Journal of Mathematical Inequalities
Volume 15, Number 1 (2021), 261–275 doi:10.7153/jmi-2021-15-20

PARAMETRIC LITTLEWOOD–PALEY OPERATORS ON VARIABLE HERZ–MORREY SPACES

WENHUA WANG AND QINGDONG GUO*

(*Communicated by Y. Sawano*)

Abstract. Let $\alpha(\cdot)\in L^{\infty}(\mathbb{R}^n)$ and $p(\cdot):\mathbb{R}^n\to(0,\infty)$ be variable exponent functions satisfying the globally log-Hölder continuous condition. In this paper, the authors obtain the boundedness of parametric Littlewood-Paley operators and their commutators generated by BMO functions on variable Herz-Morrey spaces. All these results are still new even when the exponent function $\alpha(\cdot)$ is α.

1. Introduction

Suppose that S^{n-1} is the unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$ $(n\geqslant 2)$. Let Ω be a homogeneous function of degree zero on $\mathbb{R}^n$ which is locally integrable and satisfies the cancellation condition

$$\int_{S^{n-1}}\Omega(x')\,d\sigma(x')=0, \tag{1.1}$$

where $d\sigma$ is the Lebesgue measure and $x':=x/|x|$ for any $x\neq\mathbf{0}$. For a function f on $\mathbb{R}^n$, the parametric area integrals $\mu^{\rho}_{\Omega,S}$ and Littlewood-Paley operators $\mu^{\rho,*}_{\Omega,\lambda}$ are, respectively, defined by setting, for any $x\in\mathbb{R}^n$,

$$\mu^{\rho}_{\Omega,S}(f)(x):=\left(\int_0^{\infty}\int_{|y-x|<t}\left|\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2},$$

and

$$\mu^{\rho,*}_{\Omega,\lambda}f(x)$$
$$:=\left[\int_0^{\infty}\int_{\mathbb{R}^{n+1}_+}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}(f)(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2},$$

where $\rho\in(0,\infty)$ and $\lambda\in(1,\infty)$.

Mathematics subject classification (2010): Primary 42B20; Secondary 42B25, 42B35.
Keywords and phrases: Parametric Littlewood-Paley operator, commutator, Herz-Morrey space, variable exponent, BMO function.

* Corresponding author.

Paper JMI-15-20

In 1999, the operators $\mu_{\Omega,S}^{\rho}$ and $\mu_{\Omega,\lambda}^{\rho,*}$ were first studied by Sakamoto and Yabuta [13]. They showed that if $\Omega \in \mathrm{Lip}_{\alpha}(S^{n-1})$ with $\alpha \in (0,1)$, then $\mu_{\Omega,S}^{\rho}$ and $\mu_{\Omega,\lambda}^{\rho,*}$ are bounded on $L^{p}(\mathbb{R}^{n})$ with $p \in (1,\infty)$. In 2009, Xue and Ding [18] obtained a celebrated result that $\mu_{\Omega,S}^{\rho}$ and $\mu_{\Omega,\lambda}^{\rho,*}$ are bounded on $L_{\omega}^{p}(\mathbb{R}^{n})$ with $p \in (1,\infty)$ under weaker smoothness condition of Ω, where $\omega \in A_p$ and A_p denotes the Muckenhoupt weight class.

Now let us recall the definitions of corresponding m-order commutators of the parameterized Littlewood-Paley operators. Let $b \in L_{loc}^{1}$, $m \in \mathbb{N}$, the commutators $[b^m, \mu_{\Omega,S}^{\rho}]$ and $[b^m, \mu_{\Omega,\lambda}^{\rho,*}]$ are, respectively, defined by setting, for any $x \in \mathbb{R}^n$,

$$[b^m, \mu_{\Omega,S}^{\rho}](f)(x) := \left(\int_0^{\infty} \int_{|y-x|<t} \left| \int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} [b(x)-b(z)]^m f(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right)^{1/2}$$

and

$$[b^m, \mu_{\Omega,\lambda}^{\rho,*}](f)(x) := \left[\int\int_{\mathbb{R}_{+}^{n+1}} \left(\frac{t}{t+|x-y|} \right)^{\lambda n} \left| \int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} [b(x)-b(z)]^m f(z)\, dz \right|^2 \frac{dydt}{t^{n+2\rho+1}} \right]^{1/2}.$$

In recent years, the theory of function spaces with variable exponents has developed and applying in fluid dynamics, partial differential equations, variational calculus and harmonic analysis (see [2, 3, 11, 12]). In 2010, Izuki [7] introduced the Herz-Morrey spaces with variable exponents and obtained the boundedness of fractional integrals on those spaces. In 2012, Almeida and Drihem [1] introduced the Herz spaces with two variable exponents and obtained the boundedness of some sublinear operators on those spaces. In 2014, Lu et al. [10] introduced the Herz-Morrey spaces with variable exponents and obtained the boundedness of some operators on those spaces. In 2015, Wang et al. [16] established the boundedness of parameterized Littlewood-Paley operators and their commutators generated by BMO functions on variable Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. In 2016, Wang et al. [17] obtained the boundedness of parameterized Littlewood-Paley operators and their commutators generated by BMO functions on variable Herz spaces. For more information about the variable function spaces and Littlewood-Paley operators, see [3, 4, 8, 9, 14, 15].

Inspired by the previous papers, we would like to declare that the goal of this paper is to obtain the boundedness of parameterized Littlewood-Paley operators and their commutators generated by BMO functions on Herz-Morrey spaces with variable exponents.

Precisely, this article is organized as follows.

In Section 2, we first recall some notations and definitions, the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and the variable Herz-Morrey spaces $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}(\mathbb{R}^n)$ and $MK_{q,p(\cdot)}^{\alpha(\cdot),\gamma}(\mathbb{R}^n)$. Then, motivated by Lu et al. [10] and Wang et al. [16], we obtain the boundedness of parametric Littlewood-Paley operators and their commutators generated by BMO functions on Herz-Morrey spaces with variable exponents (see Theorems 2.5–2.7 below for more details). Section 3 is devoted to proving Theorems 2.5 and 2.7. In the process of the proof of Theorems 2.5 and 2.7, it is worth pointing out that, establishing a more subtle pointwise estimate plays an important role (see Theorems 2.5 and 2.7 below for more details).

