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AN INEQUALITY ON THE DIFFERENCE POLYNOMIALS OF MEROMORPHIC FUNCTIONS AND ITS APPLICATION

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Abstract. Let $f(z)$ be a transcendental meromorphic function of finite order and $\Psi(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_n) - a(f(z))^n$ be a difference polynomials of f, where $a \in \mathbb{C}\setminus\{0\}$, $c_1, c_2, \cdots, c_n (n \in \mathbb{N}^+)$ be complex constants satisfying that at least one of them is non-zero. If $\Psi(z)$ is transcendental, the author establishes the following inequality on $\Psi(z)$:

$$nT(r,f) \leqslant nN\left(r,\frac{1}{f}\right) + 4nN(r,f) + N\left(r,\frac{1}{\Psi(z)-b}\right) + S(r,f),$$

where $b \in \mathbb{C}\setminus\{0\}$. As an application of this inequality, the author investigates the value distribution of $\Psi(z)$. Results are obtained partially solve some open questions raised by Zheng and Chen in [X. M. Zheng, Z. X. Chen, On the value distribution of some difference polynomials, J. Math. Anal. Appl. 397(2013) 814–821].

1. Introduction and main results

The main purpose of this paper is to study the value distribution of difference polynomials of meromorphic functions by using the Nevanlinna theory. Therefore, we use the standard notation of the Nevanlinna theory and assume that the reader knows these notation (see [9, 15, 18]).

Let $f(z)$ be a function meromorphic in the complex plane $\mathbb{C}$. The order of $f(z)$ is denoted by $\sigma(f)$. For any $a \in \mathbb{C}$, the exponent of convergence of zeros of $f(z)-a$ (or poles of $f(z)$) is denoted by $\lambda(f,a)$ (or $\lambda(\frac{1}{f})$). For simplicity, we denote $\lambda(f,0)$ by $\lambda(f)$. If $\lambda(f,a) < \sigma(f)$ (or $\lambda(\frac{1}{f}) < \sigma(f)$), then a (or ∞) is said to be a Borel exceptional value of $f(z)$. For any $a \in \mathbb{C}\bigcup\{\infty\}$, we denote the Nevanlinna's deficiency of f with respect to a by $\delta(a,f)$. Moreover, we use $S(r,f)$ to denote any quantity of $S(r,f) = o(T(r,f))(r \to \infty)$, possibly outside a set E with finite logarithmic measure.

Let $f(z)$ be a transcendental meromorphic function and $a \in \mathbb{C}\setminus\{0\}$, $m,n \in \mathbb{N}^+$ and $c_1, c_2, \cdots, c_m$ be complex constants satisfying that at least one of them is non-zero. Zheng and Chen [17] define and investigate the value distribution of difference polynomials

$$\Psi(z) = f(z+c_1)f(z+c_2)\cdots f(z+c_m) - a(f(z))^n. \tag{1}$$

Unless otherwise stated, this article always holds that $\Psi(z) \not\equiv 0$. In [17], Zheng and Chen have proved the following theorems.

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THEOREM A. [17] *Let f be a transcendental entire function of finite order. If $N\left(r,\frac{1}{f}\right)=S(r,f)$, $n\neq m,\min\{n,m\}\geqslant 2$, then $\Psi(z)$ assumes every non-zero value $b\in\mathbb{C}$ infinitely often.*

THEOREM B. [17] *Let f be a transcendental entire function of finite order $\sigma(f)$ with a Borel exceptional value $s\in\mathbb{C}$. Then for $1\leqslant m<n$ and every $b(\neq s^m-as^n)\in\mathbb{C}$, $\Psi(z)$ assumes the value b infinitely often and $\lambda(b,\Psi(z))=\sigma(f)$.*

In [14], Yi and Yang have proved the following theorem.

THEOREM C. [14] *Let f be meromorphic function in $\mathbb{C}$ with a positive order. If f has two distinct Borel exceptional values a_1 and a_2, then $\delta(a_1,f)=\delta(a_2,f)=1$.*

By Theorem C, we can derive that the conditions in Theorems A imply that $\delta(0,f)$ $=\delta(\infty,f)=1$, and get that the conditions in Theorems B imply that $\delta(s,f)=\delta(\infty,f)=1$. Put $s=0$, then $\Psi(z)$ assumes the nonzero value b infinitely often and $\lambda(b,\Psi(z))=\sigma(f)$ under the conditions of Theorem B.

Zheng and Chen have settled with the case $m=n$ for Theorem A, the case $n\leqslant m$ for Theorem B and the case of meromorphic functions for Theorems A, B as open questions in [17]. In this paper, we'll try to solve these open problems by using a discussion method similar to Wu and Xu [13].

In order to solve these open questions, we firstly establish the following inequality.

