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CERTAIN QUANTUM ESTIMATES ON THE PARAMETERIZED INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

TINGSONG DU*, CHUNYAN LUO AND BO YU

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Abstract. The present paper aims to study the parameterized inequalities of Hadamard–Simpson type for quantum integrals. By employing a quantum integral identity of multi-parameter, we establish novel inequalities for a class of q-differentiable mappings, which are related to s-(α,m)-convex mappings. Moreover, we acquire estimation-type results by considering the boundedness and the Lipschitz condition. As applications, we present two illustrative examples and several quantum integral inequalities for the special means.

1. Introduction and preliminaries

1.1. Classical convexities and inequalities

Throughout this paper we let $\mathscr{K} \subseteq \mathbb{R}$ be a real interval and $\mathscr{K}^{\circ}$ be the interior of $\mathscr{K}$. We evoke, now, some basic definitions as follows.

DEFINITION 1.1. A mapping $f : \mathscr{K} \to \mathbb{R}$ is said to be a convex mapping on $\mathscr{K}$ if

$$f\big(t\nu_1 + (1-t)\nu_2\big) \leqslant tf(\nu_1) + (1-t)f(\nu_2) \tag{1.1}$$

holds for all $\nu_1, \nu_2 \in \mathscr{K}$ and $t \in [0,1]$.

DEFINITION 1.2. [10] A mapping $f : [0,\infty) \to [0,\infty)$ is said to be s-convex in the second sense if

$$f\big(t\nu_1 + (1-t)\nu_2\big) \leqslant t^s f(\nu_1) + (1-t)^s f(\nu_2) \tag{1.2}$$

holds for all $\nu_1, \nu_2 \in \mathscr{K}$ and $t \in [0,1]$.

The class of s-convex mappings in the second sense is usually denoted by K_s^2. Here is an example to illustrate that some mappings could be either an s-convex mapping in the second sense or not under different conditions.

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* Corresponding author.

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EXAMPLE 1.1. [21] Let $s \in (0,1)$ and $a,b,c \in \mathbb{R}$. Define the mapping $f : [0,\infty) \to \mathbb{R}$ as

$$f(t) = \begin{cases} a, & t = 0, \\ bt^s + c, & t > 0. \end{cases}$$

It can be easily checked that
(i) if $b \geqslant 0$ and $0 \leqslant c \leqslant a$, then $f \in K_s^2$,
(ii) if $b > 0$ and $c < 0$, then $f \notin K_s^2$.

Recall that a set $\mathscr{I}$ is m-convex if for any $v_1, v_2 \in \mathscr{I}$ and $t \in [0,1]$, $tv_1 + m(1-t)v_2 \in \mathscr{I}$, or equivalently, $mtv_1 + (1-t)v_2 \in \mathscr{I}$.

DEFINITION 1.3. [49] A mapping $f : \mathscr{I} \subseteq \mathbb{R} \to \mathbb{R}$ is said to be m-convex with $m \in [0,1]$ if

$$f\big(tv_1 + m(1-t)v_2\big) \leqslant tf(v_1) + m(1-t)f(v_2) \tag{1.3}$$

holds for all $v_1, v_2 \in \mathscr{I}$ and $t \in [0,1]$, where $\mathscr{I}$ is an m-convex set.

It is worthy to mention that the above inequality is equivalent to

$$f\big(mtv_1 + (1-t)v_2\big) \leqslant mtf(v_1) + (1-t)f(v_2). \tag{1.4}$$

DEFINITION 1.4. [32] A mapping $f : [0,b^*) \to \mathbb{R}$ is called (α,m)-convex if for all $v_1, v_2 \in [0,b^*)$ with $b^* > 0$ and $t \in [0,1]$, the following inequality holds:

$$f\big(tv_1 + m(1-t)v_2\big) \leqslant t^\alpha f(v_1) + m(1-t^\alpha)f(v_2), \tag{1.5}$$

where $(\alpha,m) \in (0,1] \times (0,1]$.

Clearly, if $m = 1$, then an (α,m)-convex mapping is reduced to an α-convex mapping.

DEFINITION 1.5. [34] A mapping $f : [0,\infty) \to [0,\infty)$ is called an s-(α,m)-convex mapping in the second sense, if for all $v_1, v_2 \in [0,\infty)$ and $t \in [0,1]$, the following inequality holds:

$$f\big(tv_1 + (1-t)v_2\big) \leqslant t^{\alpha s} f(v_1) + m(1-t^\alpha)^s f\left(\frac{v_2}{m}\right), \tag{1.6}$$

where $(\alpha,m) \in (0,1] \times (0,1]$ and for certain fixed $s \in (0,1]$.

Note that for $s = 1 = m$, $\alpha = 1 = m$ as well as $s = \alpha = m = 1$, one obtains the following classes of mappings respectively: α-convex, s-convex and convex. A series of works ([8, 11, 24, 26, 37, 38, 39, 41, 44] and references therein) are devoted to convex, s-convex, α-convex, m-convex, (α,m)-convex and (α,s,m)-convex mappings and some integral inequalities are established.

In fact, in terms of different convexity, many scholars have extended different kinds of integral type inequalities, among which the Hermite–Hadamard inequality and Simpson inequality are the most famous. The Hermite–Hadamard inequality is stated as follows: If $f:\mathscr{K}\to\mathbb{R}$ is a convex mapping defined on the interval $\mathscr{K}$, $\nu_1,\nu_2\in\mathscr{K}$ with $\nu_1<\nu_2$, then

$$f\left(\frac{\nu_1+\nu_2}{2}\right)\leqslant\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}f(t)\mathrm{d}t\leqslant\frac{f(\nu_1)+f(\nu_2)}{2}. \tag{1.7}$$

Another classical inequality called Simpson's inequality is described by: If $f:\mathscr{K}\to\mathbb{R}$ is four-order differentiable on $\mathscr{K}^\circ$, where $\left\|f^{(4)}\right\|_\infty=\sup_{t\in\mathscr{K}^\circ}\left|f^{(4)}(t)\right|<\infty$, then

$$\left|\frac{1}{6}\left[f(\nu_1)+4f\left(\frac{\nu_1+\nu_2}{2}\right)+f(\nu_2)\right]-\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2}f(t)\mathrm{d}t\right|\leqslant\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(\nu_2-\nu_1)^4. \tag{1.8}$$

Considering the importance of Hermite–Hadamard and Simpson inequalities, many researches generalized and extended them. Recently, Dragomir and Nikodem [16], Khan et al. [23], Abramovich and Persson [1], Latif and Dragomir [27], Liao et al. [29], Hwang and Dragomir [22], have obtained many Hermite–Hadamard type inequalities for differentiable mappings which are strongly convex, s-convex, N-quasiconvex, preinvex, α-preinvex and r-preinvex. Further results involving these two inequalities with applications to fractional integrals have been carried out by many researchers, including Chen and Katugampola [13] in the study of the Hermite–Hadamard type inequalities using the Katugampola fractional integrals, Ahmad et al. [3] in the Hermite–Hadamard inequalities for new fractional integral operators with exponential kernel, as well as Set et al. [43] in the Simpson-type inequalities for convex mappings via Riemann–Liouville integrals. For more results related to these two inequalities, see, for example, [2, 7, 17, 19, 20, 31, 33] and the references cited therein.

1.2. Quantum integral inequalities

Quantum calculus, also known as q-calculus, refers to the study of calculus without limits. Quantum calculus started its story when Euler introduced the parameter q into Newton's infinite series. Later in the early 20th century, Jackson(1910) began to study the symmetry of q-calculus and introduced q-definite integral. This theme has been widely used in various areas of mathematics and physics including basic hypergeometric functions, combinatorics, orthogonal polynomials, quantum theory, number theory, relativity theory and mechanics. Quantum calculus has aroused great interest of many researchers because it is considered as an incorporative subject between mathematics and physics. The research of mathematical inequalities related to quantum integral operators, especially Hermite–Hadamard's inequality and Simpson-type inequality, is a current research focus. At present, q-analogues of some identities and inequalities have been obtained, see [6, 28, 40].

Now let us recall some concepts related to quantum integrals. These concepts are mainly due to Tariboon and Ntouyas [47].

DEFINITION 1.6. If the mapping $f : \mathscr{K} \to \mathbb{R}$ is continuous, then the q-derivative of f at $t \in \mathscr{K}$ is defined by the expression

$$ {}_{\nu_1}D_q f(t) = \frac{f(t) - f(qt + (1-q)\nu_1)}{(1-q)(t-\nu_1)}, \quad t \neq \nu_1. \tag{1.9} $$

Since f is a continuous mapping, one has that ${}_{\nu_1}D_q f(\nu_1) = \lim_{t\to\nu_1} {}_{\nu_1}D_q f(t)$.

DEFINITION 1.7. Let $f : \mathscr{K} \to \mathbb{R}$ be continuous, the q-integral with $t \in \mathscr{K}$ is defined as

$$ \int_{\nu_1}^{t} f(u)_{\nu_1}\mathrm{d}_q u = (1-q)(t-\nu_1)\sum_{n=0}^{\infty} q^n f\big(q^n t + (1-q^n)\nu_1\big). \tag{1.10} $$

Moreover, if $\tau \in (\nu_1, t)$, then the definite q-integral on $[\tau, t]$ is described as

$$ \int_{\tau}^{t} f(u)_{\nu_1}\mathrm{d}_q u = \int_{\nu_1}^{t} f(u)_{\nu_1}\mathrm{d}_q u - \int_{\nu_1}^{\tau} f(u)_{\nu_1}\mathrm{d}_q u. \tag{1.11} $$

In [45], Sudsutad et al. established the following lemma.

LEMMA 1.1. *Let* $f : \mathscr{K} \to \mathbb{R}$ *be a continuous and* q*-differentiable mapping with* $0 < q < 1$ *on* $\mathscr{K}^\circ$. *If* ${}_{\nu_1}D_q f$ *is integrable on* $\mathscr{K}$, *then the following equation holds:*

$$ \begin{aligned} &\frac{qf(\nu_1)+f(\nu_2)}{1+q} - \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} f(u)_{\nu_1}\mathrm{d}_q u \\ &= \frac{q(\nu_2-\nu_1)}{1+q}\int_0^1 \big((1+q)t-1\big)_{\nu_1}D_q f\big((1-t)\nu_1 + t\nu_2\big)_0\mathrm{d}_q t. \end{aligned} \tag{1.12} $$

In the same paper, Sudsutad et al. gave the following quantum estimation for the upper bound of the left hand-side of equation (1.12) via convexity.