Finally, we make some conventions on notation. Let $\mathbb{N} := \{1, 2, \ldots\}$ and $\mathbb{Z}_+ := \{0\} \cup \mathbb{N}$. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line. The *symbol* $D \lesssim F$ means that $D \leqslant CF$. If $D \lesssim F$ and $F \lesssim D$, we then write $D \sim F$. For any $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. If E is a subset of $\mathbb{R}^n$, we denote by χ_E its *characteristic function*. If there are no special instructions, any space $\mathscr{X}(\mathbb{R}^n)$ is denoted simply by $\mathscr{X}$. For instance, $L^2(\mathbb{R}^n)$ is simply denoted by L^2. For any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the *maximal integer* not larger than a.

2. Preliminaries

In this section, we first recall some notations and definitions. Now we recall that a measurable function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ is called a *variable exponent*. For any variable exponent $p(\cdot)$, let

$$p_- := \operatorname*{ess\,inf}_{x \in \mathbb{R}^n} p(x) \quad \text{and} \quad p_+ := \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} p(x). \tag{2.1}$$

Denote by $\mathscr{P}$ the set of all variable exponents $p(\cdot)$ satisfying $p_- > 1$ and $p_+ < \infty$.

Let f be a measurable function on $\mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}$. Then the *modular function* (or, for simplicity, the *modular*) $\rho_{p(\cdot)}$, associated with $p(\cdot)$, is defined by setting

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} |f(x)|^{p(x)}\, dx$$

and the *Luxemburg* (also called *Luxemburg-Nakano*) *quasi-norm* $\|f\|_{L^{p(\cdot)}}$ by

$$\|f\|_{L^{p(\cdot)}} := \inf\left\{\lambda \in (0, \infty) : \rho_{p(\cdot)}(f/\lambda) \leqslant 1\right\}.$$

Moreover, the *variable Lebesgue space* $L^{p(\cdot)}$ is defined to the set of all measurable functions f satisfying that $\rho_{p(\cdot)}(f) < \infty$, equipped with the quasi-norm $\|f\|_{L^{p(\cdot)}}$.

We recall the definition of Hardy-Littlewood maximal function $M_{\mathrm{HL}}(f)$. For any $f \in L^1_{\mathrm{loc}}$ and $x \in \mathbb{R}^n$,

$$M_{\mathrm{HL}}(f)(x) := \sup_{x \in B \subset \mathbb{R}^n} \frac{1}{|B|} \int_B |f(z)|\, dz. \tag{2.2}$$

Let $\mathscr{B}$ be the set of $p(\cdot) \in \mathscr{P}$ satisfying the condition that M_{HL} is bounded on $L^{p(\cdot)}$. It's well known that if $p(\cdot) \in \mathscr{P}$ and satisfies the following globally log-Hölder continuous then $p(\cdot) \in \mathscr{B}$.

DEFINITION 2.1. [1] Let $g(\cdot)$ be a real function on $\mathbb{R}^n$.

(1) $g(\cdot)$ is locally log-Hölder continuous, if there exists a constant $C>0$ such that

$$|g(x)-g(y)| \leqslant \frac{C}{\log(e+1/|x-y|)}$$

for any $x, y \in \mathbb{R}^n$ and $|x-y|<1/2$.

(2) $g(\cdot)$ is locally log-Hölder continuous at the origin(or has a log decay at the origin), if there exists a constant $C>0$ such that

$$|g(x)-g(0)| \leqslant \frac{C}{\log(e+1/|x|)}$$

for any $x \in \mathbb{R}^n$.

(3) $g(\cdot)$ is locally log-Hölder continuous at the infinity(or has a log decay at the infinity), if there exist $g_\infty \in \mathbb{R}$ and a constant $C>0$ such that

$$|g(x)-g_\infty| \leqslant \frac{C}{\log(e+|x|)}$$

for any $x \in \mathbb{R}^n$.

We denote by $\mathscr{P}_0^{\log}$ and $\mathscr{P}_\infty^{\log}$ the class of all variable exponents $p(\cdot) \in \mathscr{P}$, which are log-Hölder continuous at the origin and at the infinity respectively. We call $p'(\cdot)$ the conjugate exponent to $p(\cdot)$, that is $p'(\cdot)=\frac{p(\cdot)}{p(\cdot)-1}$, we know that $p(\cdot) \in \mathscr{B}$ is equivalent to $p'(\cdot) \in \mathscr{B}$.

In this paper, we denote $R_k = B_k \setminus B_{k-1}$ and denote briefly the characteristic function $\chi_{B_k \setminus B_{k-1}}$ by χ_k.

DEFINITION 2.2. [10] Let $0<q \leqslant \infty$, $p(\cdot) \in \mathscr{P}$, $0 \leqslant \gamma < \infty$ and $\alpha(\cdot) \in L^\infty$. The *homogeneous Herz-Morrey space with variable exponents* $M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}$ and the *non-homogeneous Herz-Morrey space with variable exponents* $MK_{q,p(\cdot)}^{\alpha(\cdot),\gamma}$ are defined respectively by setting,

$$M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma} := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n \setminus \mathbf{0}) : \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}} < \infty \right\}$$

and

$$MK_{q,p(\cdot)}^{\alpha(\cdot),\gamma} := \left\{ f \in L_{\mathrm{loc}}^{p(\cdot)}(\mathbb{R}^n) : \|f\|_{MK_{q,p(\cdot)}^{\alpha(\cdot),\gamma}} < \infty \right\},$$

where

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}} := \sup_{L \in \mathbb{Z}} 2^{-L\gamma} \left\{ \sum_{k=-\infty}^{L} \left\| 2^{\alpha(\cdot)k} f \chi_k \right\|_{L^{p(\cdot)}}^q \right\}^{1/q}$$

and

$$\|f\|_{MK^{\alpha(\cdot),\gamma}_{q,p(\cdot)}} := \sup_{L\in\mathbb{Z}_+} 2^{-L\gamma}\left\{\sum_{k=0}^{L}\left\|2^{\alpha(\cdot)k}f\chi_k\right\|^q_{L^{p(\cdot)}}\right\}^{1/q}.$$

Here, there is the usual modification when $q=\infty$.

