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ON THE COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF WEAKLY DEPENDENT RANDOM VARIABLES

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Abstract. The authors investigate the complete moment convergence for weighted sums of identically distributed ρ^*-mixing random variables. The obtained results improve the corresponding ones in Li et al. [Li, W., Chen, P. Y., Sung, S. H., 2017. Remark on convergence rate for weighted sums of ρ^*-mixing random variables. Rev. R. Acad. Cienc. Exactas Fs. Nat., Ser. A Mat. (RACSAM), 111: 507–513] and Wu and Peng [Wu, Y. F., Peng, J. Y., 2014. Strong convergence for weighted sums of ρ^*-mixing random variables. Glas. Mat., 49, 221–234]. The authors establish some much stronger conclusions under the same conditions of Li et al. (2017) and Wu and Peng (2014).

1. Introduction

Let $\{X_n, n \geqslant 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathscr{F}, P)$ and the σ-algebras are denoted as

$$\mathscr{F}_n^m = \sigma(X_k, n \leqslant k \leqslant m), \quad n \leqslant m \leqslant \infty.$$

Moore (1963) introduced the following concept of ρ^*-mixing random variables.

DEFINITION 1.1. A sequence of random variables $\{X_n, n \geqslant 1\}$ is called ρ^*-mixing if for some integer $k \geqslant 1$ the mixing coefficient

$$\rho^*(k) = \sup\sup\{\mathrm{Corr}(X,Y) : X \in \mathscr{L}^2(\mathscr{F}_S), Y \in \mathscr{L}^2(\mathscr{F}_T)\} < 1,$$

where $\mathscr{F}_S = \sigma\{X_i, i \in S\}$, $\mathscr{L}^2(\mathscr{F})$ is the class of all $\mathscr{F}$-measurable random variables with the finite second moment, and the outside sup is taken over all pairs of nonempty finite sets S, T of integers, such that $\min\{|s-t|, s \in S, t \in T\} \geqslant k$.

As far as we know, Bradley (1990) was the first who studied the limit theorems for ρ^*-mixing random variables. From then on, many authors investigated the limit theory for sequences ρ^*-mixing random variables, such as Bryc and Smoleński (1993), Peligrad and Gut (1999), Utev and Peligrad (2003), Cai (2006, 2008), Kuczmaszewska (2007), An and Yuan (2008), Wu and Jiang (2008), Zhou et al. (2011), Sung (2013),

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Wu et al. (2012), Wang et al. (2012), Guo and Zhu (2013), Wu et al. (2014), Wu and Peng (2014) and Li et al. (2017).

A sequence of random variables $\{U_n, n \geqslant 1\}$ converges completely to the constant a if

$$\sum_{n=1}^{\infty} P(|U_n - a| > \varepsilon) < \infty \quad \text{for all } \varepsilon > 0.$$

This concept of complete convergence was given firstly by Hsu and Robbins (1947).

By the Borel-Cantelli lemma, we know that the above result implies that $U_n \to a$ almost surely. Therefore, many authors applied the complete convergence to establish almost sure convergence of summation of random variables.

Let $\{Z_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0$, $b_n > 0$, $q > 0$. If

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|Z_n| - \varepsilon\}_+^q < \infty \text{ for some or all } \varepsilon > 0,$$

then the above result was called the complete moment convergence by Chow (1988). It will be shown that the complete moment convergence is the more general version of the complete convergence (see Remark 2.1 and 2.2).

A sequence of random variables $\{X_n, n \geqslant 1\}$ is said to be stochastically dominated by a random variable X (write $\{X_n\} \prec X$) if there exists a constant $C > 0$ such that

$$\sup_{n \geqslant 1} P(|X_n| > x) \leqslant CP(|X| > x), \ \forall x > 0.$$

Obviously stochastic dominance of $\{X_n, n \geqslant 1\}$ by the random variable X implies $E|X_n|^p \leqslant CE|X|^p$ if the p-moment of $|X|$ exists, i. e., if $E|X|^p < \infty$.

We first state the following complete convergence result for weighted sums of identically distributed ρ^*-mixing random variables.

THEOREM A. *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX_1 = 0$, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n) \tag{1.1}$$

for some $1 < \alpha \leqslant 2$. If for some $\gamma > 0$,

$$\begin{cases} E|X_1|^{\alpha} < \infty, & \text{for } \alpha > \gamma, \\ E|X_1|^{\alpha} \log|X| < \infty, & \text{for } \alpha = \gamma, \\ E|X_1|^{\gamma} < \infty, & \text{for } \alpha < \gamma, \end{cases} \tag{1.2}$$

then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} a_{ni} X_i \right| > b_n \varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0, \tag{1.3}$$

where $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$.

Zhou et al. (2011) proved the above theorem for the case $\alpha > \gamma$. Sung (2013) pointed out that Zhou et al. (2011) left an open problem whether the case $\alpha = \gamma$ of

Theorem A remains true. Sung (2013) solved the open problem but he also presented an new open problem, that is, whether the case $\alpha < \gamma$ of Theorem A remains true? Wu et al. (2014) applied some new methods to solve the open problem posed by Sung (2013).

Li et al. (2017) extended the result of Chen and Sung (2014) from negatively associated random variables to ρ^*-mixing random variables, and obtained the following theorem.

THEOREM B. *Let $\{X_n, n \geqslant 1\}$ be a sequence of identically distributed ρ^*-mixing random variables with $EX_1 = 0$, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying (1.1) for some $1 < \alpha \leqslant 2$. Set $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$, where $0 < \gamma < \alpha$. If $E|X_1|^\alpha/(\log|X_1|)^{\alpha/\gamma-1} < \infty$, then (1.3) holds.*

Wu and Peng (2014) presented the following complete convergence theorem, which extended and improve the corresponding one in Bai and Cheng (2000).

