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THE APPLICATIONS OF SOME BASIC MATHEMATICAL INEQUALITIES ON THE CONVERGENCE OF THE PRIMITIVE EQUATIONS OF MOIST ATMOSPHERE

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Abstract. In this paper, we show the applications of some basic mathematical inequalities in partial differential equations. By using the differential inequality technique, the convergence of the primitive equations of moist atmosphere is obtained

1. Introduction

Inequalities played a very important role in various fields and solved many practical problems. It is well known that the following Sobolev inequality holds. Letting $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$ and supposing $\omega \in C_0^1(\Omega)$, we have

$$\int_\Omega |\omega|^4 dx \leqslant \Lambda_1 \Big(\int_\Omega |\nabla \omega|^2 dx \Big)^2, \tag{1}$$

where Λ_1 is a positive constant. The proof of inequality (1) can be found in [1, 2]). However, if ω does not vanish on $\partial\Omega$, the inequality (1) can not hold. Lin and Payne [3] assumed that ω satisfied nonhomogeneous condition on $\partial\Omega$. They obtained a slightly more complicated result.

LEMMA 1. [3] (B17) *Assuming that Ω is a bounded, simply connected domain with boundary $\partial\Omega$ of bounded curvature. Then*

$$\Big(\int_\Omega |\omega|^4 dx \Big)^{\frac{1}{2}} \leqslant \Lambda_2 \Big[(1+\frac{\delta}{4}) \int_\Omega |\omega|^2 dx + \frac{3}{4}\delta^{-\frac{1}{3}} \int_\Omega |\nabla \omega|^2 dx \Big], \tag{2}$$

where Λ_2 is a positive constant and $\delta > 0$.

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Lemma 1 played a key role in many problems. The purpose of this paper is to apply some basic mathematical inequalities to the primitive equations of moist atmosphere. Besides lemma 1, we will also use the Young inequality, the Hölder inequality, the arithmetic geometric mean inequality, the Cauchy-Schwarz inequality and the following well-known inequality.

LEMMA 2. ([4, 5]) *If* $\omega(z) \in C^1(0,1)$ *and* $\omega(1) = \omega(0) = 0$, *then*

$$\int_0^1 \omega^2 dz \leqslant \frac{1}{\pi^2} \int_0^1 \left(\frac{d\omega}{dz}\right)^2 dz. \tag{3}$$

In the next section, we give a brief introduction and preparation of the primitive equations of moist atmosphere. In the third section, we show how to use Sobolev inequalities to prove the convergence of the primitive equations of moist atmosphere. In section 4, we use Sobolev inequalities to derive a priori bounds of the solutions. Finally, we make a conclusion in section 5.

2. The primitive equations of large-scale moist atmosphere

The primitive equations are mathematical models which are used to understand the mechanism of long-term weather prediction and climate changes. It was Lions, Teman and Wang (see [6, 7, 8, 9]) who first started the mathematical study of the primitive equations. Then a large number of scholars began to pay attention to the primitive equations, but their results mostly focused on the well-posedness of the solutions (see [10, 11, 12, 13, 14, 15]). At that time, the primitive equations were too complicated to be studied theoretically or to be solved numerically. To overcome this difficulty, one began to simplify equations by various means. However, errors were inevitable in the process of simplification. It is necessary to know whether a small change in a coefficient in an equation, or in the boundary data, or in the equations themselves, will induce a dramatic change in the solutions. This type of study has earned the name structural stability, and is different from continuous dependence on the initial data (see [16]).

Recently, we began to study the structural stability of large-scale primitive equations. [17] obtained the continuous dependence on the viscosity coefficient of the solutions of the three-dimensional viscous primitive equations of the ocean. By using the energy analysis methods, [18] proved that the primitive equations of the coupled atmosphere-ocean continuously dependent on the boundary parameters. In the present paper we consider the following three dimensional viscous primitive equations of large-scale moist atmosphere in the pressure coordinate system system (see [19, 20])

$$\frac{\partial v}{\partial t} + (v\cdot\nabla_2)v + W(v)\frac{\partial v}{\partial z} + \nabla_2\Phi_s + \int_z^1 \frac{bP}{p(\zeta)}\nabla_2[(1+aq)T]d\zeta + \frac{1}{R_0}fv^{\perp} - \Delta v = 0, \tag{4}$$

$$\frac{\partial T}{\partial t} + v\cdot\nabla_2 T + W(v)\frac{\partial T}{\partial z} - \frac{bP}{p}(1+aq)W(v) - \Delta T = Q_1, \tag{5}$$

$$\frac{\partial q}{\partial t} + v\cdot\nabla_2 q + W(v)\frac{\partial q}{\partial z} - \Delta q = Q_2, \tag{6}$$

