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ON THE PRODUCT OF CONTINUOUS PRIME NUMBERS

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Abstract. Basing on some deep conclusions of the prime number estimation, we give a fine from of the product of continuous prime numbers. These conclusions further improve the corresponding inequality of Panaitopol and more accurate conclusion is obtained.

1. Introduction

Let $p_1 \cdots p_n$ be the product of the n first prime number. In 1907, by an elementary method, Bonse [3] obtains

$$p_1 p_2 \cdots p_n > p_{n+1}^2 \ (n \geqslant 4)$$

and

$$p_1 p_2 \cdots p_n > p_{n+1}^3 \ (n \geqslant 5),$$

and a stronger results of the same nature is given by J. Sandór [16] as the following,

$$p_1 p_2 \cdots p_n > p_{n+5}^2 + p_{[n/2]}^2 \ (n \geqslant 24).$$

Shortly after, Pósa [11] improved the above result and proved that $\forall k \geqslant 1$ there is an n_k such that

$$p_1 p_2 \cdots p_n > p_{n+1}^k \ (n \geqslant n_k). \tag{1}$$

Later, Mamangakis [8], Reich [13], Betts [1], studied the inequalities involving prime products, and obtained a series of related results. For more details and their applications, one can see [2], [7] and [12].

In the same context, we are interested, in this paper to improve an inequality given by Panaitopol [10] where it proves that

$$p_1 p_2 \cdots p_n > p_{n+1}^{n-\pi(n)}, \tag{2}$$

where $n \geqslant 2$ and $\pi(x)$ denotes the prime counting function. Moreover, the authors in [10] showed that for any integer $k \geqslant 1$ and $n \geqslant 2k$,

$$p_1 p_2 \cdots p_n > p_{n+1}^k. \tag{3}$$

Our main theorem is the following:

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THEOREM 1.1. *For $n \geqslant 8$, we have*

$$p_{n+1}^{k_1(n)} < p_1 p_2 \cdots p_n < p_{n+1}^{k_2(n)}, \tag{4}$$

where

$$k_1(n) = n - \pi(n) + \frac{\pi(n)}{\pi(\log n)} - 2\pi(\pi(n)),$$

$$k_2(n) = n - \pi(n) + \frac{\pi(n)}{\pi(\log n)} + 2\pi(\pi(n)).$$

REMARK 1.2. According to PNT, we have

$$\pi(n) \sim \frac{n}{\log n}, \quad \frac{\pi(n)}{\pi(\log n)} \sim \frac{n \log\log n}{\log^2 n}, \quad \pi(\pi(n)) \sim \frac{n}{\log^2 n},$$

so the inequality (4) is a further improvement for all the above inequalities. Meanwhile, the inequality on the right side of (4) shows that the coefficient of $\pi(n)$ cannot be improved to be any real number smaller than 1, and the coefficient of $\frac{\pi(n)}{\pi(\log n)}$ cannot be modified to be any real number larger than 1 either.

2. Some Lemmas

To prove Theorem 1.1, some lemmas are given first.

LEMMA 2.1. *For $n \geqslant 1$, we have*

$$p_n \geqslant n\left(\log n + \log\log n - 1 + \frac{\log\log n - 2.1}{\log n}\right) \ (n \geqslant 3), \tag{5}$$

and

$$p_n \leqslant n\left(\log n + \log\log n - 1 + \frac{\log\log n - 1}{\log n}\right) \ (n \geqslant 210). \tag{6}$$

Proof. From Proposition 5.16 of [5], (5) is immediately. For (6), using (1.5) of [4], we can get

$$p_n \leqslant n\left(\log n + \log\log n - 1 + \frac{\log\log n - 1.8}{\log n}\right) \ (n \geqslant 27076).$$

For the case $n \geqslant 27076$, the equation (6) is clearly true. For $210 \leqslant n < 27076$, one can get (6) using mathematical software to test directly. This completes the proof. □

LEMMA 2.2. *We have*

$$\pi(x) \geqslant \frac{x}{\log x} \ (x \geqslant 17), \quad \pi(x) \geqslant \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) \ (x \geqslant 599), \tag{7}$$

and

$$\pi(x) \leqslant \frac{1.2551x}{\log x} \ (x \geqslant 2), \quad \pi(x) \leqslant \frac{x}{\log x}\left(1+\frac{1.2762}{\log x}\right) \ (x \geqslant 2), \tag{8}$$
$$\pi(x) \leqslant \frac{x}{\log x}\left(1+\frac{1}{\log x}+\frac{2.53816}{\log^2 x}\right) \ (x \geqslant 2).$$

Proof. The details can be seen in Corollary 5.2 [5], which improves the conclusion of [15]. □

LEMMA 2.3. *We have*

$$\theta(p_n) \geqslant n\left(\log n+\log\log n-1+\frac{\log\log n-2.1454}{\log n}\right) \ (n \geqslant 3), \tag{9}$$

and

$$\theta(p_n) \leqslant n\left(\log n+\log\log n-1+\frac{\log\log n-2}{\log n}\right) \ (n \geqslant 198). \tag{10}$$

Where $\theta(x)=\sum_{p\leqslant x}\log p$, *the sum being taken after primes* p.