THEOREM C. *Suppose $1/p = 1/\alpha + 1/\beta$ for $1 < \alpha, \beta < \infty$ and $1 < p < 2$. Let $\{X_n, n \geqslant 1\}$ be a sequence of ρ^*-mixing random variables with $EX_n = 0$ and $\{X_n\} \prec X$, and let $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ be an array of constants satisfying (1.1). Then the following statements hold:*
(i) If $\alpha < \beta$, then $E|X|^\beta < \infty$ implies

$$\sum_{n=1}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > n^{1/p}\varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{1.4}$$

(ii) If $\alpha = \beta$, then $E|X|^\beta \log^+|X| < \infty$ implies (1.4).
(iii) If $\alpha > \beta$, then $E|X|^\alpha < \infty$ implies (1.4).

In this paper, the authors study the complete moment convergence for weighted sums of ρ^*-mixing random variables. The obtained results improve Theorem B and Theorem C in two directions, namely:

(i) Under the same conditions of Theorem B, we get (2.2) which is much stronger than (1.3).

(ii) Under the same conditions of Theorem C for $\alpha > \beta$, we get (2.3) and (2.4) which are also much stronger than (1.4).

Throughout this paper, the symbol C always stands for a generic positive constant which may differ from one place to another. $\log x = \max\{1, \ln x\}$. The symbol $I(A)$ denotes the indicator function of the event A.

2. Preliminaries and main results

To prove our main results, we need the following lemmas.

LEMMA 2.1. (Utev and Peligrad, 2003) *Suppose N is a positive integer, $0 \leqslant r < 1$, and $q \geqslant 2$. Then there exists a positive constant $C = C(N, r, q)$ such that the following statement holds:*

If $\{X_k, k \geqslant 1\}$ *is a sequence of random variables such that* $\rho_N^* \leqslant r$*, and such that* $EX_k = 0$ *and* $E|X_k|^q < \infty$ *for every* $k \geqslant 1$*, then for all* $n \geqslant 1$*,*

$$E\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_k\right|\right\}^q \leqslant C\left\{\sum_{k=1}^{n} E|X_k|^q + \left(\sum_{k=1}^{n} EX_k^2\right)^{q/2}\right\}.$$

By means of Lemma 2.1, Chen and Sung (2016) presented the following Marcinkiewicz-Zygmund inequality with exponent p $(1 < p < 2)$ for ρ^*-mixing random variables.

LEMMA 2.2. *Suppose* N *is a positive integer,* $0 \leqslant r < 1$*, and* $1 < p < 2$*. Then there exists a positive constant* $C = C(N, r, p)$ *such that the following statement holds:*

If $\{X_k, k \geqslant 1\}$ *is a sequence of random variables such that* $\rho_N^* \leqslant r$*, and such that* $EX_k = 0$ *and* $E|X_k|^p < \infty$ *for every* $k \geqslant 1$*, then for all* $n \geqslant 1$*,*

$$E\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_k\right|\right\}^p \leqslant C\sum_{k=1}^{n} E|X_k|^p.$$

LEMMA 2.3. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of random variables with* $\{X_n\} \prec X$*. Then there exists a constant* C *such that, for all* $q > 0$ *and* $x > 0$*,*

(i) $E|X_k|^q I(|X_k| \leqslant x) \leqslant C\{E|X|^q I(|X| \leqslant x) + x^q P(|X| > x)\}$,

(ii) $E|X_k|^q I(|X_k| > x) \leqslant CE|X|^q I(|X| > x)$.

This lemma can be easily proved by using integration by parts. Therefore, we omit the details. Now we state our main results and the proofs will be presented in next section.

THEOREM 2.1. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of identically distributed* ρ^**-mixing random variables with* $EX_1 = 0$*, and let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying (1.1) for some* $1 < \alpha \leqslant 2$*. Set* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$*, where* $0 < \gamma < \alpha$*. If*

$$E|X_1|^\alpha / (\log|X_1|)^{\alpha/\gamma - 1} < \infty, \tag{2.1}$$

then for $0 < \theta < \alpha$*,*

$$\sum_{n=1}^{\infty} n^{-1} E\left\{b_n^{-1} \max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon\right\}_+^\theta < \infty \quad \text{for all } \varepsilon > 0. \tag{2.2}$$

REMARK 2.1. Noting that the conditions of Theorem 2.1 are the same as those of Theorem B and

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{-1} E\left\{b_n^{-1} \max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon\right\}_+^\theta \\
&= \sum_{n=1}^{\infty} n^{-1} \int_0^\infty P\left(b_n^{-1} \max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon > t^{1/\theta}\right) \mathrm{d}t \\
&= \int_0^\infty \sum_{n=1}^{\infty} n^{-1} P\left(b_n^{-1} \max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon + t^{1/\theta}\right) \mathrm{d}t.
\end{aligned}$$

Hence Theorem 2.2 improves Theorem B.

THEOREM 2.2. *Suppose* $1/p = 1/\alpha + 1/\beta$ *for* $1 < \beta < \alpha < \infty$ *and* $1 < p < 2$. *Let* $\{X_n, n \geqslant 1\}$ *be a sequence of* ρ^**-mixing random variables with* $EX_n = 0$ *and* $\{X_n\} \prec X$, *and let* $\{a_{ni}, 1 \leqslant i \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying (1.1). Then* $E|X|^\alpha < \infty$ *implies*

$$\sum_{n=1}^{\infty} n^{\mu-\alpha/p} E\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - n^{1/p}\varepsilon\right\}_+^{\alpha} < \infty \quad \text{for all } \varepsilon > 0, \tag{2.3}$$

where $0 \leqslant \mu < \alpha/\beta - 1$. *Moreover, we have*

$$\sum_{n=1}^{\infty} n^{\mu} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > n^{1/p}\varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0. \tag{2.4}$$

REMARK 2.2. Similar to Remark 2.1, we know that

$$\begin{aligned}
\infty > & \sum_{n=1}^{\infty} n^{\mu-\alpha/p} E\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - n^{1/p}\varepsilon\right\}_+^{\alpha} \\
= & \sum_{n=1}^{\infty} n^{\mu} \int_0^{\infty} P\left(n^{-1/p}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon > t^{1/\alpha}\right) \mathrm{d}t \\
\geqslant & \sum_{n=1}^{\infty} n^{\mu} \int_0^{\varepsilon^{\alpha}} P\left(n^{-1/p}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon + t^{1/\alpha}\right) \mathrm{d}t \\
\geqslant & \sum_{n=1}^{\infty} n^{\mu} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > 2n^{1/p}\varepsilon\right).
\end{aligned}$$

Therefore, Theorem 2.2 improves Theorem C for the case $\alpha > \beta$ under the same conditions.