$$\int_0^1 \nabla_2 \cdot v(x,y,\zeta,t)d\zeta = 0, \tag{7}$$

where the horizontal velocity field $v = (v_1, v_2)$, the temperature T, the mixing ratio of water vapor in the air q, the geopotential Φ_s and the pressure p are the unknowns. Here $W(v) = \int_z^1 \nabla_2 \cdot v(x,y,\zeta,t)d\zeta$, $v^\perp = (-v_2, v_1)$, $f = 2\cos\theta_0$ is the Coriolis parameter, R_0 is the Rossby number, P is an approximate value of pressure at the surface of the earth, p_0 represents the pressure of the upper atmosphere and $p_0 > 0$, the variable z satisfies $p = (P - P_0)z + P_0 (0 < P_0 \leqslant p \leqslant P)$, Q_1, Q_2 are given functions, $a \approx 0.618$. $\nabla_2 = (\partial_x, \partial_y)$ is the horizontal gradient operator and $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$.

The region of (4)–(7) is defined as

$$\Omega = M \times (0,1),$$

where M is a smooth bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary ∂M. The boundary value conditions are

$$\begin{aligned} &\frac{\partial v}{\partial z}\Big|_{z=0,1} = 0,\ v \cdot \overrightarrow{n}\Big|_{\partial M \times (0,1)} = \frac{\partial v}{\partial \overrightarrow{n}} \times \overrightarrow{n}\Big|_{\partial M \times (0,1)} = 0 \\ &\Big(\frac{\partial T}{\partial z} + \alpha T\Big)\Big|_{z=1} = \Big(\frac{\partial q}{\partial z} + \beta q\Big)\Big|_{z=1} = 0,\ \frac{\partial T}{\partial z}\Big|_{z=0} = \frac{\partial q}{\partial z}\Big|_{z=0} = 0, \\ &\frac{\partial T}{\partial \overrightarrow{n}}\Big|_{\partial M \times (0,1)} = \frac{\partial q}{\partial \overrightarrow{n}}\Big|_{\partial M \times (0,1)} = 0, \end{aligned} \tag{8}$$

where $\overrightarrow{n}$ is the normal vector of $\partial M \times (0,1)$. The initial conditions are

$$v(x,y,z,0) = v_0(x,y,z),\ T(x,y,z,0) = T_0(x,y,z),\ q(x,y,z,0) = q_0(x,y,z). \tag{9}$$

In order to establish continuous dependence on the given functions Q_1 and Q_2, we assume that $(v^*, T^*, q^*, \Phi_s^*)$ are solutions of (4)–(8), but with $Q_1 = Q_2 = 0$. If we let

$$\widetilde{v} = v - v^*,\ \widetilde{T} = T - T^*,\ \widetilde{q} = q - q^*,\ \pi_s = \Phi_s - \Phi_s^*, \tag{10}$$

then $(\widetilde{v}, \widetilde{T}, \widetilde{q}, \pi_s)$ satisfy

$$\begin{aligned} &\frac{\partial \widetilde{v}}{\partial t} + (\widetilde{v} \cdot \nabla_2)v + W(\widetilde{v})\frac{\partial v}{\partial z} + (v^* \cdot \nabla_2)\widetilde{v} + w(v^*)\frac{\partial \widetilde{v}}{\partial z} + \nabla \pi_s \\ &\quad + \int_z^1 \frac{abP}{p(\zeta)} \nabla_2[\widetilde{q}(x,y,\zeta,t)T(x,y,\zeta,t)]d\zeta \\ &\quad + \int_z^1 \frac{bP}{p(\zeta)} \nabla_2[(1 + aq^*)(x,y,\zeta,t))\widetilde{T}(x,y,\zeta,t)]d\zeta + f\widetilde{v}^\perp - \Delta \widetilde{v} = 0, \end{aligned} \tag{11}$$

$$\begin{aligned} &\frac{\partial \widetilde{T}}{\partial t} + \widetilde{v} \cdot \nabla_2 T + W(\widetilde{v})\frac{\partial T}{\partial z} + v^* \cdot \nabla_2 \widetilde{T} + W(v^*)\frac{\partial \widetilde{T}}{\partial z} - \frac{bP}{p} a\widetilde{q}W(v) \\ &\quad - \frac{bP}{p}(1 + aq^*)W(\widetilde{v}) - \Delta \widetilde{T} = Q_1, \end{aligned} \tag{12}$$

$$\frac{\partial \widetilde{q}}{\partial t}+\widetilde{v}\cdot\nabla_2 q+W(\widetilde{v})\frac{\partial q}{\partial z}+v^*\cdot\nabla_2\widetilde{q}+W(v^*)\frac{\partial \widetilde{q}}{\partial z}-\Delta\widetilde{q}=Q_2, \tag{13}$$

