}x_0)+c_0\rho(y^{-1}x_0)\leqslant 2c_0r_0$. Since $0<\alpha<Q$, we get $r_0^{\alpha-Q}\leqslant C\rho(x^{-1}y)^{\alpha-Q}$. Therefore

$$\begin{aligned}
|b,I_\alpha|\chi_{D_0}(x) &= \int_{D_0}|b(x)-b(y)|\rho(x^{-1}y)^{\alpha-Q}dy \geqslant Cr_0^{\alpha-Q}\int_{D_0}|b(x)-b(y)|dy \\
&\geqslant Cr_0^{\alpha-Q}\left|\int_{D_0}(b(x)-b(y))dy\right| = Cr_0^\alpha|b(x)-b_{D_0}|. \qquad \square
\end{aligned}$$

THEOREM 5.8. *Let* $1<p<q<\infty$, $0<\alpha<\frac{Q}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{Q}$, $w\in A_{p,q}(\mathbb{G})$, $b\in BMO(\mathbb{G})\setminus\{const\}$ *and* φ_1, φ_2 *positive measurable functions on* $\mathbb{G}\times(0,\infty)$.

1. *Then the condition* (5.17) *is sufficient for the boundedness of the operator* $[b,I_\alpha]$ *from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$.
2. *If the function* $\varphi_1\in\mathcal{G}_w^p$, *then the condition*
$$t^\alpha\varphi_1(t)\leqslant C\varphi_2(t) \quad \textit{for all } t>0, \tag{5.18}$$
where $C>0$ *does not depend on* t, *is necessary for the boundedness of* $[b,I_\alpha]$ *from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$.
3. *If the function* $\varphi_1\in\mathcal{G}_w^p$ *satisfies the regularity condition*
$$\int_t^\infty \ln\Big(e+\frac{t}{r}\Big)\frac{\varphi_1(r)w^p(D(x,r))^{\frac{1}{p}}}{w^q(D(x,r))^{\frac{1}{q}}}\frac{dr}{r}\leqslant Ct^\alpha\varphi_1(t)$$
for all $t>0$, *where* $C>0$ *does not depend on* t, *then condition* (5.18) *is necessary and sufficient for the boundedness of* $[b,I_\alpha]$ *from* $M_{p,\varphi_1}(w^p)$ *to* $M_{q,\varphi_2}(w^q)$.

Proof. The first part of the theorem proved in Theorem 5.7. We shall now prove the second part.

Note that $\|b\|_*$ is only a seminorm, $\|b\|_* = 0$ if and only if b is constant (a.e.). Therefore, if $b \in BMO(\mathbb{G}) \setminus \{const\}$, then $\|b\|_* > 0$. For all $r > 0$, there exists $x_0 \in D$ such that $\|b(\cdot) - b_{D(x_0,r_0)}\|_{L_1(D(x_0,r_0))} > 0$, otherwise b equals to the constant.

By the Lemma 5.5 $w(x)\, r_0^{\alpha}\, |b(x) - b_{D_0}| \lesssim w(x)\, |b, I_\alpha| \chi_{D_0}(x)$.

Then

$$r_0^{\alpha} \Big(\int_{D_0} w^q(x)\, |b(x) - b_{D_0}|^q dx \Big)^{\frac{1}{q}} \lesssim \Big(\int_{D_0} w^q(x) \Big(|b, I_\alpha| \chi_{D_0}(x) \Big)^q dx \Big)^{\frac{1}{q}}.$$

Therefore,

$$\begin{aligned} \frac{r_0^{\alpha}}{w^q(D_0)^{\frac{1}{q}}} \Big(\int_{D_0} w^q(x)\, |b(x) - b_{D_0}|^q dx \Big)^{\frac{1}{q}} &\lesssim \frac{1}{w^q(D_0)^{\frac{1}{q}}} \| |b, I_\alpha| \chi_{D_0} \|_{L_{q,w^q}(D_0)} \\ &\lesssim \varphi_2(D_0)\, \| |b, I_\alpha| \chi_{D_0} \|_{M_{q,\varphi_2}(w^q)} \\ &\lesssim \varphi_2(D_0)\, \|b\|_* \|\chi_{D_0}\|_{M_{p,\varphi_1}(w^p)}. \end{aligned}$$

Then

$$\frac{r_0^{\alpha}}{w^q(D_0)^{\frac{1}{q}}} \Big(\int_{D_0} w^q(x)\, |b(x) - b_{D_0}|^q dx \Big)^{\frac{1}{q}} \lesssim \varphi_2(D_0)\, \|b\|_* \|\chi_{D_0}\|_{M_{p,\varphi_1}(w^p)}.$$

Using the Lemmas 3.1 and 5.4 gives

$$r_0^{\alpha} \lesssim \varphi_2(D_0)\, \|\chi_{D_0}\|_{M_{p,\varphi_1}(w^p)} \lesssim \frac{\varphi_2(D_0)}{\varphi_1(D_0)}$$

is true for all D_0.