3. Proofs of main results

Proof of Theorem 2.1. For any given $\varepsilon > 0$, we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} n^{-1} E\left\{b_n^{-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon\right\}_+^{\theta} \\
= & \sum_{n=1}^{\infty} n^{-1} \int_0^{\infty} P\left(b_n^{-1}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon > t^{1/\theta}\right) \mathrm{d}t \\
\leqslant & \sum_{n=1}^{\infty} n^{-1} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > b_n\varepsilon\right) \\
& + \sum_{n=1}^{\infty} n^{-1} \int_1^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > b_n t^{1/\theta}\right) \mathrm{d}t \\
=: & I_1 + I_2.
\end{aligned}$$

From Theorem B, we get directly $I_1 < \infty$. To prove (2.2), we need only to show $I_2 < \infty$. For all $t \geqslant 1$, let

$$Y_{ni} = a_{ni}X_iI(|X_i| \leqslant h(n)t^{1/\theta}),\ \ Z_{ni} = a_{ni}X_i - Y_{ni} = a_{ni}X_iI(|X_i| > h(n)t^{1/\theta}),$$

where $h(x)$ is an increasing function defined on $[0, \infty)$ such that $h(0) = 0$ and $h(n) = n^{1/\alpha}(\log n)^{1/\gamma-1/\alpha}$. Then

$$\begin{aligned} I_2 &\leqslant \sum_{n=1}^{\infty} n^{-1}\sum_{i=1}^{n}\int_1^{\infty} P\big(|X_i| > h(n)t^{1/\theta}\big)\mathrm{d}t \\ &\quad + \sum_{n=1}^{\infty} n^{-1}\int_1^{\infty} P\bigg(\max_{1\leqslant j\leqslant n}\bigg|\sum_{i=1}^{j} Y_{ni}\bigg| > b_n t^{1/\theta}\bigg)\mathrm{d}t \\ &=: I_3 + I_4. \end{aligned}$$

Observe that

$$\int_1^{\infty} P\big(|X_i| > h(n)t^{1/\theta}\big)\mathrm{d}t \leqslant (h(n))^{-\theta}E|X_i|^{\theta}I(|X_i| > h(n))$$

and

$$\sum_{n=1}^{m}(h(n))^{-\theta} \leqslant Cm^{1-\theta/\alpha}(\log m)^{-\theta(1/\gamma-1/\alpha)}.$$

Hence

$$\begin{aligned} I_3 &\leqslant \sum_{n=1}^{\infty} n^{-1}(h(n))^{-\theta}\sum_{i=1}^{n} E|X_i|^{\theta}I(|X_i| > h(n)) \\ &= \sum_{n=1}^{\infty}(h(n))^{-\theta}E|X_1|^{\theta}I(|X_1| > h(n)) \\ &= \sum_{n=1}^{\infty}(h(n))^{-\theta}\sum_{m=n}^{\infty} E|X_1|^{\theta}I(h(m) < |X_1| \leqslant h(m+1)) \\ &= \sum_{m=1}^{\infty} E|X_1|^{\theta}I(h(m) < |X_1| \leqslant h(m+1))\sum_{n=1}^{m}(h(n))^{-\theta} \\ &\leqslant C\sum_{m=1}^{\infty} m^{1-\theta/\alpha}(\log m)^{-\theta(1/\gamma-1/\alpha)}E|X_1|^{\theta}I(h(m) < |X_1| \leqslant h(m+1)). \end{aligned}$$

Obviously $x^{\alpha-\theta}/(\log x)^{\alpha/\gamma-1}$ is an increasing function on $[T, \infty]$ for some large $T > 0$. So if $h(m) < |X_1|$, we can obtain

$$\frac{(h(m))^{\alpha-\theta}}{(\log h(m))^{\alpha/\gamma-1}} \leqslant \frac{|X_1|^{\alpha-\theta}}{(\log|X_1|)^{\alpha/\gamma-1}},$$

that is

$$|X_1|^{\theta} \leqslant \frac{(\log h(m))^{\alpha/\gamma-1}}{(h(m))^{\alpha-\theta}}\frac{|X_1|^{\alpha}}{(\log|X_1|)^{\alpha/\gamma-1}}. \tag{3.1}$$

Then

$$\begin{aligned}
I_3 &\leqslant C\sum_{m=1}^{\infty} m^{1-\theta/\alpha}(\log m)^{-\theta(1/\gamma-1/\alpha)}\frac{(\log h(m))^{\alpha/\gamma-1}}{(h(m))^{\alpha-\theta}} \\
&\quad \times E|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1}I(h(m)<|X_1|\leqslant h(m+1)) \\
&= C\sum_{m=1}^{\infty}\left[\frac{\log m+(\alpha/\gamma-1)\log\log m}{\alpha\log m}\right]^{\alpha/\gamma-1} \\
&\quad \times E|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1}I(h(m)<|X_1|\leqslant h(m+1)) \\
&\leqslant C\sum_{m=1}^{\infty} E|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1}I(h(m)<|X_1|\leqslant h(m+1)) \\
&\leqslant CE|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1}<\infty.
\end{aligned}$$