$$\int_0^1 \nabla_2\cdot\widetilde{v}(x,y,\zeta,t)d\zeta=0. \tag{14}$$

The boundary conditions can be written as

$$\begin{gathered}\frac{\partial\widetilde{v}}{\partial z}\Big|_{z=0,1}=0,\ \widetilde{v}\cdot\overrightarrow{n}\Big|_{\partial M\times(0,1)}=\frac{\partial\widetilde{v}}{\partial\overrightarrow{n}}\times\overrightarrow{n}\Big|_{\partial M\times(0,1)}=0\\ \Big(\frac{\partial\widetilde{T}}{\partial z}+\alpha\widetilde{T}\Big)\Big|_{z=1}=\Big(\frac{\partial\widetilde{q}}{\partial z}+\beta\widetilde{q}\Big)\Big|_{z=1}=0,\ \frac{\partial\widetilde{T}}{\partial z}\Big|_{z=0}=\frac{\partial\widetilde{q}}{\partial z}\Big|_{z=0}=0,\\ \frac{\partial\widetilde{T}}{\partial\overrightarrow{n}}\Big|_{\partial M\times(0,1)}=\frac{\partial\widetilde{q}}{\partial\overrightarrow{n}}\Big|_{\partial M\times(0,1)}=0,\end{gathered} \tag{15}$$

and the initial conditions

$$\widetilde{v}=\widetilde{T}=\widetilde{q}=0. \tag{16}$$

To get the main result, we firstly give a usefull lemma.

LEMMA 3. *If* $\psi_1\times\overrightarrow{n}|_{\partial M}=0,\ W(\psi_2)|_{z=0,1}=0$ *and* $\phi=\phi(x,y,t)\in C^\infty(M)$, *then*

$$\int_\Omega[\psi_2\nabla_2\psi_1+W(\psi_2)\frac{\psi_1}{\partial z}]\psi_1 dxdydz=0,$$

$$\int_\Omega\nabla_2\phi\,\psi_2 dxdydz=0.$$

By using the divergence theorem, the lemma 3 can be easily proved.

3. The main result and its proof

Based on previous preparations, we give the main results and the proof in this section.

THEOREM 1. (Main) *Let* (u,T,q) *be solutions of* (4)–(10) *and* (u^*,T^*,q^*) *be solutions of* (4)–(10) *with* $Q_1=Q_2=0$. *If* $T_0,q_0,v_0\in L^2(\Omega)$, $Q_1,Q_2\in H^1(\Omega)$, *then*

$$(u,T,q)\to(u^*,T^*,q^*), when\ Q_1,\ Q_2\to 0. \tag{17}$$

The differences of the two solutions satisfy

$$\begin{aligned}||\widetilde{v}||_2^2+||\widetilde{T}||_2^2+||\widetilde{q}||_2^2\leqslant 2a_1(t)\int_0^t\exp\Big\{2\int_s^t a_1(\eta)d\eta\Big\}\int_0^s[||Q_1||_2^2+||Q_2||_2^2]d\eta ds\\ +\int_0^t[||Q_1||_2^2+||Q_2||_2^2]d\eta,\end{aligned}$$

which demonstrates convergence on the water vapor source and the given heat source. Here $a_1(t)$ *is a positive function.*

Proof. We take the inner product of (11) with $\widetilde{v}$ in $L^2(\Omega)$ and use lemma 3 to obtain

$$\begin{aligned}
&\frac{1}{2}||\widetilde{v}||_2^2+\int_0^t ||\nabla\widetilde{v}||_2^2 d\eta \\
&=-\int_0^t\int_\Omega\Big[(\widetilde{v}\cdot\nabla_2)v+W(\widetilde{v})\frac{\partial v}{\partial z}\Big]\widetilde{v}dxdydzd\eta \\
&\quad-\int_0^t\int_\Omega\Big[\int_z^1\frac{abP}{p(\zeta)}\nabla_2[\widetilde{q}T]d\zeta\Big]\widetilde{v}dxdydzd\eta \\
&\quad-\int_0^t\int_\Omega\Big[\int_z^1\frac{bP}{p(\zeta)}\nabla_2[(1+aq^*)\widetilde{T}]d\zeta\Big]\widetilde{v}dxdydzd\eta \\
&\doteq\sum_{i=1}^3 A_i.
\end{aligned}\tag{18}$$

We take the inner product of Eq. (12) with $\widetilde{T}$ in $L^2(\Omega\times(0,t))$ and use lemma 3 to find

$$\begin{aligned}
\frac{1}{2}||\widetilde{T}||_2^2&+\int_0^t ||\nabla\widetilde{T}(\eta)||_2^2 d\eta+\alpha\int_0^t ||\widetilde{T}(z=1)||_{L^2(M)}^2 d\eta \\
&=\int_0^t\int_\Omega Q_1\widetilde{T}dxdydzd\eta \\
&\quad-\int_0^t\int_\Omega\Big[\widetilde{v}\cdot\nabla_2 T+W(\widetilde{v})\frac{\partial T}{\partial z}\Big]\widetilde{T}dxdydzd\eta \\
&\quad+\int_0^t\int_\Omega\frac{abP}{p}\widetilde{q}W(v)\widetilde{T}dxdydzd\eta \\
&\quad+\int_0^t\int_\Omega\frac{bP}{p}(1+aq^*)W(\widetilde{v})\widetilde{T}dxdydzd\eta \\
&\doteq\sum_{i=1}^4 B_i.
\end{aligned}\tag{19}$$