THEOREM 1.1. *Let* $f : \mathscr{K} \to \mathbb{R}$ *be a continuous mapping with* $0 < q < 1$. *If* $|{}_{\nu_1}D_q f|^\sigma$, $\sigma \geqslant 1$, *is convex and integrable on* $\mathscr{K}^\circ$, *then the following inequality holds:*

$$ \begin{aligned} &\left|\frac{qf(\nu_1)+f(\nu_2)}{1+q} - \frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_1} f(u)_{\nu_1}\mathrm{d}_q u\right| \\ &\leqslant \frac{q^2(q^2+q+2)(\nu_2-\nu_1)}{(q+1)^4}\left[\frac{(q^2+4q+1)\left|{}_{\nu_1}D_q f(\nu_2)\right|^\sigma + (2q^3+3q^2+1)\left|{}_{\nu_1}D_q f(\nu_1)\right|^\sigma}{(q^2+q+1)(q^3+q+2)}\right]^{\frac{1}{\sigma}}. \end{aligned} $$

In [50], Tunç and Balgeçti presented the following lemma and developed the corresponding quantum estimates.

LEMMA 1.2. *Let $f : \mathscr{K} \to \mathbb{R}$ be a continuous and q-differentiable mapping with $0 < q < 1$ on $\mathscr{K}^\circ$. If ${}_{\nu_1}D_q f$ is integrable on $\mathscr{K}$, then the following equation holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(\nu_1)+4f\left(\frac{\nu_1+\nu_2}{2}\right)+f(\nu_2)\right]-\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} f(u)_{\nu_1}\mathrm{d}_q u \\
&= (\nu_2-\nu_1)\left\{\int_0^{\frac{1}{2}}\left(qt-\frac{1}{6}\right)_{\nu_1}D_q f\big((1-t)\nu_1+t\nu_2\big)_0\mathrm{d}_q t \right.\\
&\quad \left. +\int_{\frac{1}{2}}^{1}\left(qt-\frac{5}{6}\right)_{\nu_1}D_q f\big((1-t)\nu_1+t\nu_2\big)_0\mathrm{d}_q t\right\}.
\end{aligned}$$

THEOREM 1.2. *Let $f : \mathscr{K} \to \mathbb{R}$ be a continuous mapping with $0 < q < 1$. If $|{}_{\nu_1}D_q f|$ is convex and integrable on $\mathscr{K}^\circ$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(\nu_1)+4f\left(\frac{\nu_1+\nu_2}{2}\right)+f(\nu_2)\right]-\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} f(u)_{\nu_1}\mathrm{d}_q u\right| \\
&\leqslant \frac{\nu_2-\nu_1}{12}\left[\frac{1+4q+4q^2+6q^3}{3(1+2q+2q^2+q^3)}\left|{}_{\nu_1}D_q f(\nu_1)\right|+\frac{1+2q+2q^2}{1+2q+2q^2+q^3}\left|{}_{\nu_1}D_q f(\nu_2)\right|\right].
\end{aligned}$$

THEOREM 1.3. *Let $f : \mathscr{K} \to \mathbb{R}$ be a q-differentiable mapping on $\mathscr{K}^\circ$ with $0 < q < 1$. If $|{}_{\nu_1}D_q f|^\rho$ is convex and integrable on $\mathscr{K}^\circ$ where $\rho, \rho > 1$, $\frac{1}{\rho}+\frac{1}{\rho} = 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(\nu_1)+4f\left(\frac{\nu_1+\nu_2}{2}\right)+f(\nu_2)\right]-\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} f(u)_{\nu_1}\mathrm{d}_q u\right| \\
&\leqslant \frac{(\nu_2-\nu_1)}{2^{\frac{1}{\rho}}}\left(\frac{(1-q)}{6^{\rho+1}q(1-q^{\rho+1})}\right)^{\frac{1}{\rho}} \\
&\quad\times\left\{\left(1+(3q-1)^{\rho+1}\right)^{\frac{1}{\rho}}\left(\left|{}_{\nu_1}D_q f(\nu_1)\right|^\rho+\left|{}_{\nu_1}D_q f\left(\frac{\nu_1+\nu_2}{2}\right)\right|^\rho\right)^{\frac{1}{\rho}}\right. \\
&\quad\left.+\left[(5-3q)^{\rho+1}+(6q-5)^{\rho+1}\right]^{\frac{1}{\rho}}\left(\left|{}_{\nu_1}D_q f(\nu_2)\right|^\rho+\left|{}_{\nu_1}D_q f\left(\frac{\nu_1+\nu_2}{2}\right)\right|^\rho\right)^{\frac{1}{\rho}}\right\}.
\end{aligned}$$

In 2018, Alp et al. extended the Hermite–Hadamard's inequality to the version of quantum integrals as follows.

THEOREM 1.4. ([5]) *Let $f : \mathscr{K} \to \mathbb{R}$ be convex and q-differentiable on $\mathscr{K}$ with $0 < q < 1$. Then we have that*

$$f\left(\frac{q\nu_1+\nu_2}{1+q}\right)\leqslant\frac{1}{\nu_2-\nu_1}\int_{\nu_1}^{\nu_2} f(u)_{\nu_1}\mathrm{d}_q u\leqslant\frac{qf(\nu_1)+f(\nu_2)}{1+q}. \tag{1.13}$$

The family of quantum integral inequalities associated with different kinds of convex mappings is an interesting topic. For example, Liu and Zhuang [30], Noor et al. [36], Zhuang et al. [54], Riahi et al. [42], and Zhang et al. [53], have presented some quantum integral inequalities of Hermite–Hadamard type for differentiable mappings which are convex, strongly convex, quasi-convex, s-convex and (α,m)-convex. Further results involving other forms of quantum integral inequalities have been studied by many researchers, including Chen and Yang [12] in the study of Chebyshev type quantum integral inequalities on finite intervals, Sudsutad et al. [46] in the fractional quantum integral inequalities for the new q-shifting operator ${}_a\Phi_q(m) = qm + (1-q)a$, Yang [52] in the quantum integral inequalities of Fejér type on finite intervals, as well as Bin-Mohsin et al. [9] in the quantum Hermite–Hadamard inequalities of the q-Jackson integral operator in terms of harmonic convexity. For more results related to the quantum integral inequalities, the interested reader is directed to [35, 25, 48] and the references cited therein.

Inspired by the results mentioned above, especially the results developed in [45] and [50], we notice that it is possible to treat these results uniformly through quantum integral operators. For this purpose, we will establish a general quantum integral identity for q-differentiable mappings. Using this quantum integral identity, we derive certain parameterized quantum integral inequalities, which unifies the Simpson-type inequality, the averaged midpoint–trapezoid inequality, as well as the trapezoid-like inequality. This is the main contribution of this work.

The remainder of the paper is organized as follows. Some new quantum integral inequalities, including Simpson-type, averaged midpoint–trapezoid, as well as trapezoid-like inequalities for s-(α,m)-convex mappings are established in Section 2, where a quantum integral identity of multi-parameter is utilized. Several further estimation-type results, by considering the boundedness and the Lipschitz condition of ${}_aD_qf(x)$, are obtained in Section 3. Two illustrative examples are presented in Section 4 and several quantum integral inequalities for some special means are given in Section 5. Finally, a conclusion is drawn in Section 6.

2. Main results

The principal goal of this section is to prove the Hadamard–Simpson type inequalities for s-(α,m)-convex mappings through quantum integrals. To this end, we present the following lemma.

LEMMA 2.1. *Let $f:\mathscr{K}\to\mathbb{R}$ be a continuous and q-differentiable mapping on $\mathscr{K}^\circ$ with $0<q<1$. If ${}_aD_qf$ is an integrable mapping on $\mathscr{K}$, then for each $u\in[a,b]$ the following identity*

$$\lambda f(a)+(1-k)f(b)+(k-\lambda)f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)\,{}_a\mathrm{d}_qu \tag{2.1}$$

$$=(b-a)\bigg\{\int_0^{\frac12}(qt-\lambda)\,{}_aD_qf\big((1-t)a+tb\big)\,{}_0\mathrm{d}_qt+\int_{\frac12}^1(qt-k)\,{}_aD_qf\big((1-t)a+tb\big)\,{}_0\mathrm{d}_qt\bigg\}$$

holds for all $k,\lambda\in\mathbb{R}$.

Proof. On one hand, using Definition 1.6 and Definition 1.7, one has that

$$
\begin{aligned}
&\int_0^{\frac{1}{2}}(qt-\lambda)_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt\\
&=\int_0^{\frac{1}{2}}qt_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt-\lambda\int_0^{\frac{1}{2}}{}_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt\\
&=\int_0^{\frac{1}{2}}q\frac{f\big((1-t)a+tb\big)-f\big((1-qt)a+qtb\big)}{(1-q)(b-a)}{}_0\mathrm{d}_qt\\
&\quad-\lambda\int_0^{\frac{1}{2}}\frac{f\big((1-t)a+tb\big)-f\big((1-qt)a+qtb\big)}{t(1-q)(b-a)}{}_0\mathrm{d}_qt\\
&=\frac{1}{2}\sum_{n=0}^{\infty}q^{n+1}\frac{f\big((1-\frac{1}{2}q^n)a+\frac{1}{2}q^nb\big)}{b-a}-\frac{1}{2}\sum_{n=0}^{\infty}q^{n+1}\frac{f\big((1-\frac{1}{2}q^{n+1})a+\frac{1}{2}q^{n+1}b\big)}{b-a}\\
&\quad-\lambda\sum_{n=0}^{\infty}\frac{f\big((1-\frac{1}{2}q^n)a+\frac{1}{2}q^nb\big)}{b-a}+\lambda\sum_{n=0}^{\infty}\frac{f\big((1-\frac{1}{2}q^{n+1})a+\frac{1}{2}q^{n+1}b\big)}{b-a}\\
&=\frac{1}{2}q\sum_{n=0}^{\infty}q^n\frac{f\big((1-\frac{1}{2}q^n)a+\frac{1}{2}q^nb\big)}{b-a}-\frac{1}{2}\sum_{n=1}^{\infty}q^n\frac{f\big((1-\frac{1}{2}q^n)a+\frac{1}{2}q^nb\big)}{b-a}\\
&\quad-\frac{\lambda}{b-a}\sum_{n=0}^{\infty}f\Big(\Big(1-\frac{1}{2}q^n\Big)a+\frac{1}{2}q^nb\Big)+\frac{\lambda}{b-a}\sum_{n=1}^{\infty}f\Big(\Big(1-\frac{1}{2}q^n\Big)a+\frac{1}{2}q^nb\Big)\\
&=-\frac{1-q}{2(b-a)}\sum_{n=0}^{\infty}q^nf\Big(\Big(1-\frac{1}{2}q^n\Big)a+\frac{1}{2}q^nb\Big)+\frac{1}{2(b-a)}f\Big(\frac{a+b}{2}\Big)\\
&\quad-\frac{\lambda}{(b-a)}\Big[f\Big(\frac{a+b}{2}\Big)-f(a)\Big]\\
&=-\frac{1}{b-a}\int_0^{\frac{1}{2}}f\big(tb+(1-t)a\big)_0\mathrm{d}_qt+\Big(\frac{1}{2}-\lambda\Big)\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)+\frac{\lambda}{b-a}f(a).
\end{aligned}
\tag{2.2}
$$