Since this is true for every D_0, we are done.

The third statement of the theorem follows from first and second parts of the theorem. □

REMARK 5.6. Note that, in the case $w \equiv 1$, Theorem 5.8 was proved in [9], see also [23].

6. Some applications

It is known that (see [1]) if $\rho(\cdot)$ is a homogeneous norm on $\mathbb{G}$, then there exists a positive constant C_0 such that $\Gamma(x) = C_0 \rho(x)^{2-Q}$ is the fundamental solution of $\mathscr{L}$.

From Theorem 4.5, one easily obtains an inequality extending the classical Sobolev embedding theorem to the Carnot groups.

THEOREM 6.9. (Sobolev-Stein embedding on generalized weighted Morrey space) *Let* $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, $w \in A_{p,q}(\mathbb{G})$, *and* (φ_1, φ_2) *satisfy the condition* (4.11). *Then*

$$\|u\|_{M_{q,\varphi_2}(w^q)} \lesssim \|\nabla_{\mathscr{L}} u\|_{M_{p,\varphi_1}(w^p)}, \ \textit{for every } u \in C_0^{\infty}(\mathbb{G}).$$

Proof. Let $u \in C_0^\infty(\mathbb{G})$. By using the integral representation formula for the fundamental solution (see [1]), we have

$$u(g) = \int_{\mathbb{G}} \Gamma(g^{-1}y)\mathscr{L}u(y)dy \tag{6.19}$$

Keeping in mind that $\mathscr{L} = \sum_{i=1}^{2n} X_i^2$ and $X_i^* = -X_i$, by integrating by parts at the right-hand side (6.19), we obtain

$$u(g) = \int_{\mathbb{G}} (\nabla_{\mathscr{L}}\Gamma)(g^{-1}y)\nabla_{\mathscr{L}}u(y)dy. \tag{6.20}$$

On the other hand, out of the origin, we have

$$\nabla_{\mathscr{L}}\Gamma(x) = C_0\nabla_{\mathscr{L}}\left(\rho(x)^{2-Q}\right) = (2-Q)C_0\rho(x)^{1-Q}\nabla_{\mathscr{L}}\rho(x),$$

so that, since $\nabla_{\mathscr{L}}\rho(\cdot)$ is smooth in $\mathbb{G}\setminus\{0\}$ and δ_λ-homogeneous of degree zero,

$$\nabla_{\mathscr{L}}\Gamma(x) \leqslant C\rho(x)^{1-Q},$$

for a suitable constant $C > 0$ depending only on $\mathscr{L}$. Using this inequality in (6.20), we get

$$|u(x)| \leqslant C\int_{\mathbb{G}} |\nabla_{\mathscr{L}}u(y)|\,\rho(x^{-1}y)^{1-Q}dy = CI_1(|\nabla_{\mathscr{L}}u|)(x). \tag{6.21}$$

Then, by Theorem 4.5,

$$\|u\|_{M_{q,\varphi_2}(w^q)} \leqslant C\|I_1(|\nabla_{\mathscr{L}}u|)\|_{M_{q,\varphi_2}(w^q)} \leqslant C\|\nabla_{\mathscr{L}}u\|_{M_{p,\varphi_1}(w^p)}. \qquad \square$$

In the following theorem we prove the boundedness of $\mathscr{I}_\alpha$ in the generalized weighted Besov-Morrey spaces on $\mathbb{G}$

$$BM^s_{p\theta,\varphi}(\mathbb{G},w) = \Big\{f : \|f\|_{BM^s_{p\theta,\varphi}(\mathbb{G},w)} = \|f\|_{M_{p,\varphi}(\mathbb{G},w)} + \Big(\int_{\mathbb{G}} \frac{\|f(x\cdot)-f(\cdot)\|^\theta_{M_{p,\varphi}(w)}}{\rho(x)^{Q+s\theta}}dx\Big)^{\frac{1}{\theta}} < \infty\Big\} \tag{6.22}$$

where $1 \leqslant p,\theta \leqslant \infty$ and $0 < s < 1$.