We take the inner product of Eq. (13) with $\widetilde{q}$ in $L^2(\Omega\times(0,t))$ and use lemma 3 to find

$$\begin{aligned}
\frac{1}{2}||\widetilde{q}||_2^2+\int_0^t ||\nabla\widetilde{q}(\eta)||_2^2 d\eta+\alpha\int_0^t ||\widetilde{q}(z=1)||_{L^2(M)}^2 d\eta&=\int_0^t\int_\Omega Q_2\widetilde{q}dxdydzd\eta \\
&\quad-\int_0^t\int_\Omega\Big(\widetilde{v}\cdot\nabla_2 q+W(\widetilde{v})\frac{\partial q}{\partial z}\Big)\widetilde{q}dxdydzd\eta \\
&\doteq C_1+C_2.
\end{aligned}\tag{20}$$

Using the Hölder inequality, the Cauchy-Schwarz inequality, lemma 1, lemma 2 and the Young inequality, we have

$$\begin{aligned}
A_1\leqslant&\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t ||\widetilde{v}||_4^4 d\eta\Big)^{\frac{1}{2}} \\
&+\Big(\int_0^t ||W(\widetilde{v})||_2^2 d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t ||\widetilde{v}||_4^4 d\eta\Big)^{\frac{1}{4}}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{4}}
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \Lambda_2\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}\Big[(1+\frac{1}{4}\delta_1)\int_0^t ||\widetilde{v}||_2^2 d\eta+\frac{3}{4}\delta_1^{-3}\int_0^t ||\nabla\widetilde{v}||_2^2 d\eta\Big]\\
&\quad+\frac{\sqrt{\Lambda_2}}{\pi}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{4}}\Big(\int_0^t ||\nabla_2\widetilde{v}||_2^2 d\eta\Big)^{\frac{1}{2}}\\
&\quad\cdot\Big[(1+\frac{1}{4}\delta_1)\int_0^t ||\widetilde{v}||_2^2 d\eta+\frac{3}{4}\delta_1^{-3}\int_0^t ||\nabla\widetilde{v}||_2^2 d\eta\Big]^{\frac{1}{2}} \qquad (21)\\
&\leqslant \Big[\Lambda_2\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}+\frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big]\\
&\quad\cdot\Big[(1+\frac{1}{4}\delta_1)\int_0^t ||\widetilde{v}||_2^2 d\eta+\frac{3}{4}\delta_1^{-3}\int_0^t ||\nabla\widetilde{v}||_2^2 d\eta\Big]+\frac{1}{8}\int_0^t ||\nabla_2\widetilde{v}||_2^2 d\eta,
\end{aligned}
$$

$$
\begin{aligned}
B_2 &\leqslant \frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 T||_2^2 d\eta\Big)^{\frac{1}{2}}\Big[(1+\frac{1}{4}\delta_2)\int_0^t ||\widetilde{v}||_2^2 d\eta+\frac{3}{4}\delta_2^{-3}\int_0^t ||\nabla\widetilde{v}||_2^2 d\eta\Big]\\
&\quad+\Big[\frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 T||_2^2 d\eta\Big)^{\frac{1}{2}}+\frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial T}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big] \qquad (22)\\
&\quad\cdot\Big[(1+\frac{1}{4}\delta_3)\int_0^t ||\widetilde{T}||_2^2 d\eta+\frac{3}{4}\delta_3^{-3}\int_0^t ||\nabla\widetilde{T}||_2^2 d\eta\Big]+\frac{1}{8}\int_0^t ||\nabla_2\widetilde{v}||_2^2 d\eta,
\end{aligned}
$$

$$
\begin{aligned}
C_2 &\leqslant \frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 q||_2^2 d\eta\Big)^{\frac{1}{2}}\Big[(1+\frac{1}{4}\delta_4)\int_0^t ||\widetilde{v}||_2^2 d\eta+\frac{3}{4}\delta_4^{-3}\int_0^t ||\nabla\widetilde{v}||_2^2 d\eta\Big]\\
&\quad+\Big[\frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 q||_2^2 d\eta\Big)^{\frac{1}{2}}+\frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial q}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big] \qquad (23)\\
&\quad\cdot\Big[(1+\frac{1}{4}\delta_5)\int_0^t ||\widetilde{q}||_2^2 d\eta+\frac{3}{4}\delta_5^{-3}\int_0^t ||\nabla\widetilde{q}||_2^2 d\eta\Big]+\frac{1}{8}\int_0^t ||\nabla_2\widetilde{v}||_2^2 d\eta,
\end{aligned}
$$

where $\delta_i,(i=1,2,\dots,5)$ are positive constants to be determined later.