On the other hand, according to equality (1.11), we have that

$$
\begin{aligned}
&\int_{\frac{1}{2}}^{1}(qt-k)_aD_qf\big(tb+(1-t)a\big)_0\mathrm{d}_qt\\
&=\int_0^1(qt-k)_aD_qf\big(tb+(1-t)a\big)_0\mathrm{d}_qt-\int_0^{\frac{1}{2}}(qt-k)_aD_qf\big(tb+(1-t)a\big)_0\mathrm{d}_qt.
\end{aligned}
\tag{2.3}
$$

Similar technique yields that

$$
\begin{aligned}
&\int_0^1(qt-k)_aD_qf\big(tb+(1-t)a\big)_0\mathrm{d}_qt\\
&=-\frac{1}{b-a}\int_0^1f\big(tb+(1-t)a\big)_0\mathrm{d}_qt+(1-k)\frac{1}{b-a}f(b)+\frac{k}{b-a}f(a)
\end{aligned}
\tag{2.4}
$$

and

$$\begin{aligned}&\int_0^{\frac{1}{2}}(qt-k)_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt\\&=-\frac{1}{b-a}\int_0^{\frac{1}{2}}f\big(tb+(1-t)a\big)_0\mathrm{d}_qt+\Big(\frac{1}{2}-k\Big)\frac{1}{b-a}f\Big(\frac{a+b}{2}\Big)+\frac{k}{b-a}f(a).\end{aligned}\tag{2.5}$$

Now, let us substitute (2.4) and (2.5) into (2.3), and then add the corresponding results to (2.2), finally we have that

$$\begin{aligned}&\int_0^{\frac{1}{2}}(qt-\lambda)_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt+\int_{\frac{1}{2}}^1(qt-k)_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt\\&=\frac{\lambda}{b-a}f(a)+\frac{1-k}{b-a}f(b)+\frac{k-\lambda}{b-a}f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_0^1f\big(tb+(1-t)a\big)_0\mathrm{d}_qt\\&=\frac{\lambda}{b-a}f(a)+\frac{1-k}{b-a}f(b)+\frac{k-\lambda}{b-a}f\Big(\frac{a+b}{2}\Big)-\frac{1}{(b-a)^2}\int_a^bf(u)_a\mathrm{d}_qu,\end{aligned}$$

which proves the desired result in (2.1). □

REMARK 2.1. Consider Lemma 2.1.

(i) Taking $\lambda=\frac{1}{6}$ and $k=\frac{5}{6}$, we have Lemma 3 presented by Tunç and Balgeçti in [50].

(ii) Taking $\lambda=k=\frac{q}{1+q}$, we have Lemma 3.1 provided by Sudsutad et al. in [45].

(iii) Taking $\lambda=\frac{1}{4}$ and $k=\frac{3}{4}$, we have that

$$\begin{aligned}&\frac{1}{4}\Big[f(a)+2f\Big(\frac{a+b}{2}\Big)+f(b)\Big]-\frac{1}{b-a}\int_a^bf(u)_a\mathrm{d}_qu\\&=(b-a)\Big\{\int_0^{\frac{1}{2}}\Big(qt-\frac{1}{4}\Big)_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt\\&\quad+\int_{\frac{1}{2}}^1\Big(qt-\frac{3}{4}\Big)_aD_qf\big((1-t)a+tb\big)_0\mathrm{d}_qt\Big\},\end{aligned}$$

which is a new form of Lemma 2.1.

REMARK 2.2. If we take $q\to1^-$ on both sides of equation (2.1), then we have Lemma 2.1 established by Du et al. in [18] for the case of $m=1$.

Considering s-(α,m)-convexity, Lemma 2.1 can be applied to obtain our first parameterized bound as follows.

THEOREM 2.1. *Let* $f : \mathscr{K} \to \mathbb{R}$ *be a continuous function with* $0 < q < 1$. *If* $|_aD_qf|$ *is an* s-(α,m)*-convex function on* $\mathscr{K}^\circ$*, then the following inequality*

$$\left|\lambda f(a)+(1-k)f(b)+(k-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,{}_a\mathrm{d}_qu\right|$$
$$\leqslant (b-a)\left\{\left(2^{1-s}\mathscr{T}_1(\lambda;q)-\mathscr{T}_3(\alpha,s,\lambda;q)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+\mathscr{T}_3(\alpha,s,\lambda;q)\left|{}_aD_qf(b)\right|\right.$$
$$\left.+\left(2^{1-s}\mathscr{T}_2(k;q)-\mathscr{T}_4(\alpha,s,k;q)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+\mathscr{T}_4(\alpha,s,k;q)\left|{}_aD_qf(b)\right|\right\} \tag{2.6}$$

holds for $k,\lambda \in [0,1]$*, where*

$$\mathscr{T}_1(\lambda;q)=\int_0^{\frac{1}{2}}|qt-\lambda|\,{}_0\mathrm{d}_qt=\begin{cases}\frac{8\lambda^2-2\lambda(1+q)+q}{4(1+q)}, & 0\leqslant\frac{\lambda}{q}\leqslant\frac{1}{2},\\ \frac{2\lambda(1+q)-q}{4(1+q)}, & \frac{1}{2}<\frac{\lambda}{q},\end{cases} \tag{2.7}$$

$$\mathscr{T}_2(k;q)=\int_{\frac{1}{2}}^{1}|qt-k|\,{}_0\mathrm{d}_qt=\begin{cases}\frac{3q-2k(1+q)}{4(1+q)}, & 0\leqslant\frac{k}{q}\leqslant\frac{1}{2},\\ \frac{8k^2-6k(1+q)+5q}{4(1+q)}, & \frac{1}{2}<\frac{k}{q}\leqslant 1,\\ \frac{2k(1+q)-3q}{4(1+q)}, & 1<\frac{k}{q},\end{cases} \tag{2.8}$$

$$\begin{aligned}\mathscr{T}_3(\alpha,s,\lambda;q)&=\int_0^{\frac{1}{2}}|qt-\lambda|t^{\alpha s}\,{}_0\mathrm{d}_qt\\ &=\begin{cases}\frac{2\lambda^{\alpha s+2}(1-q)^2}{(1-q^{\alpha s+1})(1-q^{\alpha s+2})}\\ \quad+\frac{(1-q)(q-\lambda)+(q-1)(1-\lambda)q^{\alpha s+2}}{2^{\alpha s+2}(1-q^{\alpha s+1})(1-q^{\alpha s+2})}, & 0\leqslant\frac{\lambda}{q}\leqslant\frac{1}{2},\\ -\frac{(1-q)(q-\lambda)+(q-1)(1-\lambda)q^{\alpha s+2}}{2^{\alpha s+2}(1-q^{\alpha s+1})(1-q^{\alpha s+2})}, & \frac{1}{2}<\frac{\lambda}{q}\end{cases}\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}\mathscr{T}_4(\alpha,s,k;q)&=\int_{\frac{1}{2}}^{1}|qt-k|t^{\alpha s}\,{}_0\mathrm{d}_qt\\ &=\begin{cases}\frac{k(1-q)(1-2^{\alpha s+1})}{2^{\alpha s+1}(1-q^{\alpha s+1})}+\frac{q(1-q)(2^{\alpha s+2}-1)}{2^{\alpha s+2}(1-q^{\alpha s+2})}, & 0\leqslant\frac{k}{q}\leqslant\frac{1}{2},\\ -\frac{k(1-q)(1+2^{\alpha s+1})}{2^{\alpha s+1}(1-q^{\alpha s+1})}+\frac{q(1-q)(1+2^{\alpha s+2})}{2^{\alpha s+2}(1-q^{\alpha s+2})}\\ \quad+\frac{2k^{\alpha s+2}(1-q)^2}{(1-q^{\alpha s+1})(1-q^{\alpha s+2})}, & \frac{1}{2}<\frac{k}{q}\leqslant 1,\\ -\frac{k(1-q)(1-2^{\alpha s+1})}{2^{\alpha s+1}(1-q^{\alpha s+1})}+\frac{q(1-q)(1-2^{\alpha s+2})}{2^{\alpha s+2}(1-q^{\alpha s+2})}, & 1<\frac{k}{q}.\end{cases}\end{aligned} \tag{2.10}$$

Proof. Combining Lemma 2.1 with the s-(α,m)-convexity of $|{}_aD_qf|$ on $\mathscr{K}^\circ$ yields that

$$\begin{aligned}
&\left|\lambda f(a)+(1-k)f(b)+(k-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,{}_a\mathrm{d}_qu\right| \\
&\leqslant (b-a)\left\{\int_0^{\frac12}\left|qt-\lambda\right|\left|{}_aD_qf\big((1-t)a+tb\big)\right|{}_0\mathrm{d}_qt\right. \\
&\qquad\left.+\int_{\frac12}^1\left|qt-k\right|\left|{}_aD_qf\big((1-t)a+tb\big)\right|{}_0\mathrm{d}_qt\right\} \\
&\leqslant (b-a)\left\{\int_0^{\frac12}\left|qt-\lambda\right|\left[m(1-t^\alpha)^s\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+t^{\alpha s}\left|{}_aD_qf(b)\right|\right]{}_0\mathrm{d}_qt\right. \\
&\qquad\left.+\int_{\frac12}^1\left|qt-k\right|\left[m(1-t^\alpha)^s\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+t^{\alpha s}\left|{}_aD_qf(b)\right|\right]{}_0\mathrm{d}_qt\right\}.
\end{aligned}$$

Using the inequality $(1-t^\alpha)^s\leqslant 2^{1-s}-t^{\alpha s}$ for $t\in[0,1]$ with certain fixed $\alpha\in(0,1]$ and $s\in(0,1]$, we get that

$$\int_0^{\frac12}\left|qt-\lambda\right|(1-t^\alpha)^s{}_0\mathrm{d}_qt\leqslant\int_0^{\frac12}\left|qt-\lambda\right|\left(2^{1-s}-t^{\alpha s}\right){}_0\mathrm{d}_qt.$$

Similarly,

$$\int_{\frac12}^1\left|qt-k\right|(1-t^\alpha)^s{}_0\mathrm{d}_qt\leqslant\int_{\frac12}^1\left|qt-k\right|\left(2^{1-s}-t^{\alpha s}\right){}_0\mathrm{d}_qt.$$

This ends the proof. □

Particular cases are stated as follows.