THEOREM 1. *Suppose that f is a transcendental meromorphic function of finite order and $\Psi(z)$ is a difference polynomial of the form (1) and $\Psi(z)$ is transcendental, where $m=n$. Then, for any $b\in\mathbb{C}\setminus\{0\}$, we have*

$$nT(r,f)\leqslant nN\left(r,\frac{1}{f}\right)+4nN(r,f)+N\left(r,\frac{1}{\Psi(z)-b}\right)+S(r,f)$$

Theorem 1 can be seen as an difference counterpart of Milloux inequality (see [12, 15]). On the case of $a=0$ in $\Psi(z)$, Wu and Xu [13] give a detailed discussion. As an application of Theorem 1, we shall prove the following theorem, which partly answers the open questions of Zheng and Chen [17].

THEOREM 2. *Let $f(z)$ be a transcendental meromorphic function of finite order, and assume that $\delta(\infty,f)=1,m=n$. Then,*

(i) for $\delta(0,f)>0$, $\Psi(z)$ assumes every non-zero value b infinitely often and $\lambda(b,\Psi(z))=\sigma(f)$.

(ii) for $\delta(0,f)=1$, $\Psi(z)$ assumes every non-zero value b infinitely often and

$$T(r,\Psi)\sim nT(r,f)\sim N\left(r,\frac{1}{\Psi-b}\right)$$

as $r\notin E,r\to\infty$, where E is a possible exception set of r with finite logarithmic measure.

If $m=n=1$ and $a=1$, then $\Psi(z)$ becomes the forward difference $\Delta_c f(z)$ (see [1]), i.e.

$$\Psi = f(z+c)-f(z)=\Delta_c f(z).$$

Therefore, we can get the following Corollary from Theorem 2.

COROLLARY 1. *Let $f(z)$ be a transcendental meromorphic function of finite order, and assume that $\delta(\infty,f)=1$. Then,*

(i) for $\delta(0,f)>0$, $\Delta_c f(z)$ assumes every non-zero value b infinitely often and $\lambda(b,\Delta_c f(z))=\sigma(f)$;

(ii) for $\delta(0,f)=1$, $\Psi(z)$ assumes every non-zero value b infinitely often and

$$T(r,\Delta_c f)\sim T(r,f)\sim N\left(r,\frac{1}{(\Delta_c f)-b}\right)$$

as $r\notin E, r\to\infty$, where E is a possible exception set of r with finite logarithmic measure.

If $m=n=1$ and $a=-1$, then $\Psi(z)=f(z+c)+f(z)$. We define

$$\nabla_c f(z)=f(z+c)+f(z).$$

Then, we can get the following Corollary from Theorem 2.

COROLLARY 2. *Let $f(z)$ be a transcendental meromorphic function of finite order, and assume that $\delta(\infty,f)=1$. Then,*

(i) for $\delta(0,f)>0$, $\nabla_c f(z)$ assumes every non-zero value b infinitely often and $\lambda(b,\nabla_c f(z))=\sigma(f)$;

(ii) for $\delta(0,f)=1$, $\nabla_c f(z)$ assumes every non-zero value b infinitely often and

$$T(r,\nabla_c f(z))\sim T(r,f)\sim N\left(r,\frac{1}{(\nabla_c f(z))-b}\right)$$

as $r\notin E, r\to\infty$, where E is a possible exception set of r with finite logarithmic measure.

If $c_1=c\neq 0$, $c_2=c_3=\cdots=c_m=0$, then,

(i) for $a=1$ and $m=n\in\mathbb{N}^+$,

$$\Psi(z)=f^n(f(z+c)-f(z))=f^n\Delta_c f(z).$$

(ii) for $a=-1$ and $m=n\in\mathbb{N}^+$,

$$\Psi(z)=f^n(f(z+c)+f(z))=f^n\nabla_c f(z).$$

Therefore, we can get the following Corollary from Theorem 2.

COROLLARY 3. *Let $f(z)$ be a transcendental meromorphic function of finite order, and assume that $\delta(\infty, f) = 1$. Then,*

(i) for $\delta(0, f) > 0$, both $f^n\Delta_c f(z)$ and $f^n\nabla_c f(z)$ assume every non-zero value b infinitely often and

$$\lambda(b, f^n\Delta_c f(z)) = \lambda(b, f^n\nabla_c f(z)) = \sigma(f).$$

(ii) for $\delta(0, f) = 1$, both $f^n\Delta_c f(z)$ and $f^n\nabla_c f(z)$ assume every non-zero value b infinitely often and

$$\begin{aligned} &T(r, f^n\Delta_c f(z)) \sim T(r, f^n\nabla_c f(z)) \\ &\sim N\left(r, \tfrac{1}{(f^n\Delta_c f(z))-b}\right) \sim N\left(r, \tfrac{1}{(f^n\nabla_c f(z))-b}\right) \\ &\sim (n+1)T(r, f) \end{aligned}$$

as $r \notin E, r \to \infty$, where E is a possible exception set of r with finite logarithmic measure.