For A_2 and B_3, we integrate by parts and use the Hölder inequality, lemma 1, lemma 2 and the Young inequality, we have

$$
\begin{aligned}
A_2 &= -\int_0^t\int_\Omega \frac{abP}{p(\zeta)}\widetilde{q}TW(\widetilde{v})dxdydzd\eta\\
&\leqslant \frac{abP}{\pi P_0}\Big(\int_0^t ||\nabla_2\widetilde{v}||_2^2 d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t ||\widetilde{q}||_4^4 d\eta\Big)^{\frac{1}{4}}\Big(\int_0^t ||T||_4^4 d\eta\Big)^{\frac{1}{4}} \qquad (24)\\
&\leqslant (\frac{abP}{\pi P_0})^2\Lambda_2\Big(\int_0^t ||T||_4^4 d\eta\Big)^{\frac{1}{2}}\\
&\quad\cdot\Big[(1+\frac{1}{4}\delta_6)\int_0^t ||\widetilde{q}||_2^2 d\eta+\frac{3}{4}\delta_6^{-3}\int_0^t ||\nabla\widetilde{q}||_2^2 d\eta\Big]+\frac{1}{8}\int_0^t ||\nabla_2\widetilde{v}||_2^2 d\eta,
\end{aligned}
$$

$$
\begin{aligned}
B_3 &\leqslant \frac{abP}{\pi P_0}\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}\Big(\int_0^t ||\widetilde{q}||_4^4 d\eta\Big)^{\frac{1}{4}}\Big(\int_0^t ||\widetilde{T}||_4^4 d\eta\Big)^{\frac{1}{4}}\\
&\leqslant \frac{abhP\Lambda_2}{\pi P_0}\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}\Big[(1+\frac{1}{4}\delta_7)\int_0^t ||\widetilde{q}||_2^2 d\eta+\frac{3}{4}\delta_7^{-3}\int_0^t ||\nabla\widetilde{q}||_2^2 d\eta\Big]\\
&\quad+\frac{abP\Lambda_2}{\pi P_0}\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}\Big[(1+\frac{1}{4}\delta_8)\int_0^t ||\widetilde{T}||_2^2 d\eta+\frac{3}{4}\delta_8^{-3}\int_0^t ||\nabla\widetilde{T}||_2^2 d\eta\Big], \qquad (25)
\end{aligned}
$$

where δ_6, δ_7 and δ_8 are positive constants to be determined later.

Using the Hölder inequality and the Young inequality, we have

$$B_1 \leqslant \frac{1}{2}\int_0^t ||\widetilde{T}||_2^2 d\eta + \frac{1}{2}\int_0^t ||Q_1||_2^2 d\eta, \tag{26}$$

$$C_1 \leqslant \frac{1}{2}\int_0^t ||\widetilde{q}||_2^2 d\eta + \frac{1}{2}\int_0^t ||Q_2||_2^2 d\eta. \tag{27}$$

Based on integration by parts, we can easily obtain $A_3 + B_4 = 0$. Choosing suitable δ_i, $(i = 1, 2, \ldots, 8)$ such that

$$\begin{aligned}
&\Big[\Lambda_2\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}} + \frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big]\frac{3}{4}\delta_1^{-3} + \frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 T||_2^2 d\eta\Big)^{\frac{1}{2}}\frac{3}{4}\delta_2^{-3} \\
&\quad + \frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 q||_2^2 d\eta\Big)^{\frac{1}{2}}\frac{3}{4}\delta_4^{-3} \leqslant \frac{1}{2}, \\
&\Big[\frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 T||_2^2 d\eta\Big)^{\frac{1}{2}} + \frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big]\frac{3}{4}\delta_3^{-3} + \frac{abP\Lambda_2}{\pi P_0}\frac{3}{4}\delta_8^{-3} \leqslant 1, \\
&\Big[\frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 q||_2^2 d\eta\Big)^{\frac{1}{2}} + \frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big]\frac{3}{4}\delta_5^{-3} \\
&\quad + \Big(\frac{abP}{\pi P_0}\Big)^2\Lambda_2\Big(\int_0^t ||T||_4^4 d\eta\Big)^{\frac{1}{2}}\frac{3}{4}\delta_6^{-3} + \frac{abhP\Lambda_2}{\pi P_0}\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}\frac{3}{4}\delta_7^{-3} \leqslant 1,
\end{aligned}$$