COROLLARY 2.1. *Consider Theorem 2.1.*
(i) Taking $\lambda=\frac16$ and $k=\frac56$, we obtain the Simpson-type integral inequality

$$\begin{aligned}
&\left|\frac16\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)\,{}_a\mathrm{d}_qu\right| \\
&\leqslant (b-a)\left\{\left(2^{1-s}\mathscr{T}_1\left(\frac16;q\right)-\mathscr{T}_3\left(\alpha,s,\frac16;q\right)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+\mathscr{T}_3\left(\alpha,s,\frac16;q\right)\left|{}_aD_qf(b)\right|\right. \\
&\quad\left.+\left(2^{1-s}\mathscr{T}_2\left(\frac56;q\right)-\mathscr{T}_4\left(\alpha,s,\frac56;q\right)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+\mathscr{T}_4\left(\alpha,s,\frac56;q\right)\left|{}_aD_qf(b)\right|\right\}.
\end{aligned}$$

In particular, if $s=\alpha=m=1$ and $q\to1^-$, then we have that

$$\left|\frac16\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\leqslant\frac{5(b-a)}{72}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

which is established by Alomari et al. in [4, Corollary 1].

(*ii*) *Taking* $\lambda=\frac{1}{4}$ *and* $k=\frac{3}{4}$, *we obtain the averaged midpoint-trapezoid integral inequality*

$$\begin{aligned}&\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right|\\&\leqslant(b-a)\left\{\left(2^{1-s}\mathscr{T}_1\left(\frac{1}{4};q\right)-\mathscr{T}_3\left(\alpha,s,\frac{1}{4};q\right)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+\mathscr{T}_3\left(\alpha,s,\frac{1}{4};q\right)\left|{}_aD_qf(b)\right|\right.\\&\quad\left.+\left(2^{1-s}\mathscr{T}_2\left(\frac{3}{4};q\right)-\mathscr{T}_4\left(\alpha,s,\frac{3}{4};q\right)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|+\mathscr{T}_4\left(\alpha,s,\frac{3}{4};q\right)\left|{}_aD_qf(b)\right|\right\}.\end{aligned}$$

In particular, if $s=\alpha=m=1$ *and* $q\to1^-$, *then we have that*

$$\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right|\leqslant\frac{(b-a)}{16}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

which is established by Xi and Qi in [51, Corollary 3.4].

(*iii*) *Taking* $\lambda=k=\frac{q}{1+q}$, *we obtain the trapezoid-like integral inequality*

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right|\\&\leqslant(b-a)\left\{\left(2^{1-s}\mathscr{T}_1\left(\frac{q}{1+q};q\right)-\mathscr{T}_3\left(\alpha,s,\frac{q}{1+q};q\right)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\right.\\&\quad+\left(2^{1-s}\mathscr{T}_2\left(\frac{q}{1+q};q\right)-\mathscr{T}_4\left(\alpha,s,\frac{q}{1+q};q\right)\right)m\left|{}_aD_qf\left(\frac{a}{m}\right)\right|\\&\quad\left.+\mathscr{T}_3\left(\alpha,s,\frac{q}{1+q};q\right)\left|{}_aD_qf(b)\right|+\mathscr{T}_4\left(\alpha,s,\frac{q}{1+q};q\right)\left|{}_aD_qf(b)\right|\right\}.\end{aligned}$$

In particular, if $s=\alpha=m=1$, *then we have that*

$$\begin{aligned}&\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right|\\&\leqslant(b-a)\left\{\frac{q^2(1+3q^2+2q^3)}{(1+q)^4(1+q+q^2)}\left|{}_aD_qf(a)\right|+\frac{q^2(1+4q+q^2)}{(1+q)^4(1+q+q^2)}\left|{}_aD_qf(b)\right|\right\},\end{aligned}$$

which is established by Sudsutad et al. in [45, Theorem 4.1].

A similar result is available for $e_1>1$ and $|{}_aD_qf|^{e_1}$ being s-(α,m)-convex on $\mathscr{K}$.

THEOREM 2.2. *Let* $f:\mathscr{K}\to\mathbb{R}$ *be continuous and* q*-differentiable on* $\mathscr{K}^\circ$ *and* ${}_aD_qf$ *be integrable on* $\mathscr{K}$. *If* $|{}_aD_qf|^{e_1}$ *with* $e_1>1$ *is* s-(α,m)*-convex on* $\mathscr{K}$ *and* e_2

is the conjugate index of e_1, *i.e.,* $e_1^{-1}+e_2^{-1}=1$, *then the following inequality*

$$\begin{aligned}
&\left|\lambda f(a)+(1-k)f(b)+(k-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right|\\
&\leqslant (b-a)\Bigg\{\Phi_3^{\frac{1}{e_1}}(\lambda,q;e_1)\left[m\Big(2^{-s}-\Phi_1(\alpha,s,q)\Big)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^{e_2}\right.\\
&\quad\left.+\Phi_1(\alpha,s,q)\left|{}_aD_qf(b)\right|^{e_2}\right]^{\frac{1}{e_2}}\\
&\quad+\Phi_4^{\frac{1}{e_1}}(k,q;e_1)\left[m\Big(2^{-s}-\Phi_2(\alpha,s,q)\Big)\left|{}_aD_qf\left(\frac{a}{m}\right)\right|^{e_2}+\Phi_2(\alpha,s,q)\left|{}_aD_qf(b)\right|^{e_2}\right]^{\frac{1}{e_2}}\Bigg\}
\end{aligned} \tag{2.11}$$

holds for $k,\lambda\in[0,1]$, *where*

$$\Phi_1(\alpha,s,q)=\frac{1-q}{2^{\alpha s+1}(1-q^{\alpha s+1})},$$

$$\Phi_2(\alpha,s,q)=\frac{(1-q)(2^{\alpha s+1}-1)}{2^{\alpha s+1}(1-q^{\alpha s+1})},$$

$$\begin{aligned}
\Phi_3(\lambda,q;e_1)&=\int_0^{\frac{1}{2}}|qt-\lambda|^{e_1}{}_0\mathrm{d}_qt\\
&=\begin{cases}\frac{1-q}{2}\sum\limits_{n=0}^{\infty}q^n\left(\frac{q^{n+1}}{2}-\lambda\right)^{e_1}\\ \qquad+\frac{2(1-q)\lambda^{e_1+1}}{q}\sum\limits_{n=0}^{\infty}q^n\Big(1-q^n\Big)^{e_1}, & 0\leqslant\frac{\lambda}{q}\leqslant\frac{1}{2},\\ \frac{1-q}{2}\sum\limits_{n=0}^{\infty}q^n\left(\lambda-\frac{q^{n+1}}{2}\right)^{e_1}, & \frac{1}{2}\leqslant\frac{\lambda}{q}\end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\Phi_4(k,q;e_1)&=\int_{\frac{1}{2}}^{1}|qt-k|^{e_1}{}_0\mathrm{d}_qt\\
&=\begin{cases}(1-q)\sum\limits_{n=0}^{\infty}q^n(q^{n+1}-k)^{e_1}-\frac{1-q}{2}\sum\limits_{n=0}^{\infty}q^n\left(\frac{q^{n+1}}{2}-k\right)^{e_1}, & 0\leqslant\frac{k}{q}\leqslant\frac{1}{2},\\ \frac{2(1-q)k^{e_1+1}}{q}\sum\limits_{n=0}^{\infty}q^n\Big(1-q^n\Big)^{e_1}+(1-q)\sum\limits_{n=0}^{\infty}q^n(q^{n+1}-k)^{e_1}\\ \qquad+\frac{1-q}{2}\sum\limits_{n=0}^{\infty}q^n\left(\frac{q^{n+1}}{2}-k\right)^{e_1}, & \frac{1}{2}\leqslant\frac{k}{q}\leqslant 1,\\ (1-q)\sum\limits_{n=0}^{\infty}q^n(k-q^{n+1})^{e_1}+\frac{1-q}{2}\sum\limits_{n=0}^{\infty}q^n\left(\frac{q^{n+1}}{2}-k\right)^{e_1}, & 1\leqslant\frac{k}{q}.\end{cases}
\end{aligned}$$

Proof. Evoking Lemma 2.1 and Hölder inequality, we have that

$$\begin{aligned}&\left|\lambda f(a)+(1-k)f(b)+(k-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)\,{}_a\mathrm{d}_q u\right|\\&\leqslant (b-a)\left\{\left(\int_0^{\frac12}|qt-\lambda|^{e_1}\,{}_0\mathrm{d}_q t\right)^{\frac{1}{e_1}}\left(\int_0^{\frac12}\left|{}_aD_q f\big((1-t)a+tb\big)\right|^{e_2}{}_0\mathrm{d}_q t\right)^{\frac{1}{e_2}}\right.\\&\quad\left.+\left(\int_{\frac12}^1|qt-k|^{e_1}\,{}_0\mathrm{d}_q t\right)^{\frac{1}{e_1}}\left(\int_{\frac12}^1\left|{}_aD_q f\big((1-t)a+tb\big)\right|^{e_2}{}_0\mathrm{d}_q t\right)^{\frac{1}{e_2}}\right\}.\end{aligned}$$

Considering the s-(α,m)-convexity of $|{}_aD_q f|^{e_2}$ and using the inequality $(1-t^{\alpha})^s \leqslant 2^{1-s}-t^{\alpha s}$, for $t\in[0,1]$ with certain fixed $\alpha\in(0,1]$ and $s\in(0,1]$, we get that

$$\begin{aligned}&\int_0^{\frac12}\left|{}_aD_q f\big((1-t)a+tb\big)\right|^{e_2}{}_0\mathrm{d}_q t\\&\leqslant \int_0^{\frac12}\left[m(1-t^{\alpha})^s\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^{e_2}+t^{\alpha s}\left|{}_aD_q f(b)\right|^{e_2}\right]{}_0\mathrm{d}_q t\\&\leqslant m\left(2^{-s}-\frac{1-q}{2^{\alpha s+1}(1-q^{\alpha s+1})}\right)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^{e_2}+\frac{1-q}{2^{\alpha s+1}(1-q^{\alpha s+1})}\left|{}_aD_q f(b)\right|^{e_2}\end{aligned}$$

and that

$$\begin{aligned}&\int_{\frac12}^1\left|{}_aD_q f\big((1-t)a+tb\big)\right|^{e_2}{}_0\mathrm{d}_q t\\&\leqslant m\left(2^{-s}-\frac{(1-q)(2^{\alpha s+1}-1)}{2^{\alpha s+1}(1-q^{\alpha s+1})}\right)\left|{}_aD_q f\left(\frac{a}{m}\right)\right|^{e_2}+\frac{(1-q)(2^{\alpha s+1}-1)}{2^{\alpha s+1}(1-q^{\alpha s+1})}\left|{}_aD_q f(b)\right|^{e_2}.\end{aligned}$$

Thus, the proof is completed. □

REMARK 2.3. By taking suitable choices of the special parameter λ and k in Theorem 2.2, we get several new results, for example, on Simpson-type, averaged midpoint-trapezoid, and trapezoid-like integral inequalities.