LEMMA 2.3. [5] *Let $p(\cdot)\in\mathscr{B}$. Then there exist $0<\delta_1,\delta_2<1$ depending only on $p(\cdot)$ and n such that for all $B,S\subset\mathbb{R}^n$ and $S\subset B$,*

$$\frac{\|\chi_S\|_{L^{p(\cdot)}}}{\|\chi_B\|_{L^{p(\cdot)}}}\leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_1} \text{ and } \frac{\|\chi_S\|_{L^{p'(\cdot)}}}{\|\chi_B\|_{L^{p'(\cdot)}}}\leqslant C\left(\frac{|S|}{|B|}\right)^{\delta_2}.$$

Here and hereafter, we always assume that Ω is homogeneous of degree zero and satisfies (1.1). To obtain the main results, we need follows condition:

$$\int_0^1\frac{\omega_q(\delta)}{\delta}(1+|\log\delta|)^{\sigma}\,d\delta<\infty,\ \sigma>2. \tag{2.3}$$

where $\omega_q(\delta)$ is the integral modulus of continuity of order q of Ω defined by setting, for any $\delta\in(0,1]$,

$$\omega_q(\delta) := \sup_{\|\gamma\|<\delta}\left(\int_{S^{n-1}}|\Omega(\gamma x')-\Omega(x')|^q\,d\sigma(x')\right)^{1/q}$$

and γ denotes a rotation on S^{n-1} with $\|\gamma\| := \sup_{y'\in S^{n-1}}|\gamma y'-y'|$.

The main results of this paper are as follows.

THEOREM 2.4. *Let $\alpha(\cdot)\in L^\infty\cap\mathscr{P}_0^{\log}\cap\mathscr{P}_\infty^{\log}$, $p(\cdot)\in\mathscr{P}_0^{\log}\cap\mathscr{P}_\infty^{\log}$, $q\in(0,\infty)$, $0<\gamma<n$, and $\rho>n/2$. Suppose that $\Omega\in L^2(S^{n-1})$ satisfies (2.3). If $\gamma-n\delta_1<\alpha_-\leqslant\alpha_+<n\delta_2$, where δ_1,δ_2 are the constant in Lemma 2.3, then there exists a positive constant C independent of f such that*

$$\left\|\mu^{\rho}_{\Omega,S}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.$$

THEOREM 2.5. *Let $\alpha(\cdot)\in L^\infty\cap\mathscr{P}_0^{\log}\cap\mathscr{P}_\infty^{\log}$, $p(\cdot)\in\mathscr{P}_0^{\log}\cap\mathscr{P}_\infty^{\log}$, $q\in(0,\infty)$, $0<\gamma<n$, $\lambda>2$ and $\rho>n/2$. Suppose that $\Omega\in L^2(S^{n-1})$ satisfies (2.3). If $\gamma-n\delta_1<\alpha_-\leqslant\alpha_+<n\delta_2$, where δ_1,δ_2 are the constant in Lemma 2.3, then there exists a positive constant C independent of f such that*

$$\left\|\mu^{\rho,*}_{\Omega,\lambda}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}\leqslant C\|f\|_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.$$

THEOREM 2.6. *Let* $\alpha(\cdot) \in L^{\infty} \cap \mathscr{P}_0^{\log} \cap \mathscr{P}_{\infty}^{\log}$, $p(\cdot) \in \mathscr{P}_0^{\log} \cap \mathscr{P}_{\infty}^{\log}$, $q \in (0,\infty)$, $\rho > n/2$, *and* $m \in \mathbb{N}$. *Suppose that* $b \in BMO$ *and* $\Omega \in L^2(S^{n-1})$ *satisfies* (2.3). *If* $\gamma - n\delta_1 < \alpha_- \leqslant \alpha_+ < n\delta_2$, *where* δ_1, δ_2 *are the constant in Lemma* 2.3, *then there exists a positive constant* C *independent of* f *such that*

$$\left\|[b^m, \mu_{\Omega,S}^{\rho}](f)\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}} \leqslant C\|b\|_{BMO}^m \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}.$$

THEOREM 2.7. *Let* $\alpha(\cdot) \in L^{\infty} \cap \mathscr{P}_0^{\log} \cap \mathscr{P}_{\infty}^{\log}$, $p(\cdot) \in \mathscr{P}_0^{\log} \cap \mathscr{P}_{\infty}^{\log}$, $q \in (0,\infty)$, $\rho > n/2$, $0 < \gamma < n$, $\lambda > 2$ *and* $m \in \mathbb{N}$. *Suppose that* $b \in BMO$ *and* $\Omega \in L^2(S^{n-1})$ *satisfies* (2.3). *If* $\gamma - n\delta_1 < \alpha_- \leqslant \alpha_+ < n\delta_2$, *where* δ_1, δ_2 *are the constant in Lemma* 2.3, *then there exists a positive constant* C *independent of* f *such that*

$$\left\|[b^m, \mu_{\Omega,\lambda}^{\rho,*}](f)\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}} \leqslant C\|b\|_{BMO}^m \|\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}.$$

REMARK 2.8. (i) It's easy to see that, for any $x \in \mathbb{R}^n$, $m \in \mathbb{N}$,

$$\mu_{\Omega,S}^{\rho}(f)(x) \lesssim \mu_{\Omega,\lambda}^{\rho,*}(f)(x) \text{ and } [b^m, \mu_{\Omega,S}^{\rho}](f)(x) \lesssim [b^m, \mu_{\Omega,\lambda}^{\rho,*}](f)(x).$$

Therefore, we only need to prove Theorems 2.5 and 2.7.

(ii) All these results for non-homogeneous Herz-Morrey spaces with variable exponents can also be proved. The arguments are similar, so the details are omitted here.

3. Proofs of Theorems 2.5 and 2.7

To prove the main results, we need the following technical lemmas.

LEMMA 3.1. [8] *Let* $p(\cdot) \in \mathscr{P}$. *If* $f \in L^{p(\cdot)}$ *and* $g \in L^{p'(\cdot)}$, *then* fg *is integrable on* $\mathbb{R}^n$ *and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|\,dx \leqslant C_p \|f\|_{L^{p(\cdot)}} \|g\|_{L^{p'(\cdot)}},$$

where $C_p = 1 + 1/p_- + 1/p_+$.