Besov spaces $B^s_{p\theta}(\mathbb{G})$ in the setting Lie groups G were studied by many authors (see, for example [4, 6, 11, 38, 41]).

THEOREM 6.10. *Let* $1 < p < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$, *and* (φ_1,φ_2) *satisfy the condition* (4.11). *If* $1 \leqslant \theta \leqslant \infty$ *and* $0 < s < 1$, *then the operator* $\mathscr{I}_\alpha$ *is bounded from the spaces* $BM^s_{p\theta,\varphi_1}(\mathbb{G},w^p)$ *to* $BM^s_{q\theta,\varphi_2}(\mathbb{G},w^q)$. *More precisely, there is a constant* $C > 0$ *such that*

$$\|\mathscr{I}_\alpha f\|_{BM^s_{q\theta,\varphi_2}(\mathbb{G},w^q)} \leqslant C\|f\|_{BM^s_{p\theta,\varphi_1}(\mathbb{G},w^p)}$$

holds for all $f \in BM^s_{p\theta,\varphi_1}(\mathbb{G},w^p)$.

Proof. By the definition of the generalized weighted Besov-Morrey spaces on $\mathbb{G}$ it suffices to show that

$$\|\tau_h\mathscr{I}_\alpha f - \mathscr{I}_\alpha f\|_{M_{q,\varphi}(\mathbb{G},w^q)} \leqslant C\|\tau_h f - f\|_{M_{p,\varphi}(\mathbb{G},w^p)},$$

where $\tau_h f(x) = f(hx)$.

It is easy to see that $\tau_h f$ commutes with $\mathscr{I}_\alpha$, i.e., $\tau_h \mathscr{I}_\alpha f = \mathscr{I}_\alpha(\tau_h f)$. Hence we obtain

$$|\tau_h \mathscr{I}_\alpha f - \mathscr{I}_\alpha f| = |\mathscr{I}_\alpha(\tau_h f) - \mathscr{I}_\alpha f| \leqslant \mathscr{I}_\alpha(|\tau_h f - f|).$$

Taking $M_{p,\varphi}$-norm on both sides of the last inequality, we obtain the desired result by using the boundedness of $\mathscr{I}_\alpha$ from $M_{p,\varphi}(\mathbb{G}, w^p)$ to $M_{q,\varphi}(\mathbb{G}, w^q)$.

Thus the proof of the Theorem 6.10 is completed. □

From Theorem 6.10 we obtain the following Sobolev-Stein embedding inequality on generalized weighted Besov-Morrey space.

THEOREM 6.11. (Sobolev-Stein embedding on generalized weighted Besov-Morrey space) *Let* $1 < p < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, $w \in A_{p,q}(\mathbb{G})$, (φ_1, φ_2) *satisfy the condition* (4.11), $1 \leqslant \theta \leqslant \infty$ *and* $0 < s < 1$. *Then*

$$\|u\|_{BM^s_{q\theta,\varphi_2}(\mathbb{G},w^q)} \lesssim \|\nabla_{\mathscr{L}} u\|_{BM^s_{p\theta,\varphi_1}(\mathbb{G},w^p)} \text{ for every } u \in C_0^\infty(\mathbb{G}).$$

The Dirichlet problem for the Kohn-Laplacian on $\mathbb{G}$ belongs to Folland [3, 4]. In particular, our results lead to the following apriori estimate for the sub-Laplacian equation $\mathscr{L} f = g$.

THEOREM 6.12. *Let* $1 < p < q < \infty$, $w \in A_{p,q}(\mathbb{G})$, $0 < s < 1$, $1 \leqslant \theta \leqslant \infty$, $g \in BM^s_{p\theta,w^p}(\mathbb{G})$ *and* $\mathscr{L} f = g$

1) If $\frac{1}{q} = \frac{1}{p} - \frac{2}{Q}$, *and* (φ_1, φ_2) *satisfy the condition* (4.11), *then*

$$\|f\|_{BM^s_{q\theta,\varphi_2}(\mathbb{G},w^q)} \lesssim \|g\|_{BM^s_{p\theta,\varphi_1}(\mathbb{G},w^p)}.$$

2) If $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, *and* (φ_1, φ_2) *satisfy the condition* (4.11), *then*

$$\|X_i f\|_{BM^s_{q\theta,\varphi_2}(\mathbb{G},w^q)} \lesssim \|g\|_{BM^s_{p\theta,\varphi_1}(\mathbb{G},w^p)}, \quad i = 1, 2, \ldots, n.$$

The proof of Theorems 6.11 and 6.12 are similar to Theorem 6.9.

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