and combining (18)–(27), we have

$$\begin{aligned}
||\widetilde{v}||_2^2 + ||\widetilde{T}||_2^2 + ||\widetilde{q}||_2^2 \leqslant 2a_1(t)\int_0^t [||\widetilde{v}(\eta)||_2^2 + ||\widetilde{T}(\eta)||_2^2 + ||\widetilde{q}(\eta)||_2^2] d\eta \\
+ \int_0^t [||Q_1||_2^2 + ||Q_2||_2^2] d\eta,
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
a_1(t) = \max\Big\{&\Big[\Lambda_2\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}} + \frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big] + \Big(1 + \frac{1}{4}\delta_1\Big) \\
&+ \frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 T||_2^2 d\eta\Big)^{\frac{1}{2}}\Big(1 + \frac{1}{4}\delta_2\Big) + \frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 q||_2^2 d\eta\Big)^{\frac{1}{2}}\Big(1 + \frac{1}{4}\delta_4\Big)\Big\}, \\
&\frac{1}{2} + \Big[\frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 T||_2^2 d\eta\Big)^{\frac{1}{2}} + \frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial T}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big]\Big(1 + \frac{1}{4}\delta_3\Big) \\
&+ \frac{abP\Lambda_2}{\pi P_0}\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac{1}{2}}\Big(1 + \frac{1}{4}\delta_8\Big), \frac{1}{2} + \Big[\frac{1}{2}\Lambda_2\Big(\int_0^t ||\nabla_2 q||_2^2 d\eta\Big)^{\frac{1}{2}} \\
&+ \frac{2\Lambda_2}{\pi^2}\Big(\int_0^t ||\frac{\partial q}{\partial z}||_4^4 d\eta\Big)^{\frac{1}{2}}\Big]\Big(1 + \frac{1}{4}\delta_5\Big) + \Big(\frac{abP}{\pi P_0}\Big)^2\Lambda_2\Big(\int_0^t ||T||_4^4 d\eta\Big)^{\frac{1}{2}}\Big(1 + \frac{1}{4}\delta_6\Big).
\end{aligned}$$

With Gronwall inequality in (28), the proof of Theorem 1 is completed. □

REMARK 1. Since the definition of $a_1(t)$ involves $\int_0^t ||\nabla_2 v||_2^2 d\eta$, $\int_0^t ||\nabla_2 T||_2^2 d\eta$, $\int_0^t ||\nabla_2 q||_2^2 d\eta, \int_0^t ||T||_4^4 d\eta$ and $\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta$, to make our result meaningful, we must derive their explicit upper bounds. This process is somewhat complicated, so we will give the derivations in the next section.

4. Bounds for $\int_0^t ||\nabla_2 v||_2^2 d\eta, \int_0^t ||\nabla_2 T||_2^2 d\eta, \int_0^t ||\nabla_2 q||_2^2 d\eta, \int_0^t ||T||_4^4 d\eta$ and $\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta$

In this section, we use the Sobolev inequalities to derive the bounds for $\int_0^t ||\nabla_2 v||_2^2 d\eta$, $\int_0^t ||\nabla_2 T||_2^2 d\eta, \int_0^t ||\nabla_2 q||_2^2 d\eta, \int_0^t ||T||_4^4 d\eta$ and $\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta$.

1. Bounds for $\int_0^t ||\nabla_2 v||_2^2 d\eta$ and $\int_0^t ||\nabla_2 T||_2^2 d\eta$

Taking the inner product of Eq. (6) with q in $L^2(\Omega)$, by lemma 3 we obtain

$$\frac{1}{2}\frac{d}{dt}||q||_2^2 + ||\nabla q||_2^2 + \beta ||q(z=1)||_{L^2(M)}^2 = \int_\Omega Q_2 q dxdydz. \tag{29}$$

By the Young inequality we have

$$\frac{d}{dt}||q||_2^2 + 2||\nabla q||_2^2 + 2\beta ||q(z=1)||_{L^2(M)}^2 \leqslant ||Q_2||_2^2 + ||q||_2^2. \tag{30}$$

Integrating (30) from 0 to t, we have

$$\begin{aligned} ||q||_2^2 + 2\int_0^t ||\nabla q||_2^2 d\eta + 2\beta \int_0^t ||q(z=1)||_{L^2(M)}^2 d\eta \\ \leqslant \int_0^t e^{t-\eta} ||Q_2||_2^2 d\eta + ||q_0||_2^2 \doteq F_1(t). \end{aligned} \tag{31}$$

Taking the inner product of Eq. (4) with v in $L^2(\Omega)$, by lemma 3 we obtain

$$\frac{1}{2}\frac{d}{dt}||v||_2^2 + ||\nabla v||_2^2 = -\int_\Omega \Big(\int_z^1 \frac{bP}{p(\zeta)} \nabla_2[(1+aq)T] d\zeta\Big) \cdot v dxdydz. \tag{32}$$