LEMMA 3.2. [5] *Let* $p(\cdot) \in \mathscr{B}$. *Then there exists a positive constant* $C > 0$ *such that for all* $B \subset \mathbb{R}^n$,

$$\frac{1}{|B|}\|\chi_B\|_{L^{p(\cdot)}} \|\chi_B\|_{L^{p'(\cdot)}} \leqslant C.$$

LEMMA 3.3. [16] *Let* $p(\cdot) \in \mathscr{B}$, $\rho > n/2$, $\lambda > 2$ *and* $\Omega \in L^2(S^{n-1})$ *satisfying* (2.3). *Then there exists a positive constant* C *independent of* f *such that*

$$\left\|\mu_{\Omega,S}^{\rho}(f)\right\|_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}} \text{ and } \left\|\mu_{\Omega,\lambda}^{\rho,*}(f)\right\|_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

LEMMA 3.4. [6] *Let* $p(\cdot) \in \mathscr{B}$, *m be a positive integer and* B *be a ball in* $\mathbb{R}^n$. *Then we have that for any* $b \in BMO$ *and any* $i, j \in \mathbb{Z}$ *with* $i < j$,

$$\frac{1}{C}\|b\|_{BMO}^m \leqslant \sup_B \frac{1}{\|\chi_B\|_{L^{p(\cdot)}}}\|(b-b_B)^m \chi_B\|_{L^{p(\cdot)}} \leqslant C\|b\|_{BMO}^m,$$

$$\|(b-b_{B_i})^m \chi_{B_j}\|_{L^{p(\cdot)}} \leqslant C(j-i)^m \|b\|_{BMO}^m \|\chi_{B_j}\|_{L^{p(\cdot)}},$$

where $B_i = \{x \in \mathbb{R}^n : |x| \leqslant 2^i, i \in \mathbb{N}\}$.

LEMMA 3.5. [16] *Let* $b \in BMO$ *and* $m \in \mathbb{N}$. *Suppose that* $p(\cdot) \in \mathscr{B}$, $\rho > n/2$, $\lambda > 2$ *and* $\Omega \in L^2(S^{n-1})$ *satisfying* (2.3). *Then there exists a positive constant* C *independent of* f *such that*

$$\left\|[b^m, \mu_{\Omega,S}^{\rho}](f)\right\|_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}} \text{ and } \left\|[b^m, \mu_{\Omega,\lambda}^{\rho,*}](f)\right\|_{L^{p(\cdot)}} \leqslant C\|f\|_{L^{p(\cdot)}}.$$

LEMMA 3.6. [1] *Let* $\alpha \in L^\infty$ *and* $r_1 > 0$. *If* $\alpha(x)$ *is log-Hölder continuous both at origin and at infinity, then*

$$r_1^{\alpha(x)} \lesssim r_2^{\alpha(y)} \times \begin{cases} \left(\frac{r_1}{r_2}\right)^{\alpha_+}, & 0 < r_2 \leqslant \frac{r_1}{2}, \\ 1, & \frac{r_1}{2} < r_2 \leqslant 2r_1, \\ \left(\frac{r_1}{r_2}\right)^{\alpha_-}, & r_2 > 2r_1, \end{cases}$$

for any $x \in B(0,r_1) \setminus B(0,r_1/2)$ *and* $y \in B(0,r_2) \setminus B(0,r_2/2)$, *with the implicit constant not depending on* x, y, r_1 *and* r_2.

LEMMA 3.7. [10] *Let* $q \in (0,\infty)$, $p(\cdot) \in \mathscr{P}$, $\gamma \in [0,\infty)$ *and* $\alpha(\cdot) \in L^\infty$. *If* $\alpha(\cdot)$ *is log-Hölder continuous both at origin and at infinity, then*

$$\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}^q = C\max\Bigg\{\sup_{L<0, L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L} 2^{kq\alpha(0)}\|f\chi_k\|_{L^{p(\cdot)}}^q,$$
$$\sup_{L\geqslant 0, L\in\mathbb{Z}}\left[2^{-L\gamma q}\sum_{k=-\infty}^{-1} 2^{kq\alpha(0)}\|f\chi_k\|_{L^{p(\cdot)}}^q + 2^{-L\gamma q}\sum_{k=0}^{L} 2^{kq\alpha_\infty}\|f\chi_k\|_{L^{p(\cdot)}}^q\right]\Bigg\}.$$

Now we prove Theorems 2.5 and 2.7.

Proof of Theorem 2.5. Let $f \in M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}$. We write

$$f = \sum_{j\in\mathbb{Z}} f(x)\chi_j(x) =: \sum_{j=-\infty}^{k-2} f_j(x) + \sum_{j=k-1}^{k+1} f_j(x) + \sum_{j=k+2}^{\infty} f_j(x).$$

By Definition 2.2, we obtain

$$\begin{aligned}
\left\|\mu_{\Omega,\lambda}^{\rho,*}(f)\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}^{q} &= \sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\mu_{\Omega,\lambda}^{\rho,*}(f)\chi_k\right\|_{L^{p(\cdot)}}^{q} \\
&\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^{q} \\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\sum_{j=k-1}^{k+1}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^{q} \\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^{q} \\
&=: C(I_1+I_2+I_3).
\end{aligned}$$

For I_1, we need to estimate $2^{\alpha(x)}\mu_{\Omega,\lambda}^{\rho,*}(f_j)(x)$ with $j\leqslant k-2$ and $x\in R_k$. By Minkowski's inequality, we get

$$\begin{aligned}
&2^{k\alpha(x)}\mu_{\Omega,\lambda}^{\rho,*}(f_j)(x) \\
=&2^{k\alpha(x)}\left[\int_0^\infty\int_{\mathbb{R}_+^{n+1}}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left|\int_{|y-z|\leqslant t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_j(z)\,dz\right|^2\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
\lesssim&2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left[\int_0^{|x|}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\int_{|y-z|\leqslant t}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}dz \\
&+2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left(\int_{|x|}^\infty\cdots\right)^{1/2}dz.
\end{aligned}$$