Next we prove that $I_4<\infty$. We first show

$$\sup_{t\geqslant 1} b_n^{-1}t^{-1/\theta}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right|\to 0 \quad \text{as } n\to\infty. \tag{3.2}$$

Without loss of generality, we assume that $\sum_{i=1}^{n}|a_{ni}|^{\alpha}\leqslant n$. Then by the Hölder inequality, one can easily obtain that $\sum_{i=1}^{n}|a_{ni}|\leqslant n$. From $EX_i=0$, $\sum_{i=1}^{n}|a_{ni}|\leqslant n$, and (3.1) for $\theta=1$, we have

$$\begin{aligned}
&\sup_{t\geqslant 1} b_n^{-1}t^{-1/\theta}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right| = \sup_{t\geqslant 1} b_n^{-1}t^{-1/\theta}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EZ_{ni}\right| \\
&\leqslant \sup_{t\geqslant 1} b_n^{-1}t^{-1/\theta}\sum_{i=1}^{n}|a_{ni}|E|X_i|I(|X_i|>h(n)t^{1/\theta}) \\
&\leqslant nb_n^{-1}E|X_1|I(|X_1|>h(n)) \\
&\leqslant nb_n^{-1}\frac{(\log h(n))^{\alpha/\gamma-1}}{(h(n))^{\alpha-1}}E|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1}I(|X_1|>h(n)) \\
&= (\log n)^{-1/\alpha}\left[\frac{\log n+(\alpha/\gamma-1)\log\log n}{\alpha\log n}\right]^{\alpha/\gamma-1}E|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1}I(|X_1|>h(n)) \\
&\quad \to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

Hence while n is sufficiently large,

$$\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}EY_{ni}\right|\leqslant b_nt^{1/\theta}/2 \tag{3.3}$$

holds uniformly for $t\geqslant 1$. Therefore, by (3.3), Lemma 2.2 and C_r inequality, we have

$$\begin{aligned}
I_4 &\leqslant \sum_{n=1}^{\infty} n^{-1}\int_1^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(Y_{ni}-EY_{ni})\right|>b_nt^{1/\theta}/2\right)\mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} n^{-1}b_n^{-\alpha}\sum_{i=1}^{n}\int_1^{\infty} t^{-\alpha/\theta}E|Y_{ni}|^{\alpha}\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} \int_1^{\infty} t^{-\alpha/\theta} E|X_i|^{\alpha} I(|X_i| \leqslant h(n)) \mathrm{d}t \\
&\quad + C\sum_{n=1}^{\infty} n^{-1} b_n^{-\alpha} \sum_{i=1}^{n} |a_{ni}|^{\alpha} \int_1^{\infty} t^{-\alpha/\theta} E|X_i|^{\alpha} I(h(n) < |X_i| \leqslant h(n) t^{1/\theta}) \mathrm{d}t \\
&=: I_5 + I_6.
\end{aligned}
$$

From $\alpha > \theta$ and (1.1), we have

$$
\begin{aligned}
I_5 &\leqslant C\sum_{n=1}^{\infty} b_n^{-\alpha} E|X_1|^{\alpha} I(|X_1| \leqslant h(n)) \\
&= C\sum_{n=1}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} \sum_{m=1}^{n} E|X_1|^{\alpha} I(h(m-1) < |X_1| \leqslant h(m)) \\
&= C\sum_{m=1}^{\infty} E|X_1|^{\alpha} I(h(m-1) < |X_1| \leqslant h(m)) \sum_{n=m}^{\infty} n^{-1} (\log n)^{-\alpha/\gamma} \\
&\leqslant C\sum_{m=1}^{\infty} (\log m)^{1-\alpha/\gamma} E|X_1|^{\alpha} I(h(m-1) < |X_1| \leqslant h(m)).
\end{aligned}
$$

As stated in the proof of $I_3 < \infty$, $x^{\alpha-1}/(\log x)^{\alpha/\gamma-1}$ is an increasing function on $[T, \infty]$ for some large $T > 0$. Hence, while m is sufficiently large, we have $m^{\alpha-1} \geqslant (\log m)^{\alpha/\gamma-1}$, that is $m \geqslant h(m)$. Therefore,

$$
\begin{aligned}
I_5 &\leqslant C\sum_{m=1}^{\infty} E|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1} I(h(m-1) < |X_1| \leqslant h(m)) \\
&\leqslant CE|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1} < \infty.
\end{aligned}
$$

Finally we will show that $I_6 < \infty$. Observe that $\sum_{m=s}^{\infty} m^{-\alpha/\theta} \leqslant \alpha/(\alpha-\theta) s^{1-\alpha/\theta}$ and $(s+1)/s \leqslant 2$ for $\alpha > \theta$ and all $s \geqslant 1$. Then

$$
\begin{aligned}
I_6 &\leqslant C\sum_{n=1}^{\infty} b_n^{-\alpha} \int_1^{\infty} t^{-\alpha/\theta} E|X_1|^{\alpha} I(h(n) < |X_1| \leqslant h(n) t^{1/\theta}) \mathrm{d}t \\
&= C\sum_{n=1}^{\infty} b_n^{-\alpha} \sum_{m=1}^{\infty} \int_m^{m+1} t^{-\alpha/\theta} E|X_1|^{\alpha} I(h(n) < |X_1| \leqslant h(n) t^{1/\theta}) \mathrm{d}t \\
&\leqslant C\sum_{n=1}^{\infty} b_n^{-\alpha} \sum_{m=1}^{\infty} m^{-\alpha/\theta} E|X_1|^{\alpha} I(h(n) < |X_1| \leqslant h(n)(m+1)^{1/\theta}) \\
&= C\sum_{n=1}^{\infty} b_n^{-\alpha} \sum_{m=1}^{\infty} m^{-\alpha/\theta} \sum_{s=1}^{m} E|X_1|^{\alpha} I(h(n)s < |X_1| \leqslant h(n)(s+1)^{1/\theta}) \\
&= C\sum_{n=1}^{\infty} b_n^{-\alpha} \sum_{s=1}^{\infty} E|X_1|^{\alpha} I(h(n)s < |X_1| \leqslant h(n)(s+1)^{1/\theta}) \sum_{m=s}^{\infty} m^{-\alpha/\theta} \\
&\leqslant C\frac{\alpha}{\alpha-\theta} \sum_{n=1}^{\infty} b_n^{-\alpha} \sum_{s=1}^{\infty} s^{1-\alpha/\theta} E|X_1|^{\alpha} I(h(n)s < |X_1| \leqslant h(n)(s+1)^{1/\theta})
\end{aligned}
$$