Taking the inner product of Eq. (5) with T in $L^2(\Omega)$, by lemma 3 we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||T||_2^2 + ||\nabla T||_2^2 + \alpha ||T(z=1)||_{L^2(M)}^2 = \int_\Omega \frac{bP}{p}(1+aq)W(v)T dxdydz \\ + \int_\Omega Q_1 T dxdydz, \end{aligned} \tag{33}$$

By integrating by parts, we have

$$\int_\Omega \frac{bP}{p}(1+aq)W(v)T dxdydz - \int_\Omega \Big(\int_z^1 \frac{bP}{p(\zeta)} \nabla_2[(1+aq)T] d\zeta\Big) \cdot v dxdydz = 0.$$

From (32) and (33), by the Young inequality we have

$$\frac{d}{dt}[||v||_2^2+||T||_2^2]+2[||\nabla v||_2^2+||\nabla T||_2^2]+2\alpha||T(z=1)||_{L^2(M)}^2 \leqslant ||T||_2^2+||Q_1||_2^2 \tag{34}$$

Integrating (34) from 0 to t, we have

$$\begin{aligned} &||v||_2^2+||T||_2^2+2\int_0^t[||\nabla v||_2^2+||\nabla T||_2^2]d\eta+2\alpha\int_0^t||T(z=1)||_{L^2(M)}^2 d\eta \\ &\qquad \leqslant \int_0^t e^{t-\eta}||Q_1||_2^2 d\eta+||v_0||_2^2+||T_0||_2^2 \doteq F_2(t). \end{aligned} \tag{35}$$

Noting that $\int_0^t||\nabla_2 v||_2^2 d\eta \leqslant \int_0^t||\nabla v||_2^2 d\eta$ and $\int_0^t||\nabla_2 T||_2^2 d\eta \leqslant \int_0^t||\nabla T||_2^2 d\eta$, from (31) and (35) we can conclude that $\int_0^t||\nabla_2 v||_2^2 d\eta$ and $\int_0^t||\nabla_2 T||_2^2 d\eta$ can be bounded by known data.

2. Bound for $\int_0^t||T||_4^4 d\eta$

Using lemma 1(with $\delta=1$), (35) and (31), we obtain

$$\begin{aligned} \int_0^t||T||_4^4 d\eta &\leqslant \Lambda_2\Big[\frac{5}{4}\int_0^t||T||_2^2 d\eta+\frac{3}{4}\int_0^t||\nabla T||_2^2 d\eta\Big]^2 \\ &\leqslant \Lambda_2\Big[\frac{5}{4}\int_0^t F_2(\eta)d\eta+\frac{3}{8}F_2(t)\Big]^2 \doteq F_3(t), \end{aligned} \tag{36}$$

and

$$\int_0^t||q||_4^4 d\eta \leqslant \Lambda_2\Big[\frac{5}{4}\int_0^t F_1(\eta)d\eta+\frac{3}{8}F_1(t)\Big]^2 \doteq F_4(t), \tag{37}$$

3. Bound for $\int_0^t||\frac{\partial v}{\partial z}||_4^4 d\eta$

We began from

$$\begin{aligned} &\int_0^t\int_\Omega\frac{\partial}{\partial z}\Big\{\frac{\partial v}{\partial \eta}+(v\cdot\nabla_2)v+W(v)\frac{\partial v}{\partial z}+\nabla_2\Phi_s+\frac{1}{Ro}fv^\perp \\ &\qquad +\int_z^1\frac{bP}{p(\zeta)}\nabla_2[(1+aq)T]d\zeta-\Delta v\Big\}\frac{\partial v}{\partial z}dxdydzd\eta=0. \end{aligned}$$

Integrating by parts and using lemma 3 we have

$$\begin{aligned} &\frac{1}{2}||\frac{\partial v}{\partial z}||_2^2+\int_0^t||\nabla\frac{\partial v}{\partial z}||_2^2 d\eta-\frac{1}{2}||\frac{\partial v_0}{\partial z}||_2^2 \\ &=\int_0^t\int_\Omega\frac{bP}{p(z)}T\nabla_2\frac{\partial v}{\partial z}dxdydzd\eta+\int_0^t\int_\Omega\frac{abP}{p(z)}qT\nabla_2\frac{\partial v}{\partial z}dxdydzd\eta \\ &\quad -\int_0^t\int_\Omega\Big[\Big(\frac{\partial v}{\partial z}\cdot\nabla_2\Big)v-(\nabla_2\cdot v)\frac{\partial v}{\partial z}\Big]\cdot\frac{\partial v}{\partial z}dxdydzd\eta. \end{aligned} \tag{38}$$