Proof. By Theorem 7 of [14] and Theorem B(v) of [9], one can easily get (9) and (10), respectively. □

3. The proof of main Theorem

Since $\log(p_1p_2\cdots p_n)=\theta(p_n)$, then (4) is equivalent to

$$\theta(p_n) > \left(n-\pi(n)+\frac{\pi(n)}{\pi(\log n)}-2\pi(\pi(n))\right)\log p_{n+1}, \tag{11}$$

and

$$\theta(p_n) < \left(n-\pi(n)+\frac{\pi(n)}{\pi(\log n)}+2\pi(\pi(n))\right)\log p_{n+1}. \tag{12}$$

Therefore, it is enough to prove (11) and (12).

Since $x-\frac{1}{2x^2}<\log(1+x)<x$ for $0<x<1$, we can get

$$\log(n+1)<\log n+\frac{1}{n}, \ \log(n+1)>\log n+\frac{1}{n}-\frac{1}{2n^2},$$

and

$$\log\log(n+1)<\log\left(\log n+\frac{1}{n}\right)<\log\log n+\frac{1}{n\log n}.$$

Now from Lemma 2.1 and $n \geqslant 599$, we have

$$\begin{aligned}\log p_{n+1} &< \log(n+1) + \log\Big(\log(n+1) + \log\log(n+1)\\ &\quad -1 + \frac{\log\log(n+1)-1}{\log(n+1)}\Big)\\ &< \log n + \frac{1}{n} + \log\log(n+1)\\ &\quad + \log\left(1 + \frac{\log\log(n+1)-1}{\log(n+1)} + \frac{\log\log(n+1)-1}{\log^2(n+1)}\right)\\ &< \log n + \log\log n + \frac{1}{n\log n} + \frac{1}{n}\\ &\quad + \frac{\log\log(n+1)-1}{\log n} + \frac{\log\log(n+1)-1}{\log^2 n}\\ &< \log n + \log\log n + \frac{\log\log n}{\log n}\\ &\quad - \frac{1}{\log n}\left(1 - \frac{1+\log n}{n} - \frac{\log\log n - 1}{\log n} - \frac{1+\log n}{n\log^2 n}\right)\\ &< \log n + \log\log n + \frac{\log\log n}{\log n} - \frac{0.8}{\log n},\end{aligned} \tag{13}$$

and

$$\begin{aligned}\log p_{n+1} &> \log(n+1) + \log\Big(\log(n+1) + \log\log(n+1)\\ &\quad -1 + \frac{\log\log(n+1)-2.1}{\log(n+1)}\Big)\\ &> \log n + \frac{1}{n} - \frac{1}{2n^2} + \log\log n\\ &\quad + \log\left(1 + \frac{\log\log n - 1}{\log n} + \frac{\log\log n - 2.1}{\log^2 n}\right)\\ &> \log n + \frac{1}{2n} + \log\log n + \frac{\log\log n - 1}{\log n}\\ &\quad + \frac{\log\log n - 2.1}{\log^2 n} - \frac{\log^2\log n}{\log^2 n}\\ &> \log n + \log\log n + \frac{\log\log n}{\log n} - \frac{1.2}{\log n}.\end{aligned} \tag{14}$$

Note that for any positive integer n,

$$\pi(n) > \frac{n}{\log n}\left(1 - \frac{1}{2\log n}\right).$$

Thus by (7) and $n \geqslant 599$, we can get

$$\pi(\pi(n)) > \frac{\frac{n}{\log n}}{\log(\frac{n}{\log n})} = \frac{n}{\log n(\log n - \log\log n)} > \frac{n}{\log^2 n}. \tag{15}$$