If $x\in R_k$, $z\in R_j$ and $j+2\leqslant k$, we have

$$t+|x-y|\geqslant|y-z|+|x-y|\geqslant|x|-|z|\geqslant\frac{1}{2}|x|.$$

By Lemmas 3.1, 3.6, the fact that $\rho>\frac{n}{2}$ and $\Omega\in L^2(S^{n-1})$, we conclude that

$$\begin{aligned}
&2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left[\int_0^{|x|}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\int_{|y-z|\leqslant t}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}dz \qquad (3.1)\\
\lesssim&2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left(\int_0^{|x|}\frac{1}{|x|^{\lambda n}}\int_{|y-z|\leqslant t}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+2\rho+1-\lambda n}}\right)^{1/2}dz \\
\sim&2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left(\int_0^{|x|}\frac{1}{|x|^{\lambda n}}\int_{S^{n-1}}\int_0^t\frac{u^{n-1}}{u^{2(n-\rho)}}|\Omega(u')|^2\,du\,d\sigma(u')\frac{dt}{t^{n+2\rho+1-\lambda n}}\right)^{1/2}dz \\
\lesssim&2^{k\alpha(x)}\|\Omega\|_{L^2(S^{n-1})}\int_{R_j}|f_j(z)|\left(\int_0^{|x|}\frac{1}{|x|^{\lambda n}}\int_0^t\frac{1}{u^{n-2\rho+1}}\,du\,\frac{dt}{t^{n+2\rho+1-\lambda n}}\right)^{1/2}dz
\end{aligned}$$

$$\begin{aligned}
&\sim 2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left(\int_0^{|x|}\frac{1}{|x|^{\lambda n}}\frac{dt}{t^{2n+1-\lambda n}}\right)^{1/2}dz\\
&\lesssim 2^{(k-j)\alpha_+}\frac{1}{|x|^n}\int_{R_j}2^{j\alpha(z)}|f_j(z)|\,dz\\
&\lesssim \frac{1}{|x|^n}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}}.
\end{aligned}$$

By the fact that $\rho > \frac{n}{2}$, Lemma 3.6 and a similar proof of the above inequality, we get

$$2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left[\int_{|x|}^{\infty}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\int_{|y-z|\leqslant t}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+2\rho+1}}\right]^{1/2}dz \tag{3.2}$$

$$\begin{aligned}
&\lesssim 2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left(\int_{|x|}^{\infty}\int_{|y-z|\leqslant t}\frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}}\frac{dydt}{t^{n+2\rho+1}}\right)^{1/2}dz\\
&\lesssim 2^{k\alpha(x)}\int_{R_j}|f_j(z)|\left(\int_{|x|}^{\infty}\frac{dt}{t^{2n+1}}\right)^{1/2}dz\\
&\lesssim \frac{1}{|x|^n}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}}.
\end{aligned}$$

By (3.1), (3.2), Lemmas 3.2 and 2.3, we get

$$\begin{aligned}
\left\|2^{k\alpha(\cdot)}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}} &\lesssim \frac{1}{2^{kn}}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\|\chi_{B_k}\|_{L^{p(\cdot)}}\\
&\lesssim \frac{1}{2^{kn}}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'(\cdot)}}\frac{|B_k|}{\|\chi_{B_k}\|_{L^{p'(\cdot)}}}\\
&\lesssim 2^{(j-k)(n\delta_2-\alpha_+)}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}.
\end{aligned}$$

To estimate I_1, below we consider two cases: $0<q\leqslant 1$ and $1<q<\infty$.

Case 1. If $0<q\leqslant 1$, by the fact that $n\delta_2-\alpha_+>0$, we see that

$$\begin{aligned}
I_1 &\lesssim \sup_{L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^q\\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k=-\infty}^{L}\sum_{j=-\infty}^{k-2}2^{q(j-k)(n\delta_2-\alpha_+)}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\\
&\lesssim \sup_{L\in\mathbb{Z}}2^{-L\gamma q}\sum_{j=-\infty}^{L-2}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\sum_{k=j+2}^{L}2^{q(j-k)(n\delta_2-\alpha_+)}\\
&\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}^q.
\end{aligned}$$

Case 2. If $1<q<\infty$, by Hölder's inequality and the fact that $n\delta_2-\alpha_+>0$, we obtain

$$\begin{aligned}
I_1 &\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-2}\left\|2^{k\alpha(\cdot)}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}\right)^q\\
&\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left(\sum_{j=-\infty}^{k-2}2^{q(j-k)(n\delta_2-\alpha_+)/2}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\right)\left(\sum_{j=-\infty}^{k-2}2^{(j-k)q'(n\delta_2-\alpha_+)/2}\right)^{q/q'}\\
&\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{j=-\infty}^{L-2}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q\sum_{k=j+2}^{L}2^{q(j-k)(n\delta_2-\alpha_+)/2}\\
&\lesssim \sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{j=-\infty}^{L-2}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}^q \lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}^q.
\end{aligned}$$

For I_2, by Lemmas 3.7 and 3.3, we have

$$\begin{aligned}
I_2 \sim \max\Bigg\{ &\sup_{L<0,L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(0)}\sum_{j=k-1}^{k+1}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^q,\\
&\sup_{L\geqslant 0,L\in\mathbb{Z}}\Bigg[2^{-L\gamma q}\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)}\sum_{j=k-1}^{k+1}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^q\\
&+2^{-L\gamma q}\sum_{k=0}^{L}\left\|2^{k\alpha_\infty}\sum_{j=k-1}^{k+1}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^q\Bigg]\Bigg\}\\
\lesssim \max\Bigg\{ &\sup_{L<0,L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(0)}|f_j|\chi_k\right\|_{L^{p(\cdot)}}^q,\\
&\sup_{L\geqslant 0,L\in\mathbb{Z}}\left[2^{-L\gamma q}\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)}|f_j|\chi_k\right\|_{L^{p(\cdot)}}^q+2^{-L\gamma q}\sum_{k=0}^{L}\left\|2^{k\alpha_\infty}|f_j|\chi_k\right\|_{L^{p(\cdot)}}^q\right]\Bigg\}\\
\lesssim \|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}^q.
\end{aligned}$$