Using the Hölder inequality, the Young inequality and (35), we have

$$\begin{aligned}&\int_0^t\int_\Omega \frac{bP}{p(z)}T\nabla_2\frac{\partial v}{\partial z}dxdydzd\eta\\&\leqslant \frac{bP}{p_0}\Big(\int_0^t ||T||_2^2 d\eta\Big)^{\frac12}\Big(\int_0^t ||\nabla_2\frac{\partial v}{\partial z}||_2^2 d\eta\Big)^{\frac12}\\&\leqslant 2\Big(\frac{bP}{p_0}\Big)^2\int_0^t F_2(\eta)d\eta+\frac18\int_0^t ||\nabla_2\frac{\partial v}{\partial z}||_2^2 d\eta.\end{aligned} \tag{39}$$

Using the Hölder inequality, the Young inequality, (36) and (37), we have

$$\begin{aligned}&\int_0^t\int_\Omega \frac{abP}{p(z)}qT\nabla_2\frac{\partial v}{\partial z}dxdydzd\eta\\&\leqslant \frac{abP}{p_0}\Big(\int_0^t ||q||_4^4 d\eta\Big)^{\frac14}\Big(\int_0^t ||T||_4^4 d\eta\Big)^{\frac14}\Big(\int_0^t ||\nabla_2\frac{\partial v}{\partial z}||_2^2 d\eta\Big)^{\frac12}\\&\leqslant 2\Big(\frac{abP}{p_0}\Big)^2\sqrt{F_3(t)F_4(t)}+\frac18\int_0^t ||\nabla_2\frac{\partial v}{\partial z}||_2^2 d\eta.\end{aligned} \tag{40}$$

Using the Hölder inequality, the Young inequality, (35) and lemma 3, we have

$$\begin{aligned}&-\int_0^t\int_\Omega\Big[\Big(\frac{\partial v}{\partial z}\cdot\nabla_2\Big)v-(\nabla_2\cdot v)\frac{\partial v}{\partial z}\Big]\cdot\frac{\partial v}{\partial z}dxdydzd\eta\\&\leqslant\Big(\int_0^t ||\nabla_2 v||_2^2 d\eta\Big)^{\frac12}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac12}\\&\leqslant\sqrt{\frac{F_2(t)}{2}}\Big[\Big(1+\frac14\delta\Big)\int_0^t ||\frac{\partial v}{\partial z}||_2^2 d\eta+\frac34\delta^{-3}\int_0^t ||\nabla\frac{\partial v}{\partial z}||_2^2 d\eta\Big]\\&\leqslant\sqrt{\frac{F_2(t)}{2}}\Big[\Big(1+\frac14\delta\Big)\frac{F_2(t)}{2}+\frac34\delta^{-3}\int_0^t ||\nabla\frac{\partial v}{\partial z}||_2^2 d\eta\Big],\end{aligned} \tag{41}$$

where $\delta>0$. Choosing δ such that $\sqrt{\frac{F_2(t)}{2}}\frac34\delta^{-3}=\frac14$, and then inserting (39)–(41) into (38), we have

$$\begin{aligned}&||\frac{\partial v}{\partial z}||_2^2+\int_0^t ||\nabla\frac{\partial v}{\partial z}||_2^2 d\eta\\&\leqslant 2\Big[\sqrt{\frac{F_2(t)}{2}}\Big]^3\Big(1+\frac14\delta\Big)+||\frac{\partial v_0}{\partial z}||_2^2\\&\quad+4\Big(\frac{abP}{p_0}\Big)^2\sqrt{F_3(t)F_4(t)}+4\Big(\frac{abP}{p_0}\Big)^2\sqrt{F_3(t)F_4(t)}\\&\doteq F_5(t)\end{aligned} \tag{42}$$

Finally, using lemma 3(with $\delta=1$) again and (42) we have

$$\begin{aligned}\Big(\int_0^t ||\frac{\partial v}{\partial z}||_4^4 d\eta\Big)^{\frac12}&\leqslant\Lambda_2\Big[\frac54\int_0^t ||\frac{\partial v}{\partial z}||_2^2 d\eta+\frac34\int_0^t ||\nabla\frac{\partial v}{\partial z}||_2^2 d\eta\Big]\\&\leqslant\Lambda_2\Big[\frac54\int_0^t F_5(\eta)d\eta+\frac34 F_5(t)\Big].\end{aligned} \tag{43}$$

Using a similar method, we also can bound $\int_0^t ||\frac{\partial T}{\partial z}||_4^4 d\eta$ and $\int_0^t ||\frac{\partial q}{\partial z}||_4^4 d\eta$

5. Conclusion

Obviously, lemma 1 plays a key role in this paper. If $\Omega \subset \mathbb{R}^2$, Payne [1] and Serrin [2] have also proved the following result. For Dirichlet integrable function $\omega \in C_0^1(\Omega)$, the Poincaré inequality holds, namely

$$\int_\Omega \omega^4 dA \leqslant \frac{1}{2} \int_\Omega \omega^2 dA \int_\Omega |\nabla_2 \omega|^2 dA. \tag{44}$$

Also, if ω does not vanish on the boundary of Ω, (44) can not hold. But it may be interesting and meaningful. We will study this problem and its applications in another paper.

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