And then from $n \geqslant 599$, Lemma 2.2 and (15), we can conclude that

$$
\begin{aligned}
& n-\pi(n)+\frac{\pi(n)}{\pi(\log n)}-2\pi(\pi(n)) \qquad (16)\\
&< n-\frac{n}{\log n}\Big(1+\frac{1}{\log n}\Big)+\frac{\frac{n}{\log n}(1+\frac{1}{\log n}+\frac{2.53826}{\log^2 n})}{\frac{\log n}{\log\log n}(1-\frac{1}{2\log\log n})}-\frac{2n}{\log^2 n}\\
&< n\Big(1-\frac{1}{\log n}-\frac{1}{\log^2 n}+\frac{\log\log n}{\log^2 n}\Big(1+\frac{1}{\log n}\\
&\quad +\frac{2.54}{\log^2 n}\Big)\Big(1+\frac{1}{2\log\log n}\Big)-\frac{2}{\log^2 n}\Big)\\
&< n\Big(1-\frac{1}{\log n}+\frac{\log\log n-2.5}{\log^2 n}+\frac{\log\log n+0.5}{\log^3 n}+\frac{2.54\log\log n+1.27}{\log^4 n}\Big)\\
&< n\Big(1-\frac{1}{\log n}+\frac{\log\log n-2.5}{\log^2 n}+\frac{2\log\log n}{\log^3 n}\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
& n-\pi(n)+\frac{\pi(n)}{\pi(\log n)}+2\pi(\pi(n)) \qquad (17)\\
&> n-\frac{n}{\log n}\Big(1+\frac{1}{\log n}+\frac{2.53826}{\log^2 n}\Big)+\frac{\frac{n}{\log n}(1+\frac{1}{\log n})}{\frac{\log n}{\log\log n}(1+\frac{1.2762}{\log\log n})}+\frac{2}{\log^2 n}\\
&> n\Big(1-\frac{1}{\log n}-\frac{1}{\log^2 n}-\frac{2.53826}{\log^3 n}\\
&\quad +\frac{\log\log n}{\log^2 n}\Big(1+\frac{1}{\log n}\Big)\Big(1-\frac{1.2762}{\log\log n}\Big)+\frac{2}{\log^2 n}\Big)\\
&> n\Big(1-\frac{1}{\log n}+\frac{\log\log n-0.2762}{\log^2 n}+\frac{\log\log n-3.81446}{\log^3 n}\Big)\\
&> n\Big(1-\frac{1}{\log n}+\frac{\log\log n-0.2762}{\log^2 n}-\frac{1.1\log\log n}{\log^3 n}\Big).
\end{aligned}
$$

Hence, from $n \geqslant 599$, (10), (13) and (16), we have

$$
\begin{aligned}
& \frac{1}{n}\Big(\theta(p_n)-\Big(n-\pi(n)+\frac{\pi(n)}{\pi(\log n)}-2\pi(\pi(n))\Big)\log p_{n+1}\Big)\\
&> \Big(\log n+\log\log n-1+\frac{\log\log n-2.1454}{\log n}\Big)\\
&\quad -\Big(1-\frac{1}{\log n}+\frac{\log\log n-2.5}{\log^2 n}+\frac{2\log\log n}{\log^3 n}\Big)\Big(\log n+\log\log n+\frac{\log\log n-0.8}{\log n}\Big)\\
&= \frac{1.1546}{y}-\frac{\log^2 y-1.5\log y+0.8}{y^2}-\frac{3\log^2 y-3.3\log y+2}{y^3}\\
&\quad -\frac{2\log^2 y-1.6\log y}{y^4}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{y^4}\big(1.1546y^3 - y^2\log^2 y + 1.5y^2\log y - 0.8y^2 - 3y\log^2 y + 3.3y\log y - 2y \\
&\quad - 2\log^2 y + 1.6\log y\big) \\
&= \frac{1}{y^4} f(y),
\end{aligned}
$$

where $y = \log n$. Since $n \geqslant 599$, i.e. $y > 6.395$, and then the derivative of $f(y)$

$$
\begin{aligned}
f'(y) &= 3.4638y^2 - 2y\log^2 y + y\log y - 0.1y - 3\log^2 y \\
&\quad -2.7\log y + 1.3 - \frac{2}{y} - \frac{3.2\log y}{y} \\
&> 95.42 > 0.
\end{aligned} \tag{18}
$$

This means that the function $f(y)$ is monotonically increasing, hence we have

$$
f(y) > f(\log 599) > f(6.395) > 204.38 > 0,
$$

and then $\frac{1}{y^4} f(y) > 0$. Thus (11) is proved.