$$\leqslant C\frac{\alpha}{\alpha-\theta}\sum_{n=1}^{\infty} b_n^{-\alpha}(h(n))^{\alpha-\theta}\sum_{s=1}^{\infty}\Big(\frac{s+1}{s}\Big)^{\alpha/\theta-1}E|X_1|^{\theta}I(h(n)s<|X_1|\leqslant h(n)(s+1)^{1/\theta})$$
$$\leqslant C\frac{\alpha 2^{\alpha/\theta-1}}{\alpha-\theta}\sum_{n=1}^{\infty} b_n^{-\alpha}(h(n))^{\alpha-\theta}\sum_{s=1}^{\infty}E|X_1|^{\theta}I(h(n)s<|X_1|\leqslant h(n)(s+1)^{1/\theta})$$
$$=C\sum_{n=1}^{\infty} n^{-\theta/\alpha}(\log n)^{-1-\theta(1/\gamma-1/\alpha)}E|X_1|^{\theta}I(|X_1|>h(n)).$$

Therefore, by some similar arguments as in the proof of $I_3<\infty$, we can obtain

$$\begin{aligned}I_6 &\leqslant C\sum_{n=1}^{\infty} n^{-\theta/\alpha}(\log n)^{-\theta(1/\gamma-1/\alpha)}E|X_1|^{\theta}I(|X_1|>h(n))\\ &\leqslant CE|X_1|^{\alpha}/(\log|X_1|)^{\alpha/\gamma-1}<\infty.\end{aligned}$$

The proof is completed. □

Proof of Theorem 2.2. We first prove (2.4). From assumption (1.1), without loss of generality, we may assume that $\sum_{i=1}^{n}|a_{ni}|^{\alpha}\leqslant n$. Then we may obtain that

$$\sum_{i=1}^{n}|a_{ni}|^{\gamma}\leqslant n,\quad \forall\, 1\leqslant\gamma<\alpha \ \text{ and }\ \sum_{i=1}^{n}|a_{ni}|^{\gamma}\leqslant n^{\frac{\gamma}{\alpha}},\quad \forall\,\gamma\geqslant\alpha. \tag{3.4}$$

For fixed $n\geqslant 1$, let $Y'_{ni}=a_{ni}X_iI(|a_{ni}X_i|\leqslant n^{1/p})$, $Z'_{ni}=a_{ni}X_i-Y'_{ni}$. Then

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{\mu}P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}a_{ni}X_i\Big|>n^{1/p}\varepsilon\Big)\\ &\leqslant \sum_{n=1}^{\infty} n^{\mu}P\Big(\max_{1\leqslant i\leqslant n}|a_{ni}X_i|>n^{1/p}\Big)+\sum_{n=1}^{\infty} n^{\mu}P\Big(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}Y'_{ni}\Big|>n^{1/p}\varepsilon\Big)\\ &=:I_7+I_8.\end{aligned}$$

Obviously $\sum_{i=1}^{n}|a_{ni}|^{\alpha}\leqslant n$ implies $|a_{ni}|\leqslant n^{1/\alpha}$. Hnece by $1/p=1/\alpha+1/\beta$, $0\leqslant\mu<\alpha/\beta-1$ and $E|X|^{\alpha}<\infty$, we have

$$\begin{aligned}I_7&\leqslant\sum_{n=1}^{\infty} n^{\mu}\sum_{i=1}^{n}P\big(|a_{ni}X|>n^{1/p}\big)\\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-\alpha/p}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha}I\big(|a_{ni}X|>n^{1/p}\big)\\ &=C\sum_{n=1}^{\infty} n^{\mu-\alpha/p}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha}I\big(|X|>n^{1/p}|a_{ni}|^{-1}\big)\\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-\alpha/\beta}E|X|^{\alpha}I\big(|X|>n^{1/\beta}\big)<\infty.\end{aligned}$$

Then we will show $I_8 < \infty$. We first prove that

$$n^{-1/p} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} EY'_{ni} \right| \to 0 \quad \text{as } n \to \infty. \tag{3.5}$$

By $EX_n = 0$, Lemma 2.3 and $1/p = 1/\alpha + 1/\beta$, we obtain

$$\begin{aligned} n^{-1/p} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} EY'_{ni} \right| &= n^{-1/p} \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} EZ'_{ni} \right| \\ &\leqslant n^{-1/p} \sum_{i=1}^{n} E|a_{ni}X_i| I(|a_{ni}X_i| > n^{1/p}) \\ &\leqslant n^{-\alpha/p} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X| > n^{1/p}) \\ &\leqslant C n^{-\alpha/\beta} E|X|^{\alpha} I(|X| > n^{1/\beta}) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Hence while n is sufficiently large, $\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} EY'_{ni} \right| < n^{1/p}\varepsilon/2$.