The following result presents an upper bound of q-integral inequality through the product of two s-(α,m)-convex mappings.

THEOREM 2.3. *Let $f,g:\mathscr{K}\to\mathbb{R}$ be continuous and nonnegative on $\mathscr{K}$. If f and g are s-(α_1,m)-convex and s-(α_2,m)-convex on $\mathscr{K}$, then the following q-inequality*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)g(x)\,{}_a\mathrm{d}_q x\\&\leqslant \frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}}f(b)g(b)+\left(\frac{2^{1-s}(1-q)}{1-q^{\alpha_2 s+1}}-\frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}}\right)mf\left(\frac{a}{m}\right)g(b)\end{aligned}$$

$$+\left(\frac{2^{1-s}(1-q)}{1-q^{\alpha_1 s+1}}-\frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}}\right)mf(b)g\left(\frac{a}{m}\right)$$
$$+\left(2^{2(1-s)}-\frac{2^{1-s}(1-q)}{1-q^{\alpha_1 s+1}}-\frac{2^{1-s}(1-q)}{1-q^{\alpha_2 s+1}}+\frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}}\right)m^2 f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right) \tag{2.12}$$

holds for certain fixed $\alpha_1,\alpha_2\in(0,1]$.

Proof. Considering the s-(α_1,m)-convexity of f and the s-(α_2,m)-convexity of g, respectively, we have, for all $t\in[0,1]$, that

$$f\big(tb+(1-t)a\big)\leqslant t^{\alpha_1 s}f(b)+m(1-t^{\alpha_1})^s f\left(\frac{a}{m}\right) \tag{2.13}$$

and that

$$g\big(tb+(1-t)a\big)\leqslant t^{\alpha_2 s}g(b)+m(1-t^{\alpha_2})^s g\left(\frac{a}{m}\right). \tag{2.14}$$

Multiplying both sides of (2.13) with corresponding parts of (2.14) and noticing that all these terms are nonnegative, we get that

$$f\big(tb+(1-t)a\big)g\big(tb+(1-t)a\big)$$
$$\leqslant t^{(\alpha_1+\alpha_2)s}f(b)g(b)+(1-t^{\alpha_1})^s t^{\alpha_2 s}mf\left(\frac{a}{m}\right)g(b)+(1-t^{\alpha_2})^s t^{\alpha_1 s}mf(b)g\left(\frac{a}{m}\right)$$
$$+(1-t^{\alpha_1})^s(1-t^{\alpha_2})^s m^2 f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right). \tag{2.15}$$

Taking q-integral for (2.15) with respect to t on $(0,1)$ and using the inequality $(1-t^{\alpha})^s\leqslant 2^{1-s}-t^{\alpha s}$, for $t\in(0,1)$ with certain fixed $\alpha\in(0,1]$ and $s\in(0,1]$, we obtain that

$$\int_0^1 f\big(tb+(1-t)a\big)g\big(tb+(1-t)a\big)\,{}_0\mathrm{d}_q t$$
$$\leqslant f(b)g(b)\int_0^1 t^{(\alpha_1+\alpha_2)s}\,{}_0\mathrm{d}_q t+mf\left(\frac{a}{m}\right)g(b)\int_0^1(1-t^{\alpha_1})^s t^{\alpha_2 s}\,{}_0\mathrm{d}_q t$$
$$+mf(b)g\left(\frac{a}{m}\right)\int_0^1(1-t^{\alpha_2})^s t^{\alpha_1 s}\,{}_0\mathrm{d}_q t$$
$$+m^2 f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1(1-t^{\alpha_1})^s(1-t^{\alpha_2})^s\,{}_0\mathrm{d}_q t$$
$$\leqslant f(b)g(b)\int_0^1 t^{(\alpha_1+\alpha_2)s}\,{}_0\mathrm{d}_q t+mf\left(\frac{a}{m}\right)g(b)\int_0^1(2^{1-s}-t^{\alpha_1 s})t^{\alpha_2 s}\,{}_0\mathrm{d}_q t$$
$$+mf(b)g\left(\frac{a}{m}\right)\int_0^1(2^{1-s}-t^{\alpha_2 s})t^{\alpha_1 s}\,{}_0\mathrm{d}_q t$$
$$+m^2 f\left(\frac{a}{m}\right)g\left(\frac{a}{m}\right)\int_0^1(2^{1-s}-t^{\alpha_1 s})(2^{1-s}-t^{\alpha_2 s})\,{}_0\mathrm{d}_q t$$

$$
\begin{aligned}
&= \frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}} f(b)g(b) + \left(\frac{2^{1-s}(1-q)}{1-q^{\alpha_2 s+1}} - \frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}}\right) m f\left(\frac{a}{m}\right) g(b) \\
&\quad + \left(\frac{2^{1-s}(1-q)}{1-q^{\alpha_1 s+1}} - \frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}}\right) m f(b) g\left(\frac{a}{m}\right) \\
&\quad + \left(2^{2(1-s)} - \frac{2^{1-s}(1-q)}{1-q^{\alpha_1 s+1}} - \frac{2^{1-s}(1-q)}{1-q^{\alpha_2 s+1}} + \frac{1-q}{1-q^{(\alpha_1+\alpha_2)s+1}}\right) m^2 f\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right).
\end{aligned} \tag{2.16}
$$

A simple calculation yields that

$$
\int_0^1 f\big(tb+(1-t)a\big) g\big(tb+(1-t)a\big)_0 \mathrm{d}_q t = \frac{1}{b-a} \int_a^b f(x)g(x)_a \mathrm{d}_q x. \tag{2.17}
$$

Combining (2.16) with (2.17), we deduce the desired result in (2.12). Thus, the proof is finished. □

COROLLARY 2.2. *If we take* $\alpha_1 = \alpha = \alpha_2$ *in Theorem 2.3, then we have that*

$$
\begin{aligned}
&\frac{1}{b-a} \int_a^b f(x)g(x)_a \mathrm{d}_q x \\
&\leqslant \frac{1-q}{1-q^{2\alpha s+1}} f(b)g(b) + \left(\frac{2^{1-s}(1-q)}{1-q^{\alpha s+1}} - \frac{1-q}{1-q^{2\alpha s+1}}\right) m \left[f\left(\frac{a}{m}\right) g(b) + f(b) g\left(\frac{a}{m}\right)\right] \\
&\quad + \left(2^{2(1-s)} - \frac{2^{2-s}(1-q)}{1-q^{\alpha s+1}} + \frac{1-q}{1-q^{2\alpha s+1}}\right) m^2 f\left(\frac{a}{m}\right) g\left(\frac{a}{m}\right).
\end{aligned}
$$

Specially, if $\alpha = 1 = m$, *then we have that*

$$
\begin{aligned}
&\frac{1}{b-a} \int_a^b f(x)g(x)_a \mathrm{d}_q x \\
&\leqslant \frac{1-q}{1-q^{2s+1}} f(b)g(b) + \left(\frac{2^{1-s}(1-q)}{1-q^{s+1}} - \frac{1-q}{1-q^{2s+1}}\right) \big[f(a)g(b) + f(b)g(a)\big] \\
&\quad + \left(2^{2(1-s)} - \frac{2^{2-s}(1-q)}{1-q^{s+1}} + \frac{1-q}{1-q^{2s+1}}\right) f(a)g(a).
\end{aligned}
$$

Furthermore, if we take $s = 1$, *then we get Theorem 4.3 established by Sudsutad et al. in* [45].

Our next result describes a lower bound for q-integral inequality via a product of two mappings.

THEOREM 2.4. *Let* $f : \mathscr{K} \to \mathbb{R}$ *be an* s-(α,m)*-convex mapping. If* $h : \mathscr{K} \to \mathbb{R}$ *is nonnegative and integrable on* $\mathscr{K}$ *with symmetric property about* $x = \frac{a+b}{2}$, *then the following* q*-integrable inequality holds:*

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right) \int_a^b h(x)_a \mathrm{d}_q x \\
&\leqslant 2^{-\alpha s} \int_a^b f(x)h(x)_a \mathrm{d}_q x + m\big(2^{1-s} - 2^{-\alpha s}\big) \int_a^b f\left(\frac{x}{m}\right) h(x)_a \mathrm{d}_q x.
\end{aligned} \tag{2.18}
$$

Proof. Since f is s-(α,m)-convex, for all $x,y\in[a,b]$ we have that

$$f\left(\frac{x+y}{2}\right)\leqslant 2^{-\alpha s}f(x)+m(1-2^{-\alpha})^s f\left(\frac{y}{m}\right). \tag{2.19}$$

According to the inequality $(1-t^{\alpha})^s\leqslant 2^{1-s}-t^{\alpha s}$, for $t\in[0,1]$ with certain fixed $\alpha\in(0,1]$ and $s\in(0,1]$ in (2.19), if we put $x=\frac{1-\mu}{2}a+\frac{1+\mu}{2}b$ and $y=\frac{1+\mu}{2}a+\frac{1-\mu}{2}b$ with $\mu\in[-1,1]$ and notice that h is nonnegative, then we have that

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)h\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\\
&=f\left(\frac{1}{2}\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)+\frac{1}{2}\left(\frac{1+\mu}{2}a+\frac{1-\mu}{2}b\right)\right)h\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)\\
&\leqslant\left\{2^{-\alpha s}f\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)+m\big(2^{1-s}-2^{-\alpha s}\big)f\left(\frac{\frac{1+\mu}{2}a+\frac{1-\mu}{2}b}{m}\right)\right\}\\
&\quad\times h\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right).
\end{aligned}$$

Integrating both sides of the above inequality with respect to μ over $[-1,1]$, we get that

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\int_{-1}^{1}h\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right){}_a\mathrm{d}_q\mu\\
&\leqslant 2^{-\alpha s}\int_{-1}^{1}f\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right)h\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right){}_a\mathrm{d}_q\mu\\
&\quad+m\big(2^{1-s}-2^{-\alpha s}\big)\int_{-1}^{1}f\left(\frac{\frac{1+\mu}{2}a+\frac{1-\mu}{2}b}{m}\right)h\left(\frac{1-\mu}{2}a+\frac{1+\mu}{2}b\right){}_a\mathrm{d}_q\mu.
\end{aligned}$$

Since h is symmetric about $x=\frac{a+b}{2}$, we have that

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)\frac{2}{b-a}\int_a^b h(x)\,{}_a\mathrm{d}_qx\\
&\leqslant 2^{-\alpha s}\frac{2}{b-a}\int_a^b f(x)h(x)\,{}_a\mathrm{d}_qx+m\big(2^{1-s}-2^{-\alpha s}\big)\frac{2}{b-a}\int_a^b f\left(\frac{x}{m}\right)h\big(a+b-x\big)\,{}_a\mathrm{d}_qx\\
&=2^{-\alpha s}\frac{2}{b-a}\int_a^b f(x)h(x)\,{}_a\mathrm{d}_qx+m\big(2^{1-s}-2^{-\alpha s}\big)\frac{2}{b-a}\int_a^b f\left(\frac{x}{m}\right)h(x)\,{}_a\mathrm{d}_qx.
\end{aligned}$$

The proof of Theorem 2.4 is completed. □

COROLLARY 2.3. *If we take $m=1$ in Theorem 2.4, then we get*

$$2^{s-1}f\left(\frac{a+b}{2}\right)\int_a^b h(x)\,{}_a\mathrm{d}_qx\leqslant\int_a^b f(x)h(x)\,{}_a\mathrm{d}_qx.$$

In particular, if we take $h(x)=1$ and put $q\to1^-$, then we have the left-side part of the Hermite–Hadamard's inequality for s-convex mappings established by Dragomir and Fitzpatrick in [15].