For I_3, by Lemma 3.7, we get

$$\begin{aligned}
I_3 = C\max\Bigg\{ &\sup_{L<0,L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(0)}\sum_{j=k+2}^{\infty}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^q,\\
&\sup_{L\geqslant 0,L\in\mathbb{Z}}\Bigg[2^{-L\gamma q}\sum_{k=-\infty}^{-1}\left\|2^{k\alpha(0)}\sum_{j=k+2}^{\infty}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^q\\
&+2^{-L\gamma q}\sum_{k=0}^{L}\left\|2^{k\alpha_\infty}\sum_{j=k+2}^{\infty}\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}}^q\Bigg]\Bigg\}\\
=: C\max(I_{31},I_{32}).
\end{aligned}$$

If $x \in R_k$, $z \in R_j$ and $j \geqslant k+2$, we have

$$t+|x-y| \geqslant |y-z|+|x-y| \geqslant |x-z| \geqslant |z|-|x| \geqslant \frac{1}{2}|z|.$$

By Minkowski's inequality, Lemmas 2.3, 3.1 and a similar proof of I_1, we conclude that, for any $x \in R_k$,

$$\begin{aligned}
\mu_{\Omega,\lambda}^{\rho,*}(f_j)(x) &= \left[\int_0^\infty \int_{\mathbb{R}_+^{n+1}} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \left|\int_{|y-z|\leqslant t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)\,dz\right|^2 \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} \\
&\lesssim \int_{R_j} |f_j(z)| \left[\int_0^{2^j} \left(\frac{t}{t+|x-y|}\right)^{\lambda n} \int_{|y-z|\leqslant t} \frac{|\Omega(y-z)|^2}{|y-z|^{2(n-\rho)}} \frac{dydt}{t^{n+2\rho+1}}\right]^{1/2} dz \\
&\quad + \int_{R_j} |f_j(z)| \left(\int_{2^j}^\infty \cdots\right)^{1/2} dz \\
&\lesssim \frac{1}{2^{jn}} \int_{R_j} |f_j(z)|\,dz \lesssim \frac{1}{2^{jn}} \|f_j\|_{L^{p(\cdot)}} \|\chi_{B_j}\|_{L^{p'}(\cdot)}.
\end{aligned}$$

From this, Lemmas 3.2 and 2.3, we deduce that

$$\begin{aligned}
\left\|\sum_{j=k+2}^\infty \mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}} &\lesssim \sum_{j=k+2}^\infty \left\|\mu_{\Omega,\lambda}^{\rho,*}(f_j)\chi_k\right\|_{L^{p(\cdot)}} \\
&\lesssim \sum_{j=k+2}^\infty \frac{1}{2^{jn}} \|f_j\|_{L^{p(\cdot)}} \|\chi_{B_j}\|_{L^{p'}(\cdot)} \|\chi_{B_k}\|_{L^{p(\cdot)}} \\
&\lesssim \sum_{j=k+2}^\infty \frac{1}{2^{jn}} \|f_j\|_{L^{p(\cdot)}} \|\chi_{B_k}\|_{L^{p(\cdot)}} \frac{|B_j|}{\|\chi_{B_j}\|_{L^{p(\cdot)}}} \\
&\lesssim \sum_{j=k+2}^\infty 2^{(k-j)n\delta_1} \|f_j\|_{L^{p(\cdot)}}.
\end{aligned}$$

To estimate I_3, below we consider two cases: $0<q\leqslant 1$ and $1<q<\infty$.

Case 3. If $0<q\leqslant 1$, combining the above inequality, we have

$$\begin{aligned}
I_{31} &\leqslant C \sup_{L<0,L\in\mathbb{Z}} 2^{-L\gamma q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{j=k+2}^\infty 2^{(k-j)qn\delta_1} \|f_j\|_{L^{p(\cdot)}}^q \\
&= C \sup_{L<0,L\in\mathbb{Z}} 2^{-L\gamma q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{j=k+2}^{L-1} 2^{(k-j)qn\delta_1} \|f_j\|_{L^{p(\cdot)}}^q \\
&\quad + C \sup_{L<0,L\in\mathbb{Z}} 2^{-L\gamma q} \sum_{k=-\infty}^{L} 2^{kq\alpha(0)} \sum_{j=L}^\infty 2^{(k-j)qn\delta_1} \|f_j\|_{L^{p(\cdot)}}^q \\
&=: C(I_{31}^{'} + I_{31}^{''}).
\end{aligned}$$

For $I_{31}^{'}$, by the fact that $n\delta_1+\alpha(0)\geqslant n\delta_1+\alpha_->0$, we get

$$\begin{aligned}I_{31}^{'}&\lesssim \sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{j=-\infty}^{L-1}2^{jq\alpha(0)}\|f_j\|_{L^{p(\cdot)}}^q\sum_{k=-\infty}^{j-2}2^{(k-j)q(n\delta_1+\alpha(0))}\\&\lesssim \sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{j=-\infty}^{L-1}2^{jq\alpha(0)}\|f_j\|_{L^{p(\cdot)}}^q\lesssim\|f\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.\end{aligned}$$

Next, to deal with $I_{31}^{''}$, noticing that $\gamma-n\delta_1-\alpha(0)<\gamma-n\delta_1-\alpha_-<0$, hence we have

$$\begin{aligned}I_{31}^{''}&\lesssim \sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\sum_{j=L}^{\infty}2^{q(k-j)n\delta_1}2^{-j\alpha(0)q}2^{j\gamma q}2^{-j\gamma q}\sum_{s=-\infty}^{j}2^{s\alpha(0)q}\|f_s\|^q_{L^{p(\cdot)}}\\&\lesssim \sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k=-\infty}^{L}2^{kq(\alpha(0)+n\delta_1)}\sum_{j=L}^{\infty}2^{jq(\gamma-n\delta_1-\alpha(0))}\|f\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}\\&\lesssim\|f\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.\end{aligned}$$

Case 4. If $q>1$, we have

$$\begin{aligned}I_{31}&\leqslant C\sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=k+2}^{\infty}2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}}\right)^q\\&=C\sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=k+2}^{L}2^{(k-j)qn\delta_1}\|f_j\|_{L^{p(\cdot)}}\right)^q\\&\quad+C\sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k=-\infty}^{L}2^{kq\alpha(0)}\left(\sum_{j=L+1}^{\infty}2^{(k-j)qn\delta_1}\|f_j\|_{L^{p(\cdot)}}\right)^q\\&=:C(I_{31}^1+I_{31}^2).\end{aligned}$$