Take $q > \max\{\alpha, 2, 2p(\mu+1)/(2-p)\}$. So it follows by the Markov inequality and Lemma 2.1 that

$$\begin{aligned} I_8 &\leqslant \sum_{n=1}^{\infty} n^{\mu} P\left(\max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} (Y'_{ni} - EY'_{ni}) \right| > n^{1/p}\varepsilon/2 \right) \\ &\leqslant C \sum_{n=1}^{\infty} n^{\mu - q/p} E \max_{1 \leqslant j \leqslant n} \left| \sum_{i=1}^{j} (Y'_{ni} - EY'_{ni}) \right|^q \\ &\leqslant C \sum_{n=1}^{\infty} n^{\mu - q/p} \sum_{i=1}^{n} E|Y'_{ni}|^q + C \sum_{n=1}^{\infty} n^{\mu - q/p} \left(\sum_{i=1}^{n} E(Y'_{ni})^2 \right)^{q/2} \\ &=: I_9 + I_{10}. \end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned} I_9 &\leqslant C \sum_{n=1}^{\infty} n^{\mu - q/p} \sum_{i=1}^{n} |a_{ni}|^q E|X|^q I(|a_{ni}X| \leqslant n^{1/p}) \\ &\quad + C \sum_{n=1}^{\infty} n^{\mu} \sum_{i=1}^{n} P(|a_{ni}X| > n^{1/p}) \\ &=: I'_9 + I''_9. \end{aligned}$$

The proof of $I''_9 < \infty$ is the same as $I_7 < \infty$. Hence we will prove $I'_9 < \infty$. By $q > \alpha$, we have

$$\begin{aligned} I'_9 &\leqslant C \sum_{n=1}^{\infty} n^{\mu - \alpha/p} \sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X| \leqslant n^{1/p}) \\ &\leqslant C \sum_{n=1}^{\infty} n^{\mu - \alpha/\beta} E|X|^{\alpha} < \infty. \end{aligned}$$

By Lemma 2.3 and C_r inequality, we have

$$\begin{aligned} I_{10} &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\left(\sum_{i=1}^{n} a_{ni}^2 EX^2 I(|a_{ni}X|\leqslant n^{1/p}) + n^{2/p}\sum_{i=1}^{n} P(|a_{ni}X|>n^{1/p})\right)^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\left(\sum_{i=1}^{n} a_{ni}^2 EX^2 I(|a_{ni}X|\leqslant n^{1/p})\right)^{q/2} \\ &\quad + C\sum_{n=1}^{\infty} n^{\mu}\left(\sum_{i=1}^{n} P(|a_{ni}X|>n^{1/p})\right)^{q/2} \\ &=: I'_{10} + I''_{10}. \end{aligned}$$

By $0 \leqslant \mu < \alpha/\beta - 1$ and $q > 2$, we have

$$\begin{aligned} I''_{10} &\leqslant C\sum_{n=1}^{\infty} n^{\mu}\left(n^{-\alpha/p}\sum_{i=1}^{n} |a_{ni}|^{\alpha} E|X|^{\alpha} I(|a_{ni}X|>n^{1/p})\right)^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-\frac{\alpha q}{2\beta}}\left(E|X|^{\alpha} I(|X|>n^{1/\beta})\right)^{q/2} < \infty. \end{aligned}$$

From $1/p = 1/\alpha + 1/\beta$ and $1 < p < 2$, we know that $\alpha \leqslant 2$ and $\beta \leqslant 2$ can not hold simultaneously. Since it is assumed that $\alpha > \beta$, we have $\alpha > 2$. Hence

$$\begin{aligned} I'_{10} &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\left(\sum_{i=1}^{n} a_{ni}^2 EX^2\right)^{q/2} \\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p+q/2}(EX^2)^{q/2} < \infty. \qquad (\text{by } q > 2p(\mu+1)/(2-p)) \end{aligned}$$

The proof of (2.4) is completed. □

Next we present the proof of (2.3). For any given $\varepsilon > 0$, we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\mu-\alpha/p} E\left\{\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - n^{1/p}\varepsilon\right\}_+^{\alpha} \\ =\ &\sum_{n=1}^{\infty} n^{\mu}\int_0^{\infty} P\left(n^{-1/p}\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| - \varepsilon > t^{1/\alpha}\right)\mathrm{d}t \\ \leqslant\ &\sum_{n=1}^{\infty} n^{\mu} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > n^{1/p}\varepsilon\right) \\ &+\sum_{n=1}^{\infty} n^{\mu}\int_1^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > n^{1/p}t^{1/\alpha}\right)\mathrm{d}t \\ =:\ & I_{11} + I_{12}. \end{aligned}$$

From the proof of (2.4), we can obtain $I_{11} < \infty$. Hence we need only to prove $I_{12} < \infty$. For all $t \geqslant 1$, define

$$Y''_{ni} = a_{ni}X_i I(|a_{ni}X_i| \leqslant n^{1/p}t^{1/\alpha}),\ \ Z''_{ni} = a_{ni}X_i - Y''_{ni},$$

then

$$\begin{aligned} I_{12} \;&\leqslant\; \sum_{n=1}^{\infty} n^{\mu} \int_1^{\infty} P\Big(\max_{1\leqslant j\leqslant n}\big|a_{ni}X_i\big| > n^{1/p}t^{1/\alpha}\Big)\mathrm{d}t \\ &\quad + \sum_{n=1}^{\infty} n^{\mu} \int_1^{\infty} P\bigg(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} Y''_{ni}\Big| > n^{1/p}t^{1/\alpha}\bigg)\mathrm{d}t \\ &=: I_{13}+I_{14}. \end{aligned}$$