3. Further estimation results

To obtain further estimation-type results, let us deal with the boundedness and the Lipschitz condition of ${}_aD_qf(x)$.

THEOREM 3.1. *Let $f:\mathscr{K}\to\mathbb{R}$ be continuous and q-differentiable on $\mathscr{K}^\circ$, and ${}_aD_qf$ be integrable on $\mathscr{K}$. If there exist constants r and R with $r<R$ satisfying that $-\infty<r\leqslant {}_aD_qf(x)\leqslant R<+\infty$ for all $x\in\mathscr{K}$, then the following inequality*

$$\begin{aligned}&\Bigg|\lambda f(a)+(1-k)f(b)+(k-\lambda)f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_qu\\&\quad-\frac{(b-a)(r+R)}{2}\Big(\frac{q}{1+q}-\frac{k+\lambda}{2}\Big)\Bigg|\\&\leqslant\frac{(b-a)(R-r)}{2}\Big[\mathscr{T}_1(\lambda;q)+\mathscr{T}_2(k;q)\Big]\end{aligned}\tag{3.1}$$

holds for $k,\lambda\in[0,1]$, where $\mathscr{T}_1(\lambda;q)$ and $\mathscr{T}_2(k;q)$ are defined by (2.7) and (2.8), respectively.

Proof. Using Lemma 2.1, we have that

$$\begin{aligned}&\lambda f(a)+(1-k)f(b)+(k-\lambda)f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_qu\\&=(b-a)\Bigg\{\int_0^{\frac12}(qt-\lambda)\Big[{}_aD_qf\big((1-t)a+tb\big)-\frac{r+R}{2}+\frac{r+R}{2}\Big]{}_0\mathrm{d}_qt\\&\quad+\int_{\frac12}^1(qt-k)\Big[{}_aD_qf\big((1-t)a+tb\big)-\frac{r+R}{2}+\frac{r+R}{2}\Big]{}_0\mathrm{d}_qt\Bigg\}\\&=(b-a)\Bigg\{\int_0^{\frac12}(qt-\lambda)\Big[{}_aD_qf\big((1-t)a+tb\big)-\frac{r+R}{2}\Big]{}_0\mathrm{d}_qt\\&\quad+\int_{\frac12}^1(qt-k)\Big[{}_aD_qf\big((1-t)a+tb\big)-\frac{r+R}{2}\Big]{}_0\mathrm{d}_qt\Bigg\}\\&\quad+\frac{(b-a)(r+R)}{2}\Bigg\{\int_0^{\frac12}(qt-\lambda){}_0\mathrm{d}_qt+\int_{\frac12}^1(qt-k){}_0\mathrm{d}_qt\Bigg\}\\&=(b-a)\Bigg\{\int_0^{\frac12}(qt-\lambda)\Big[{}_aD_qf\big((1-t)a+tb\big)-\frac{r+R}{2}\Big]{}_0\mathrm{d}_qt\\&\quad+\int_{\frac12}^1(qt-k)\Big[{}_aD_qf\big((1-t)a+tb\big)-\frac{r+R}{2}\Big]{}_0\mathrm{d}_qt\Bigg\}\\&\quad+\frac{(b-a)(r+R)}{2}\Big(\frac{q}{1+q}-\frac{k+\lambda}{2}\Big).\end{aligned}$$

For the convenience of expression, let us define the quantity

$$\mathscr{F} := \lambda f(a) + (1-k)f(b) + (k-\lambda) f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u - \frac{(b-a)(r+R)}{2}\Big(\frac{q}{1+q} - \frac{k+\lambda}{2}\Big).$$

Thus,

$$\mathscr{F} = (b-a)\bigg\{\int_0^{\frac{1}{2}}(qt-\lambda)\Big[{}_aD_q f\big((1-t)a+tb\big) - \frac{r+R}{2}\Big]{}_0\mathrm{d}_q t + \int_{\frac{1}{2}}^1 (qt-k)\Big[{}_aD_q f\big((1-t)a+tb\big) - \frac{r+R}{2}\Big]{}_0\mathrm{d}_q t\bigg\}.$$

Therefore,

$$\begin{aligned}|\mathscr{F}| &\leqslant (b-a)\bigg\{\int_0^{\frac{1}{2}}|qt-\lambda|\Big|{}_aD_q f\big((1-t)a+tb\big) - \frac{r+R}{2}\Big|{}_0\mathrm{d}_q t \\ &\quad + \int_{\frac{1}{2}}^1 |qt-k|\Big|{}_aD_q f\big((1-t)a+tb\big) - \frac{r+R}{2}\Big|{}_0\mathrm{d}_q t\bigg\} \\ &\leqslant \frac{(b-a)(R-r)}{2}\bigg\{\int_0^{\frac{1}{2}}|qt-\lambda|{}_0\mathrm{d}_q t + \int_{\frac{1}{2}}^1 |qt-k|{}_0\mathrm{d}_q t\bigg\}.\end{aligned}$$

Since ${}_aD_q f$ satisfies $-\infty < r \leqslant {}_aD_q f(x) \leqslant R < +\infty$, we have that

$$r - \frac{r+R}{2} \leqslant {}_aD_q f(x) - \frac{r+R}{2} \leqslant R - \frac{r+R}{2},$$

which implies that

$$\Big|{}_aD_q f(x) - \frac{r+R}{2}\Big| \leqslant \frac{R-r}{2}.$$

The proof of Theorem 3.1 is completed. □

Particular cases are stated as follows.

COROLLARY 3.1. *Consider Theorem 3.1.*
(i) Taking $\lambda = \frac{1}{6}$ and $k = \frac{5}{6}$, we obtain that

$$\begin{aligned}&\bigg|\frac{1}{6}\Big[f(a) + 4f\Big(\frac{a+b}{2}\Big) + f(b)\Big] - \frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u + \frac{(b-a)(r+R)(1-q)}{4(1+q)}\bigg| \\ &\leqslant \begin{cases} \frac{(b-a)(R-r)(1-q)}{4(1+q)}, & 0 < q \leqslant \frac{1}{3}, \\ \frac{(b-a)(R-r)(11+18q)}{72(1+q)}, & \frac{1}{3} < q \leqslant \frac{5}{6}, \\ \frac{(b-a)(R-r)(1+30q)}{72(1+q)}, & \frac{5}{6} < q < 1. \end{cases}\end{aligned}$$

(ii) Taking $\lambda = \frac{1}{4}$ *and* $k = \frac{3}{4}$*, we have that*

$$\left|\frac{1}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u+\frac{(b-a)(r+R)(1-q)}{4(1+q)}\right|$$
$$\leqslant \begin{cases} \frac{(b-a)(R-r)(1-q)}{4(1+q)}, & 0<q\leqslant \frac{1}{2}, \\ \frac{(b-a)(R-r)(3-2q)}{16(1+q)}, & \frac{1}{2}<q\leqslant \frac{3}{4}, \\ \frac{(b-a)(R-r)q}{8(1+q)}, & \frac{3}{4}<q<1. \end{cases}$$

COROLLARY 3.2. *In Theorem 3.1, if we take* $\lambda = k = \frac{q}{1+q}$*, then we get that*

$$\left|\frac{qf(a)+f(b)}{1+q}-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right| \leqslant \frac{(b-a)(R-r)q^2}{(1+q)^3}.$$

In particular, if $q \to 1^-$*, then we have that*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leqslant \frac{(b-a)(R-r)}{8}.$$

THEOREM 3.2. *Let* $f:\mathscr{K}\to\mathbb{R}$ *be continuous and* q*-differentiable on* $\mathscr{K}^\circ$*, and* ${}_aD_qf$ *be integrable on* $\mathscr{K}$*. If* ${}_aD_qf$ *satisfies Lipschitz condition for certain* $L>0$ *on* $\mathscr{K}$*, then the following* q*-integrable inequality*

$$\begin{aligned} &\left|\lambda f(a)+(1-k)f(b)+(k-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right. \\ &\quad \left.-(b-a)\left[\left(\frac{q}{4(1+q)}-\frac{\lambda}{2}\right){}_aD_qf(a)+\left(\frac{3q}{4(1+q)}-\frac{k}{2}\right){}_aD_qf(b)\right]\right| \\ &\quad \leqslant L(b-a)^2\left[\mathscr{T}_3(1,1,\lambda;q)+\mathscr{T}_2(k;q)-\mathscr{T}_4(1,1,k;q)\right] \end{aligned} \tag{3.2}$$

holds for $k,\lambda\in[0,1]$*, where* $\mathscr{T}_2(k;q)$*,* $\mathscr{T}_3(\alpha,s,\lambda;q)$ *and* $\mathscr{T}_4(\alpha,s,k;q)$ *are defined by (2.8), (2.9) and (2.10), respectively.*