For I_{31}^1, by the fact that $n\delta_1+\alpha(0)>0$ and Hölder's inequality, we obtain

$$\begin{aligned}I_{31}^1&\lesssim \sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{k-\infty}^{L}\sum_{j=k+2}^{L}2^{jq\alpha(0)}\|f_j\|^q_{L^{p(\cdot)}}2^{(k-j)q(n\delta_1+\alpha(0))/2}\\&\quad\times\left(\sum_{j=k+2}^{L}2^{(k-j)q'(n\delta_1+\alpha(0))/2}\right)^{q/q'}\\&\lesssim \sup_{L<0,L\in\mathbb{Z}}2^{-L\gamma q}\sum_{j=-\infty}^{L}2^{jq\alpha(0)}\|f_j\|^q_{L^{p(\cdot)}}\sum_{k=-\infty}^{j-2}2^{(k-j)q(n\delta_1+\alpha(0))/2}\\&\lesssim\|f\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.\end{aligned}$$

For I_{31}^2, by Hölder's inequality and a similar proof of $I_{31}^{''}$, we obtain

$$I_{31}^2\lesssim\|f\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.$$

Combining the estimates of $I_{31}^{'}$, $I_{31}^{''}$, I_{31}^{1} and I_{31}^{2}, we have

$$I_{31} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.$$

By a similar estimate of I_{31}, it's easy to obtain

$$I_{32} \lesssim \|f\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.$$

Thus, we have

$$\left\|\mu^{\rho,*}_{\Omega,\lambda}(f)\right\|_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}} \lesssim I_1+I_2+I_3 \lesssim \|f\|_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}}.$$

This finishes the proof of Theorem 2.5. □

Proof of Theorem 2.7. Let $f \in M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}$, $b \in BMO$. We write

$$f = \sum_{j\in\mathbb{Z}} f(x)\chi_j(x) =: \sum_{j=-\infty}^{k-2} f_j(x) + \sum_{j=k-1}^{k+1} f_j(x) + \sum_{j=k+2}^{\infty} f_j(x).$$

By Definition 2.2, we obtain

$$\begin{aligned}
&\left\|[b^m,\mu^{\rho,*}_{\Omega,\lambda}](f)\right\|^q_{M\dot{K}^{\alpha(\cdot),\gamma}_{q,p(\cdot)}} = \sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}[b^m,\mu^{\rho,*}_{\Omega,\lambda}](f)\chi_k\right\|^q_{L^{p(\cdot)}}\\
&\leqslant C\sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\sum_{j=-\infty}^{k-2}[b^m,\mu^{\rho,*}_{\Omega,\lambda}](f_j)\chi_k\right\|^q_{L^{p(\cdot)}}\\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\sum_{j=k-1}^{k+1}[b^m,\mu^{\rho,*}_{\Omega,\lambda}](f_j)\chi_k\right\|^q_{L^{p(\cdot)}}\\
&\quad + C\sup_{L\in\mathbb{Z}} 2^{-L\gamma q}\sum_{k=-\infty}^{L}\left\|2^{k\alpha(\cdot)}\sum_{j=k+2}^{\infty}[b^m,\mu^{\rho,*}_{\Omega,\lambda}](f_j)\chi_k\right\|^q_{L^{p(\cdot)}}\\
&=: C(J_1+J_2+J_3).
\end{aligned}$$

Now, we begin with estimating J_1, if $z \in R_j$ and $j+2 \leqslant k$, by Lemma 3.1 and the estimate of $\mu^{\rho,*}_{\Omega,\lambda}(f_j)(x)$ in the proof of Theorem 2.5, then for any $x \in R_k$, we have

$$\begin{aligned}
&\left|2^{k\alpha(x)}[b^m,\mu^{\rho,*}_{\Omega,\lambda}](f_j)(x)\right| = \left|2^{k\alpha(x)}\mu^{\rho,*}_{\Omega,\lambda}[(b(x)-b(z))^m f_j](x)\right| \qquad (3.3)\\
&\lesssim |x|^{-n}2^{k\alpha(x)}\|(b(x)-b(\cdot))^m f_j\|_{L^1}\\
&\lesssim |x|^{-n}2^{k\alpha(x)}|b(x)-b_{B_j}|^m\|f_j\|_{L^1} + |x|^{-n}\|(b_{B_j}-b(\cdot))^m f_j\|_{L^1}\\
&\lesssim 2^{-kn}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\left(|b(x)-b_{B_j}|^m\|\chi_{B_j}\|_{L^{p'(\cdot)}} + \|(b_{B_j}-b)^m\chi_{B_j}\|_{L^{p'(\cdot)}}\right).
\end{aligned}$$

From this, the fact that $\gamma - n\delta_1 < \alpha_- \leqslant \alpha_+ < n\delta_2$, Lemmas 3.2, 2.3 and 3.6, we conclude that

$$
\begin{aligned}
&\left\|2^{\alpha(\cdot)}[b^m,\mu_{\Omega,\lambda}^{\rho,*}](f_j)\chi_k\right\|_{L^{p(\cdot)}}\\
&\lesssim 2^{-kn}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\\
&\quad\times\left[\left\|(b(\cdot)-b_{B_j})^m\chi_k\right\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'}(\cdot)}+\left\|(b_{B_j}-b(\cdot))^m\chi_{B_j}\right\|_{L^{p'}(\cdot)}\|\chi_k\|_{L^{p(\cdot)}}\right]\\
&\lesssim 2^{-kn}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\\
&\quad\times\left[\left\|(b(\cdot)-b_{B_j})^m\chi_k\right\|_{L^{p(\cdot)}}\|\chi_{B_j}\|_{L^{p'}(\cdot)}+\|b\|_{BMO}^m\|\chi_{B_j}\|_{L^{p'}(\cdot)}\|\chi_k\|_{L^{p(\cdot)}}\right]\\
&\lesssim 2^{-kn}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|b\|_{BMO}^m\\
&\quad\times\left[(k-j)^m\|\chi_{B_j}\|_{L^{p'}(\cdot)}\|\chi_k\|_{L^{p(\cdot)}}+\|\chi_{B_j}\|_{L^{p'}(\cdot)}\|\chi_k\|_{L^{p(\cdot)}}\right]\\
&\sim 2^{-kn}2^{(k-j)\alpha_+}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|b\|_{BMO}^m(k-j)^m\|\chi_{B_j}\|_{L^{p'}(\cdot)}\|\chi_k\|_{L^{p(\cdot)}}\\
&\lesssim 2^{(j-k)(n\delta_2-\alpha_+)}\left\|2^{j\alpha(\cdot)}f_j\right\|_{L^{p(\cdot)}}\|b\|_{BMO}^m(k-j)^m.
\end{aligned}
$$