From $\sum_{i=1}^{n}|a_{ni}|^{\alpha} \leqslant n$, $|a_{ni}| \leqslant n^{1/\alpha}$, $1/p = 1/\alpha + 1/\beta$ and

$$\int_1^{\infty} P\big(\big|a_{ni}X\big| > n^{1/p}t^{1/\alpha}\big)\mathrm{d}t \;\leqslant\; n^{-\alpha/p}|a_{ni}|^{\alpha}E|X|^{\alpha}I(|a_{ni}X| > n^{1/p}),$$

we have

$$\begin{aligned} I_{13} &\leqslant \sum_{n=1}^{\infty} n^{\mu} \sum_{i=1}^{n} \int_1^{\infty} P\big(\big|a_{ni}X\big| > n^{1/p}t^{1/\alpha}\big)\mathrm{d}t \\ &\leqslant \sum_{n=1}^{\infty} n^{\mu-\alpha/p} \sum_{i=1}^{n} |a_{ni}|^{\alpha}E|X|^{\alpha}I(|a_{ni}X| > n^{1/p}) \\ &\leqslant \sum_{n=1}^{\infty} n^{\mu-\alpha/\beta}E|X|^{\alpha}I(|X| > n^{1/\beta}) < \infty. \end{aligned}$$

It follows from $EX_n = 0$ that

$$\begin{aligned} &\sup_{t\geqslant 1} n^{-1/p}t^{-1/\alpha} \max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} EY''_{ni}\Big| \;=\; \sup_{t\geqslant 1} n^{-1/p}t^{-1/\alpha} \max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} EZ''_{ni}\Big| \\ &\leqslant \sup_{t\geqslant 1} n^{-1/p}t^{-1/\alpha} \sum_{i=1}^{n} |a_{ni}|E|X_i|I(|a_{ni}X_i| > n^{1/p}t^{1/\alpha}) \\ &\leqslant n^{-\alpha/p} \sum_{i=1}^{n} |a_{ni}|^{\alpha}E|X|^{\alpha}I(|a_{ni}X| > n^{1/p}) \\ &\leqslant n^{-\alpha/\beta}E|X|^{\alpha}I(|X| > n^{1/\beta}) \to 0 \quad \text{as } n\to\infty. \end{aligned}$$

Therefore while n is sufficiently large,

$$\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j} EY''_{ni}\Big| \leqslant n^{1/p}t^{1/\alpha}/2 \tag{3.6}$$

holds uniformly for $t \geqslant 1$. Hence by (3.6), Lemma 2.1 and C_r inequality, we have

$$\begin{aligned} I_{14} \;&\leqslant\; \sum_{n=1}^{\infty} n^{\mu} \int_1^{\infty} P\bigg(\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}(Y''_{ni} - EY''_{ni})\Big| > n^{1/p}t^{1/\alpha}/2\bigg)\mathrm{d}t \\ &\leqslant\; C\sum_{n=1}^{\infty} n^{\mu-q/p} \int_1^{\infty} t^{-q/\alpha}E\max_{1\leqslant j\leqslant n}\Big|\sum_{i=1}^{j}(Y''_{ni} - EY''_{ni})\Big|^{q}\mathrm{d}t \end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\int_1^{\infty} t^{-q/\alpha}\sum_{i=1}^{n} E|Y''_{ni}|^q \mathrm{d}t + C\sum_{n=1}^{\infty} n^{\mu-q/p}\int_1^{\infty} t^{-q/\alpha}\left(\sum_{i=1}^{n} E|Y''_{ni}|^2\right)^{q/2}\mathrm{d}t$$
$$=: I_{15}+I_{16}.$$

By Lemma 2.3, $\alpha > 2$ amd $q > 2p(\mu+1)/(2-p)$, we have

$$\begin{aligned} I_{16} &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\int_1^{\infty} t^{-q/\alpha}\Bigg(\sum_{i=1}^{n} a_{ni}^2 E|X|^2 I(|a_{ni}X| \leqslant n^{1/p}t^{1/\alpha}) \\ &\quad + n^{2/p}t^{2/\alpha}\sum_{i=1}^{n} P(|a_{ni}X| > n^{1/p}t^{1/\alpha})\Bigg)^{q/2}\mathrm{d}t \\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\int_1^{\infty} t^{-q/\alpha}\left(\sum_{i=1}^{n} a_{ni}^2 E|X|^2\right)^{q/2}\mathrm{d}t \\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p+q/2}\left(E|X|^2\right)^{q/2} < \infty. \end{aligned}$$

By Lemma 2.3, we also get

$$\begin{aligned} I_{15} &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\sum_{i=1}^{n}\int_1^{\infty} t^{-q/\alpha}|a_{ni}|^q E|X|^q I(a_{ni}|X| \leqslant n^{1/p}t^{1/\alpha})\mathrm{d}t \\ &\quad + C\sum_{n=1}^{\infty} n^{\mu}\sum_{i=1}^{n}\int_1^{\infty} P(|a_{ni}X| > n^{1/p}t^{1/\alpha})\mathrm{d}t \\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\sum_{i=1}^{n}\int_1^{\infty} t^{-q/\alpha}|a_{ni}|^q E|X|^q I(|a_{ni}X| \leqslant n^{1/p})\mathrm{d}t \\ &\quad + C\sum_{n=1}^{\infty} n^{\mu-q/p}\sum_{i=1}^{n}\int_1^{\infty} t^{-q/\alpha}|a_{ni}|^q E|X|^q I(n^{1/p} < |a_{ni}X| \leqslant n^{1/p}t^{1/\alpha})\mathrm{d}t \\ &\quad + C\sum_{n=1}^{\infty} n^{\mu}\sum_{i=1}^{n}\int_1^{\infty} P(|a_{ni}X| > n^{1/p}t^{1/\alpha})\mathrm{d}t \\ &=: I'_{15}+I''_{15}+I'''_{15}. \end{aligned}$$

Similar to the proof of $I_{13} < \infty$, we can obtain $I'''_{15} < \infty$. Similar to the proof of $I'_9 < \infty$, we have

$$I'_{15} \leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\sum_{i=1}^{n}|a_{ni}|^q E|X|^q I(|a_{ni}X| \leqslant n^{1/p}) < \infty.$$