Proof. Utilizing Lemma 2.1, one has that

$$\begin{aligned} &\lambda f(a)+(1-k)f(b)+(k-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u \\ &\quad =(b-a)\left\{\int_0^{\frac{1}{2}}(qt-\lambda)\left[{}_aD_qf\big((1-t)a+tb\big)-{}_aD_qf(a)+{}_aD_qf(a)\right]{}_0\mathrm{d}_q t\right. \\ &\qquad \left.+\int_{\frac{1}{2}}^1(qt-k)\left[{}_aD_qf\big((1-t)a+tb\big)-{}_aD_qf(b)+{}_aD_qf(b){}_0\mathrm{d}_q t\right]\right\} \end{aligned}$$

$$
\begin{aligned}
&= (b-a)\bigg\{ \int_0^{\frac{1}{2}} \big(qt-\lambda\big)\Big[{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(a)\Big]{}_0\mathrm{d}_qt \\
&\quad + \int_{\frac{1}{2}}^1 \big(qt-k\big)\Big[{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(b)\Big]{}_0\mathrm{d}_qt \bigg\} \\
&\quad + (b-a)\bigg\{ {}_aD_qf(a)\int_0^{\frac{1}{2}} \big(qt-\lambda\big){}_0\mathrm{d}_qt + {}_aD_qf(b)\int_{\frac{1}{2}}^1 \big(qt-k\big){}_0\mathrm{d}_qt \bigg\} \\
&= (b-a)\bigg\{ \int_0^{\frac{1}{2}} \big(qt-\lambda\big)\Big[{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(a)\Big]{}_0\mathrm{d}_qt \\
&\quad + \int_{\frac{1}{2}}^1 \big(qt-k\big)\Big[{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(b)\Big]{}_0\mathrm{d}_qt \bigg\} \\
&\quad + (b-a)\bigg\{ \bigg(\frac{q}{4(1+q)} - \frac{\lambda}{2}\bigg){}_aD_qf(a) + \bigg(\frac{3q}{4(1+q)} - \frac{k}{2}\bigg){}_aD_qf(b) \bigg\}.
\end{aligned}
$$

For the convenience of expression, we define the quantity

$$
\begin{aligned}
\mathscr{T} &:= \lambda f(a) + (1-k)f(b) + (k-\lambda)f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_qu \\
&\quad - (b-a)\bigg[\bigg(\frac{q}{4(1+q)} - \frac{\lambda}{2}\bigg){}_aD_qf(a) + \bigg(\frac{3q}{4(1+q)} - \frac{k}{2}\bigg){}_aD_qf(b)\bigg].
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\mathscr{T} = (b-a)\bigg\{ &\int_0^{\frac{1}{2}} \big(qt-\lambda\big)\Big[{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(a)\Big]{}_0\mathrm{d}_qt \\
&+ \int_{\frac{1}{2}}^1 \big(qt-k\big)\Big[{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(b)\Big]{}_0\mathrm{d}_qt \bigg\}.
\end{aligned}
$$

Since ${}_aD_qf$ satisfies Lipschitz conditions for some $L > 0$, we have that

$$
\Big|{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(a)\Big| \leqslant L\Big|(1-t)a+tb-a\Big| = L|t|(b-a)
$$

and that

$$
\Big|{}_aD_qf\big((1-t)a+tb\big) - {}_aD_qf(b)\Big| \leqslant L\Big|(1-t)a+tb-b\Big| = L|1-t|(b-a).
$$

Hence,

$$
\begin{aligned}
|\mathscr{T}| &\leqslant L(b-a)^2\bigg[\int_0^{\frac{1}{2}} t\big|qt-\lambda\big|{}_0\mathrm{d}_qt + \int_{\frac{1}{2}}^1 (1-t)\big|qt-k\big|{}_0\mathrm{d}_qt\bigg] \\
&= L(b-a)^2\bigg[\int_{\frac{1}{2}}^1 \big|qt-k\big|{}_0\mathrm{d}_qt + \int_0^{\frac{1}{2}} t\big|qt-\lambda\big|{}_0\mathrm{d}_qt - \int_{\frac{1}{2}}^1 t\big|qt-k\big|{}_0\mathrm{d}_qt\bigg].
\end{aligned}
\tag{3.3}
$$

Using (2.8)–(2.10) with $\alpha = s = 1$ in (3.3), one gets the desired result in (3.2). Thus, the proof is completed. □

COROLLARY 3.3. *In Theorem 3.1, if we take $\lambda = k = \frac{q}{1+q}$, then we get that*

$$\left|\frac{qf(a)+f(b)}{1+q} - \frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right|$$
$$\leqslant L(b-a)^2\left[\frac{q-4q^2+q^3}{4(1+q)(1+q+q^2)} - \frac{q-6q^2+q^3}{4(1+q)^3} - \frac{2q^3}{(1+q)^4(1+q+q^2)}\right]$$
$$+\frac{q(b-a)}{4(1+q)}\Big[\big|{}_aD_q f(a)\big| + \big|{}_aD_q f(b)\big|\Big].$$

In particular, if $q \to 1^-$, then we have that

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(u)\mathrm{d}u\right| \leqslant \frac{b-a}{8}\Big[f'(a)+f'(b)\Big],$$

which is presented by Dragomir and Agarwal in [14, Theorem 2.2].

4. Examples

In this section, we present two examples to illustrate our main results.

EXAMPLE 4.1. Let $a = 0$, $b = 2$, $\lambda = \frac{1}{4}$, $k = \frac{3}{4}$, $s = \alpha = m = 1$, and let $f(x) = \frac{1}{q+1}x^2$, $x \in [0,2]$ with $q = \frac{1}{2}$. Then all the assumptions in Theorem 2.1 are satisfied.
The left-hand side term of (2.6) turns out to be:

$$\begin{aligned}
&\left|\lambda f(a) + (1-k)f(b) + (k-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u\right| \\
&= \left|\frac{1}{4}\times 0 + \frac{1}{4}\times\frac{8}{3} + \frac{2}{4}\times\frac{2}{3} - \frac{1}{2}\int_0^2 \left(\frac{2}{3}u^2\right)_0\mathrm{d}_{\frac{1}{2}}u\right| \\
&= \frac{11}{21} \approx 0.5238.
\end{aligned} \tag{4.1}$$

Clearly, we have ${}_0D_{\frac{1}{2}}f(x) = x$. The right-hand side term of (2.6) becomes:

$$(b-a)\Bigg\{\Big(2^{1-s}\mathscr{T}_1(\lambda;q) - \mathscr{T}_3(\alpha,s,\lambda;q)\Big)m\left|{}_aD_q f\left(\frac{a}{m}\right)\right| + \mathscr{T}_3(\alpha,s,\lambda;q)\left|{}_aD_q f(b)\right|$$
$$+\Big(2^{1-s}\mathscr{T}_2(k;q) - \mathscr{T}_4(\alpha,s,k;q)\Big)m\left|{}_aD_q f\left(\frac{a}{m}\right)\right| + \mathscr{T}_4(\alpha,s,k;q)\left|{}_aD_q f(b)\right|\Bigg\}$$

$$= (2-0)\Big\{\Big(2^{1-1}\mathscr{T}_1\Big(\frac{1}{4};\frac{1}{2}\Big)-\mathscr{T}_3\Big(1,2,\frac{1}{4};\frac{1}{2}\Big)\Big)\Big|{}_0D_{\frac{1}{2}}f(0)\Big|+\mathscr{T}_3\Big(1,1,\frac{1}{4};\frac{1}{2}\Big)\Big|{}_0D_{\frac{1}{2}}f(2)\Big|$$
$$+\Big(2^{1-1}\mathscr{T}_2\Big(\frac{3}{4};\frac{1}{2}\Big)-\mathscr{T}_4\Big(1,2,\frac{3}{4};\frac{1}{2}\Big)\Big)\Big|{}_0D_{\frac{1}{2}}f(0)\Big|+\mathscr{T}_4\Big(1,1,\frac{3}{4};\frac{1}{2}\Big)\Big|{}_0D_{\frac{1}{2}}f(2)\Big|\Big\}$$
$$=\frac{51}{84}\approx 0.6071. \tag{4.2}$$

It is clear that $0.5238 < 0.6071$, which demonstrates the results described in Theorem 2.1.

EXAMPLE 4.2. Let $a=0$, $b=2$, $\lambda=\frac{1}{6}$, $k=\frac{5}{6}$, $s=\alpha=m=1$, and let $f(x)=\frac{1}{q+1}x^2$, $x\in[0,2]$ with $q=\frac{1}{2}$. Obviously, ${}_0D_{\frac{1}{2}}f(x)=x$, $x\in[0,2]$, we have $r=0\leqslant {}_0D_{\frac{1}{2}}f(x)\leqslant 2=R$. Then all the assumptions in Theorem 3.1 are satisfied.

The left-hand side of inequality (2.6) is:

$$\Big|\lambda f(a)+(1-k)f(b)+(k-\lambda)f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(u)_a\mathrm{d}_q u$$
$$-\frac{(b-a)(r+R)}{2}\Big(\frac{q}{1+q}-\frac{k+\lambda}{2}\Big)\Big|$$
$$=\Big|\frac{1}{6}\times 0+\frac{1}{6}\times\frac{8}{3}+\frac{4}{6}\times\frac{2}{3}-\frac{1}{2}\int_0^2\Big(\frac{2}{3}u^2\Big)_0\mathrm{d}_{1/2}u-\frac{2(0+2)}{2}\Big(\frac{\frac{1}{2}}{1+\frac{1}{2}}-\frac{\frac{1}{6}+\frac{5}{6}}{2}\Big)\Big|$$
$$=\frac{19}{63}\approx 0.3016. \tag{4.3}$$

The right-hand side of inequality (2.6) turns out to be:

$$\begin{aligned}&\frac{(b-a)(R-r)}{2}\Big[\mathscr{T}_1(\lambda;q)+\mathscr{T}_2(k;q)\Big]\\&=2\Big[\mathscr{T}_1\Big(\frac{1}{6};\frac{1}{2}\Big)+\mathscr{T}_2\Big(\frac{5}{6};\frac{1}{2}\Big)\Big]\\&=2\Big[\frac{8\times(\frac{1}{6})^2-2\times\frac{1}{6}\times\frac{3}{2}+\frac{1}{2}}{4\times\frac{3}{2}}-\frac{2\times\frac{5}{6}\times\frac{3}{2}-\frac{3}{2}}{4\times\frac{3}{2}}\Big]\\&=\frac{11}{27}\approx 0.4074.\end{aligned} \tag{4.4}$$

It is clear that $0.3016 < 0.4074$, which demonstrates the results described in Theorem 3.1.

5. Applications

For positive numbers $a > 0$ and $b > 0$, we consider some applications of the obtained theorems to the following special means:

The arithmetic mean: $\mathscr{A} := \mathscr{A}(a,b) = \frac{a+b}{2}$.

The geometric mean: $\mathscr{G} := \mathscr{G}(a,b) = \sqrt{ab}$.

The generalized log-mean: $\mathscr{L}_\rho(a,b) := \left(\frac{b^{\rho+1}-a^{\rho+1}}{(\rho+1)(b-a)}\right)^{\frac{1}{\rho}}$, $\rho \in \mathbb{R}\setminus\{-1,0\}$ and $a \neq b$.

The log-mean: $\mathscr{L}(a,b) := \frac{b-a}{\ln b - \ln a}$, $a \neq b$.