For (12), similarly, from $n \geqslant 599$, (9), (14) and (17), we can get

$$
\begin{aligned}
&\frac{1}{n}\Big(\theta(p_n) - \Big(n - \pi(n) + \frac{\pi(n)}{\pi(\log n)} + 2\pi(\pi(n))\Big)\log p_{n+1}\Big) \\
&< \Big(\log n + \log\log n - 1 + \frac{\log\log n - 2}{\log n}\Big) \\
&\quad - \Big(1 - \frac{1}{\log n} + \frac{\log\log n - 0.2762}{\log^2 n} - \frac{1.1\log\log n}{\log^3 n}\Big) \\
&\quad \times \Big(\log n + \log\log n + \frac{\log\log n - 1.2}{\log n}\Big) \\
&= -\frac{0.5238}{y} - \frac{\log^2 y - 2.3762\log y - 1.2}{y^2} + \frac{0.1\log^2 y + 1.4762\log y - 0.33144}{y^3} \\
&\quad + \frac{1.1\log^2 y - 1.32\log y}{y^4} \\
&= -\frac{1}{y^4}\big(0.5238y^3 + y^2\log^2 y - 2.3762y^2\log y + 1.2y^2 - 0.1y\log^2 y - 1.4762y\log y \\
&\quad + 0.33144y - 1.1\log^2 y + 1.32\log y\big) \\
&= -\frac{1}{y^4} g(y),
\end{aligned}
$$

where $y = \log n$. Note that $n \geqslant 599$, i.e. $y > 6.395$, and then the derivative of $g(y)$

$$
\begin{aligned}
g'(y) &= 1.5714y^2 + 2y\log^2 y - 2.7524y\log y - 0.0238y - 0.1\log^2 y - 1.6762\log y \\
&\quad -1.14476 - \frac{2.2\log y}{y} + \frac{1.32}{y} \\
&> 70.76 > 0.
\end{aligned}
$$

This means that $g(y)$ is monotonically increasing, hence we have

$$g(y) > g(\log 599) > g(6.395) > 127.61 > 0,$$

and then $-\frac{1}{y^4}g(y) < 0$. Thus inequality (12) is proved.

For the case $8 \leqslant n < 599$, using mathematical software to test directly, (4) is true. This completes the proof of Theorem 1.1.

4. A Corollary

Basing on the inequality (4), we can also obtain the following

COROLLARY 4.1. *Let $k \geqslant 2$ be an integer, and*

$$n > k\left(1 + \frac{2}{\log k}\right), \tag{19}$$

then we have

$$p_1 p_2 \cdots p_n > p_{n+1}^k.$$

Proof. From Lemma 2.2, if $n \geqslant 599$, we have

$$\begin{aligned}\frac{\pi(n)}{\pi(\log n)} &> \frac{\frac{n}{\log n}(1+\frac{1}{\log n})}{\frac{\log n}{\log\log n}(1+\frac{1.2762}{\log\log n})}\\ &= \frac{n\log\log n}{\log^2 n(1+\frac{1.2762}{\log\log n})}\\ &> \frac{0.59\,n\log\log n}{\log^2 n},\end{aligned}$$

and

$$\begin{aligned}\pi(\pi(n)) &< \frac{\frac{1.2551n}{\log n}}{\log(\frac{1.2551n}{\log n})}\\ &= \frac{1.2551n}{\log n(\log 1.2551 + \log n - \log\log n)}\\ &< \frac{1.69n}{\log^2 n}.\end{aligned}$$

Hence, from the inequality on the left side of (4), we know that if $k > 460$ and $n > k(1+\frac{2}{\log k})$, then $n > 599$, and so

$$
\begin{aligned}
k_1(n) &= n-\pi(n)+\frac{\pi(n)}{\pi(\log n)}-2\pi(\pi(n)) \qquad (20)\\
&> n-\frac{n}{\log n}\Big(1+\frac{1}{\log n}+\frac{2.54}{\log^2 n}\Big)+\frac{0.59\,n\log\log n}{\log^2 n}-\frac{3.38n}{\log^2 n}\\
&> n\Big(1-\frac{1.58}{\log n}\Big)\\
&> k\Big(1+\frac{2}{\log k}\Big)\Big(1-\frac{1.58}{\log(k(1+\frac{2}{\log k}))}\Big)\\
&> k\Big(1+\frac{2}{\log k}\Big)\Big(1-\frac{1.66}{\log k)}\Big)\\
&> k.
\end{aligned}
$$

Then the inequality (3) holds. When $2 \leqslant k \leqslant 460$, using mathematical software to test directly, inequality (3) also holds if $n > k(1+\frac{2}{\log k})$.

Thus Corollary 4.1 is proved. □

REMARK 4.2. By (19), when $2 \leqslant k \leqslant 7$, then $1+\frac{2}{\log k} > 2k$, the estimated result of Panaitopol [10] is better. However, when $k \geqslant 8$, we have $1+\frac{2}{\log k} < 2k$, then Corollary 4.1 gives a more accurate estimate.

For example, when $k > 160$, we have

$$p_1p_2\cdots p_n > p_{n+1}^k \ (n > 1.2k).$$

When $k > 10^9$, we have

$$p_1p_2\cdots p_n > p_{n+1}^k \ (n > 1.1k).$$

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