Finally, we will show $I''_{15} < \infty$. Following the similar method in the proof of $I_6 < \infty$, we get

$$\begin{aligned} I''_{15} &= C\sum_{n=1}^{\infty} n^{\mu-q/p}\sum_{i=1}^{n}\sum_{m=1}^{\infty}\int_m^{m+1} t^{-q/\alpha}|a_{ni}|^q E|X|^q I(n^{1/p} < |a_{ni}X| \leqslant n^{1/p}t^{1/\alpha})\mathrm{d}t \\ &\leqslant C\sum_{n=1}^{\infty} n^{\mu-q/p}\sum_{i=1}^{n}\sum_{m=1}^{\infty} m^{-q/\alpha}|a_{ni}|^q E|X|^q I(n^{1/p} < |a_{ni}X| \leqslant n^{1/p}(m+1)^{1/\alpha}) \end{aligned}$$

$$\leqslant C\sum_{n=1}^{\infty} n^{\mu-\alpha/p}\sum_{i=1}^{n}|a_{ni}|^{\alpha}E|X|^{\alpha}I(|a_{ni}X|>n^{1/p})$$
$$\leqslant C\sum_{n=1}^{\infty} n^{\mu-\alpha/\beta}E|X|^{\alpha}I(|X|>n^{1/\beta})<\infty.$$

The proof of (2.3) is completed. To sum up, we complete the proof of Theorem 2.2.

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REFERENCES

[1] C. C. MOORE, *The degree of randomness in a stationary time series*, Ann. Math. Statist., 34, 1253–258, 1963.

[2] R. C. BRADLEY, *Equivalent mixing conditions for random fields*, Technical Report 336, Center for Stochastic Processes, Dept. Statistics, Univ. North Carolina, Chapel Hill., 1990.

[3] W. BRYC, W. SMOLEŃSKI, *Moment conditions for almost sure convergence of weakly correlated random variables*, Proc. Amer. Math. Soc., 119, 629–635, 1993.

[4] M. PELIGRAD, A. GUT, *Almost-sure results for a class of dependent random variables*, J. Theor. Probab., 12, 87–104, 1999.

[5] S. UTEV, M. PELIGRAD, *Maximal inequalities and an invariance principle for a class of weakly dependent random variables*, J. Theoret. Probab., 16, 101–115, 2003.

[6] G. H. CAI, *Marcinkiewicz strong laws for linear statistics of $\tilde{\rho}$-mixing sequences of random variables*, An. Acad. Bras. Ciênc., 78, 615–621, 2006.

[7] G. H. CAI, *Moment inequality and complete convergence of $\tilde{\rho}$-mixing sequences*, J. Sys. Sci. Math. Scis., 28, 251–256 (in Chinese), 2008.

[8] A. KUCZMASZEWSKA, *On complete convergence for arrays of rowwise dependent random variables*, Statist. Probab. Lett., 77, 1050–1060, 2007.

[9] J. AN, D. M. YUAN, *Complete convergence of weighted sums for $\tilde{\rho}$-mixing sequence of random variables*, Statist. Probab. Lett., 78, 1466–1472, 2008.

[10] Q. Y. WU, Y. Y. JIANG, *Some strong limit theorems for $\tilde{\rho}$-mixing sequences of random variables*, Statist. Probab. Lett., 78, 1017–1023, 2008.

[11] X. C. ZHOU, C. C. TAN, J. G. LIN, *On the strong laws for weighted sums of ρ^*-mixing random variables*, J. Inequal. Appl., doi:10.1155/2011/157816, 2011.

[12] S. H. SUNG, *On the strong convergence for weighted sums of ρ^*-mixing random variables*, Stat. Papers, 54, 773–781, 2013.

[13] Y. F. WU, C. H. WANG, A. VOLODIN, *Limiting behavior for arrays of rowwise ρ^*-mixing random variables*, Lith. Math. J., 52, 214–221, 2012.

[14] J. X. WANG, X. Q. LI, W. Z. YANG,, S. H. HU, *On complete convergence for arrays of rowwise weakly dependent random variables*, Appl. Math. Lett., 25, 1916–1920, 2012.

[15] M. L. GUO, D. J. ZHU, *Equivalent conditions of complete moment convergence of weighted sums for ρ^*-mixing sequence of random variables*, Statist. Probab. Lett., 83, 13–20, 2013.

[16] Y. F. WU, S. H. SUNG, A. VOLODIN, *A note on the rates of convergence for weighted sums of ρ^*-mixing random variables*, Lith. Math. J. 54, 220–228, 2014.

[17] Y. F. WU, J. Y. PENG, *Strong convergence for weighted sums of ρ^*-mixing random variables*, Glas. Mat., 49, 221–234, 2014.

[18] W. LI, P. Y. CHEN, S. H. SUNG, *Remark on convergence rate for weighted sums of ρ^*-mixing random variables*, Rev. R. Acad. Cienc. Exactas Fs. Nat., Ser. A Mat. (RACSAM), 111: 507–513, 2017.

[19] P. L. HSU, H. ROBBINS, *Complete convergence and the law of large numbers*, Proc. Nat. Acad. Sciences., 33, 25–31, 1947.

[20] Y. S. CHOW, *On the rate of moment complete convergence of sample sums and extremes*, Bull. Inst. Math., Acad. Sin., 16, 177–201, 1988.

[21] P. Y. CHEN, S. H. SUNG, *On the strong convergence for weighted sums of negatively associated random variables*, Statist. Probab. Lett., 92, 45–52, 2014.

[22] Z. D. BAI, P. E. CHENG, *Marcinkiewicz strong laws for linear statistics*, Statist. Probab. Lett., 46, 105–112, 2000.

[23] P. Y. CHEN, S. H. SUNG, *Generalized Marcinkiewicz-Zygmund type inequalities for random variables and applications*, J Math. Inequal., 10, 837–848, 2016.

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