We have propositions as follows.

PROPOSITION 5.1. *Let $0<a<b$, $0<s<1$, $0<q<1$, and $\lambda+k=1$ with $k,\lambda \in [0,1]$. Then we have that*

$$\begin{aligned}
&\left|\frac{2\lambda}{s+1}\mathscr{A}\left(a^{s+1},b^{s+1}\right)+\frac{k-\lambda}{s+1}\mathscr{A}^{s+1}(a,b)-\frac{1}{s+1}\mathbb{Q}_1(a,b;q,s)\right| \\
&\leqslant (b-a)\Big\{\left(2^{1-s}\mathscr{T}_1(\lambda;q)-\mathscr{T}_3(1,s,\lambda;q)\right)a^s+\mathscr{T}_3(1,s,\lambda;q)\mathscr{L}_s^s\left(qb+(1-q)a,b\right) \\
&\quad+\left(2^{1-s}\mathscr{T}_2(k;q)-\mathscr{T}_4(1,s,k;q)\right)a^s+\mathscr{T}_4(1,s,k;q)\mathscr{L}_s^s\left(qb+(1-q)a,b\right)\Big\},
\end{aligned} \tag{5.1}$$

where

$$\mathbb{Q}_1(a,b;q,s) := (1-q)\sum_{n=0}^{\infty} q^n\left(q^n b+(1-q^n)a\right)^{s+1},$$

$\mathscr{T}_1(\lambda;q)$, $\mathscr{T}_2(k;q)$, $\mathscr{T}_3(\alpha,s,\lambda;q)$ and $\mathscr{T}_4(\alpha,s,k;q)$ are defined in Theorem 2.1.

Proof. Applying $f(x)=\frac{x^{s+1}}{s+1}$ where $x>0$ and certain fixed $s\in(0,1)$ to Theorem 2.1 with $\alpha=1=m$, the desired result is proved. □

COROLLARY 5.1. *In Proposition 5.1, if we let $q \to 1^-$, then we have that*

$$\begin{aligned}
&\left|\frac{2\lambda}{s+1}\mathscr{A}\left(a^{s+1},b^{s+1}\right)+\frac{k-\lambda}{s+1}\mathscr{A}^{s+1}(a,b)-\frac{1}{s+1}\mathscr{L}_{s+1}^{s+1}(a,b)\right| \\
&\leqslant (b-a)\begin{cases}\Psi_2(k,\lambda,s)a^s+\Psi_1(k,\lambda,s)b^s,\ 0\leqslant\lambda\leqslant\frac{1}{2}\leqslant k\leqslant 1,\\ \Psi_4(k,\lambda,s)a^s+\Psi_3(k,\lambda,s)b^s,\ 0\leqslant k\leqslant\frac{1}{2}\leqslant\lambda\leqslant 1,\end{cases}
\end{aligned}$$

where

$$\Psi_1(k,\lambda,s)=\frac{2^{s+2}(\lambda^{s+2}+k^{s+2})+2^{s+1}\big((s+1)\lambda-k\big)-1}{2^{s+1}(s+1)(s+2)},$$

$$\Psi_2(k,\lambda,s)=2^{1-s}\left(\lambda^2+k^2-\frac{3}{2}k-\frac{1}{2}\lambda+\frac{3}{4}\right)-\Psi_1(k,\lambda,s),$$

$$\Psi_3(k,\lambda,s)=\frac{1+2^{s+1}(\lambda s+1-2k)}{2^{s+1}(s+1)(s+2)}$$

and

$$\Psi_4(k,\lambda,s)=\frac{3\lambda-k}{2^{s+1}}-\Psi_3(k,\lambda,s).$$

PROPOSITION 5.2. *Let $0<a<b$, $0<s<1$, $0<q<1$, and $\lambda+k=1$ with $k,\lambda\in[0,1]$. Then we have that*

$$\begin{aligned}&\left|2\lambda\mathscr{A}\left(a^{-s},b^{-s}\right)+(k-\lambda)\mathscr{A}^{-s}(a,b)-\mathbb{Q}_2(a,b;q,s)\right|\\&\leqslant(b-a)\left\{\Phi_3^{\frac{1}{e_1}}(\lambda,q;e_1)\left[\frac{1}{2^s}\left(\frac{s}{a^{1+s}}\right)^{e_2}+\Phi_1(1,s,q)\cdot\Delta_1\right]^{\frac{1}{e_2}}\right.\\&\quad\left.+\Phi_4^{\frac{1}{e_1}}(k,q;e_1)\left[\frac{1}{2^s}\left(\frac{s}{a^{1+s}}\right)^{e_2}+\Phi_2(1,s,q)\cdot\Delta_1\right]^{\frac{1}{e_2}}\right\},\end{aligned}\tag{5.2}$$

where $e_1^{-1}+e_2^{-1}=1$,

$$\mathbb{Q}_2(a,b;q,s):=(1-q)\sum_{n=0}^{\infty}q^n\Big(q^nb+(1-q^n)a\Big)^{-s},$$

$$\Delta_1=\left(s\mathscr{L}_{-s-1}^{-s-1}\big(qb+(1-q)a,b\big)\right)^{e_2}-\left(\frac{s}{a^{1+s}}\right)^{e_2},$$

$\Phi_1(\alpha,s,q)$, $\Phi_2(\alpha,s,q)$, $\Phi_3(\lambda,q;e_1)$ and $\Phi_4(k,q;e_1)$ are defined in Theorem 2.2.

Proof. Applying $f(x)=\frac{1}{x^s}$ with $x>0$ and certain fixed $s\in(0,1)$ to Theorem 2.2 with $\alpha=1=m$, the proof is clear. $\square$

COROLLARY 5.2. *In Proposition 5.2, if we let $q\to1^-$ and $\lambda=\frac{1}{2}=k$, then we have that*

$$\begin{aligned}&\left|\mathscr{A}\left(a^{-s},b^{-s}\right)-\mathscr{L}_{-s}^{-s}(a,b)\right|\\&\leqslant(b-a)\left\{\Theta_1^{\frac{1}{e_1}}\cdot\Upsilon_1^{\frac{1}{e_2}}(a,b,s,e_2)+\Theta_2^{\frac{1}{e_1}}\cdot\Upsilon_2^{\frac{1}{e_2}}(a,b,s,e_2)\right\},\end{aligned}$$

where $e_1^{-1}+e_2^{-1}=1$,

$$\Theta_1=\frac{1}{2^{s+2}(s+1)(s+2)},$$

$$\Theta_2 = \frac{s2^{s+1}+1}{2^{s+2}(s+1)(s+2)},$$

$$\Upsilon_1(a,b,s,e_2) = \frac{2s+1}{2^{s+1}(s+1)}\left(sa^{-1-s}\right)^{e_2} + \frac{1}{2^{s+1}(s+1)}\left(sb^{-1-s}\right)^{e_2}$$

and

$$\Upsilon_2(a,b,s,e_2) = \frac{2s+3-2^{s+1}}{2^{s+1}(s+1)}\left(sa^{-1-s}\right)^{e_2} + \frac{2^{s+1}-1}{2^{s+1}(s+1)}\left(sb^{-1-s}\right)^{e_2}.$$

PROPOSITION 5.3. *Let* $1 < \sigma_1 < \sigma_2$, $0 < s \leqslant 1$, $0 < q < 1$, *and* $\lambda + k = 1$ *with* $k, \lambda \in [0,1]$. *Then we have that*

$$\begin{aligned}&\left|2\lambda\mathscr{A}(\sigma_1,\sigma_2) + (k-\lambda)\mathscr{G}(\sigma_1,\sigma_2) - \mathbb{Q}_3(\sigma_1,\sigma_2;q,s) + \frac{(\ln\sigma_2 - \ln\sigma_1)(r+R)(1-q)}{4s(1+q)}\right| \\ &\leqslant \frac{(\ln\sigma_2 - \ln\sigma_1)(R-r)}{2s}\left[\mathscr{T}_1(\lambda;q) + \mathscr{T}_2(k;q)\right],\end{aligned} \tag{5.3}$$

where

$$\mathbb{Q}_3(\sigma_1,\sigma_2;q,s) := (1-q)\sum_{n=0}^{\infty} q^n \cdot \sigma_1^{(1-q^n)} \cdot \sigma_2^{q^n},$$

$\mathscr{T}_1(\lambda;q)$ *and* $\mathscr{T}_2(k;q)$ *are defined in Theorem* 2.1.

Proof. Applying $f(x) = e^{sx}$ with $x > 0$ and certain fixed $s \in (0,1)$ to Theorem 3.1 with $\sigma_1 = e^{sa}$, $\sigma_2 = e^{sb}$, we get the desired result. □

COROLLARY 5.3. *In Proposition* 5.3*, if we let* $q \to 1^-$*, then we have that*

$$\begin{aligned}&\left|2\lambda\mathscr{A}(\sigma_1,\sigma_2) + (k-\lambda)\mathscr{G}(\sigma_1,\sigma_2) - \mathscr{L}(\sigma_1,\sigma_2)\right| \\ &\leqslant \frac{(\ln\sigma_2 - \ln\sigma_1)(R-r)}{8s}\begin{cases} 4(\lambda^2+k^2)+\lambda-3k, & 0 \leqslant \lambda \leqslant \frac{1}{2} \leqslant k \leqslant 1, \\ 3\lambda - k, & 0 \leqslant k \leqslant \frac{1}{2} \leqslant \lambda \leqslant 1.\end{cases}\end{aligned}$$

6. Conclusion

Utilizing mappings with the property that the absolute values of their first derivatives are s-(α,m)-convex, we establish some different forms of quantum integral inequalities in terms of a new multi-parameter identity. For the quantum integral inequalities, we obtain their upper and lower bounds by considering the product of two different mappings. Furthermore, we use the boundedness and the Lipschitz condition of ${}_aD_qf(x)$ and then acquire further estimation-type results related to Hadamard–Simpson type inequality. Some minor results can be derived from our main results by choosing special parameter values for λ and k. It is worthwhile to mention that some inequalities presented in this paper generalize parts of the results given by Sudsutad et al. (2015). With these contributions, we hope to motivate the interested researchers to explore this fascinating field of the quantum integral inequality based on the techniques and ideas developed in the present paper.

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Tingsong Du
Department of Mathematics, College of Science
China Three Gorges University
Yichang 443002, P.R. China
e-mail: tingsongdu@ctgu.edu.cn

Chunyan Luo
Department of Mathematics, College of Science
China Three Gorges University
Yichang 443002, P.R. China
e-mail: luochunyanctgu@gmail.com

Bo Yu
Department of Mathematics, College of Science
China Three Gorges University
Yichang 443002, P.R. China
e-mail: yubo2003@amss.ac.cn