Thus, combining the above estimate and a similar proof of I_1, we have

$$J_1\lesssim\|b\|_{BMO}^m\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}.$$

For J_2, by Lemma 3.2 and a similar proof of I_2, we see that

$$J_2\lesssim\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}.$$

Finally, we deal with J_3, if $x\in R_k$, $z\in R_j$ and $j\geqslant k+2$, by a similar proof of (3.3), we get

$$\left|[b^m,\mu_{\Omega,\lambda}^{\rho,*}](f_j)(x)\right|\lesssim 2^{-jn}\|f_j\|_{L^{p(\cdot)}}\left(|b(x)-b_{B_k}|^m\|\chi_{B_j}\|_{L^{p'}(\cdot)}+\|(b_{B_k}-b)^m\chi_{B_j}\|_{L^{p'}(\cdot)}\right).$$

From this, Lemmas 3.4, 3.2, 2.3 and a similar estimate of I_3, we further conclude that

$$\left\|[b^m,\mu_{\Omega,\lambda}^{\rho,*}](f_j)\chi_k\right\|_{L^{p(\cdot)}}\lesssim 2^{(k-j)n\delta_1}\|f_j\|_{L^{p(\cdot)}}\|b\|_{BMO}^m(j-k)^m.$$

Combining the above inequality and a similar estimate of I_3 in the proof of Theorem 2.5, we further conclude that

$$J_3\lesssim\|b\|_{BMO}^m\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}.$$

From the estimates of J_1, J_2 and J_3, we deduce that

$$\left\|[b^m,\mu_{\Omega,\lambda}^{\rho,*}](f)\right\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}\lesssim J_1+J_2+J_3\lesssim\|f\|_{M\dot{K}_{q,p(\cdot)}^{\alpha(\cdot),\gamma}}.$$

This finishes the proof of Theorem 2.7. □

Acknowledgements. The authors would like to express their deep thanks to the referees for their very careful reading and useful comments which do improve the presentation of this article.

REFERENCES

[1] A. ALMEDIDA AND D. DRIHEM, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math. Anal. Appl. 394 (2012), 781–795.

[2] E. ACERBI AND G. MINGIONE, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal. 164 (2002), 213–259.

[3] D. CRUZ-URIBE AND A. FIORENZA, *Variable Lebesgue spaces, Foundations and harmonic analysis*, Applied and Numerical Harmonic Analysis. Birkhäuser/Springer, Heidelberg, 2013. x+312 pp.

[4] L. DIENING, P. HARJULEHTO, P. HÄSTÖ AND M. RUŽIČKA, *Lebesgue and Sobolev Spaces with Variable Exponents*, Lecture Notes in Mathematics, 2017. Springer, Heidelberg, 2011. x+509 pp.

[5] M. IZUKI, *Boundedness of sublinear operators on Herz spaces with variable exponent and application to wavelet characterization*, Anal. Math. 36 (2010), 33–50.

[6] M. IZUKI, *Boundedness of commutators on Herz spaces with variable exponent*, Rend. del Circolo Mate. di Palero, 59 (2010), 199–213.

[7] M. IZUKI, *Fractional integrals on Herz-Morrey spaces with variable exponent*, Hiroshima Math. J. 40 (2010), 343–355.

[8] O. KOVÁČIK AND J. RÁKOSNÍK, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czech. Math. J. 41 (1991), 592–618.

[9] J. LIU, F. WEISZ, D. YANG AND W. YUAN, *Variable anisotropic Hardy spaces and their appliations*, Taiwanese J. Math. 64 (2014), 969–987.

[10] Y. LU AND Y. ZHU, *Boundedness of some sublinear operators and commutators on Morrey-Herz spaces with variable exponents*, Czech. Math. J. 64 (1991), 592–618.

[11] E. NAKAI AND Y. SAWANO, *Hardy spaces with variable exponents and generalized Campanato spaces*, J. Funct. Anal. 262 (2012), 3665–3748.

[12] E. STEIN AND G. WEISS, *On the theory of harmonic functions of several variables, I*, The theory of H^p-spaces, Acta Math. 103 (1960), 25–62.

[13] M. SAKAMOTO AND K. YABUTA, *Boundedness of Marcinkiewicz functions*, Studia Math. 135 (1999), 103–142.

[14] H. WANG, *Parametric Marcinkiewicz integrals on the weighted Hardy and weak Hardy spaces*, J. Math. Inequal. 10 (2016), 373–391.

[15] H. WANG AND Z. LIU, *The Herz-type Hardy spaces with variable exponent and their applications*, Taiwanese J. Math. 16 (2012), 1363–1389.

[16] L. WANG AND S. TAO, *Boundedness for parametrized Littlewood-Paley operators and their commutators on Lebesgue spaces with variable exponent*, Anal. Theory Appl. 31 (2015), 13–24.

[17] L. WANG AND S. TAO, *Parametrized Littlewood-Paley operators and their commutators on Herz spaces with variable exponent*, Turk J Math. 40 (2016), 122–145.

[18] Q. XUE AND Y. DING, *Weighted L^p boundedness for parametrized Littlewood-Paley operators*, Taiwanese J. Math. 11 (2007), 1143–1165.

(Received January 23, 2020)

Wenhua Wang
School of Mathematics and Statistics
Wuhan University
Wuhan 430072, Hubei, P.R. China
e-mail: whshuxue@yeah.net

Qingdong Guo
School of Mathematical Sciences
Xiamen University
Xiamen 361005, P.R. China
e-mail: guoqingdongshuxin@126.com