EXAMPLE 1. Let $f(z) = e^z$. Then $\delta(0, f) = 1$, $\delta(\infty, f) = 1$. Put $c_1 = c_2 = c_3 = \cdots = c_m = 1$ and $a = -1$. Then, for $m = n \in \mathbb{N}^+$, $\Psi(z) = (e^n - 1)e^{nz} \neq 0$. Therefore, the assumption that $b \neq 0$ cannot be ommited in Theorem 2. And above all, we have

$$T(r, \Psi) \sim nT(r, f) \sim N\left(r, \frac{1}{\Psi - b}\right)$$

as $r \to \infty$.

2. Proof of Theorem 1 and Theorem 2

In order to prove Theorem 1 and Theorem 2, we need to use the following Lemmas (see [3], [4], [6], [7] et al).

LEMMA 1. [3, 16] *Let f be a transcendental meromorphic function of finite order. Then*

$$N(r, f(z+c)) = N(r, f) + S(r, f),$$
$$T(r, f(z+c)) = T(r, f) + S(r, f),$$

where $S(r, f) = o(T(r, f))(r \to \infty)$, possibly outside a set E of r with finite logarithmic measure.

LEMMA 2. [7] *Let $f(z)$ be a transcendental meromorphic function of finite order, then*

$$m\left(r, \frac{f(z+c)}{f}\right) = S(r, f).$$

LEMMA 3. [6] *Let f be a transcendental meromorphic function of finite order. Then,*

$$m\left(r, \frac{\Delta_c f(z)}{f(z)}\right) = S(r, f).$$

LEMMA 4. [14] *Suppose that* $f(z)$ *is a transcendental meromorphic function in the complex plane and* $P(z) = a_0z^n + a_1z^{n-1} + \cdots + a_n$, *where* $a_0(\not\equiv 0), a_1, \cdots, a_n$ *are constants. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

LEMMA 5. [2, 5] *Let* $F(r)$ *and* $G(r)$ *be monotone increasing function such that* $F(r) \leqslant G(r)$ *outside of exceptional set* E *that is of finite logarithmic measure. Then for any* $\alpha > 0$, *there exists* $r_0 > 1$ *such that* $F(r) \leqslant G(\alpha r)$ *for all* $r > r_0$.

2.1. Proof of Theorem 1

Since

$$T(r, \Psi(z)) \leqslant \sum_{i=1}^{n} T(r, f(z+c_i)) + T(r, (f(z))^n) + O(1). \tag{2}$$

Using Lemma 1, we can derive from (2) that

$$T(r, \Psi(z)) \leqslant 2nT(r, f) + S(r, f). \tag{3}$$

Hence $\sigma(\Psi(z)) \leqslant \sigma(f)$ and

$$S(r, \Psi(z)) = S(r, f). \tag{4}$$

Since $\Psi(z)$ is transcendental, then there is a $c \in \mathbb{C} \setminus \{0\}$ such that $\Delta_c\Psi(z) = \Delta_c(\Psi(z) - b) \not\equiv 0$. Note that

$$\begin{aligned} \frac{1}{f^n} &= \frac{\Psi(z)}{bf^n} - \frac{\Delta_c(\Psi(z)-b)}{bf^n} \frac{\Psi(z)-b}{\Delta_c(\Psi(z)-b)} \\ &= \frac{\Psi(z)}{bf^n} - \frac{\Delta_c\Psi(z)}{bf^n} \frac{\Psi(z)-b}{\Delta_c(\Psi(z)-b)}, \end{aligned} \tag{5}$$

where

$$\begin{aligned} \frac{\Delta_c\Psi(z)}{bf^n} &= \frac{f(z+c_1+c)f(z+c_2+c)\cdots f(z+c_n+c) - a(f(z+c))^n}{bf^n} - \frac{\Psi(z)}{bf^n}, \\ \frac{\Psi(z)}{bf^n} &= \frac{f(z+c_1)f(z+c_2)\cdots f(z+c_n) - a(f(z))^n}{bf^n}. \end{aligned}$$

It follows from Lemma 1 that

$$m\left(r, \frac{\Psi(z)}{bf^n}\right) = S(r, f), \tag{6}$$

$$m\left(r, \frac{\Delta_c\Psi(z)}{bf^n}\right) = S(r, f). \tag{7}$$

From (5)–(7), we get

$$m\left(r, \frac{1}{f^n}\right) \leqslant m\left(r, \frac{\Psi(z)-b}{\Delta_c(\Psi(z)-b)}\right) + S(r, f).$$

Therefore

$$T\left(r,\frac{1}{f^n}\right)\leqslant N\left(r,\frac{1}{f^n}\right)+m\left(r,\frac{\Psi(z)-b}{\Delta_c(\Psi(z)-b)}\right)+S(r,f)$$
$$\leqslant nN\left(r,\frac{1}{f}\right)+m\left(r,\frac{\Psi(z)-b}{\Delta_c(\Psi(z)-b)}\right)+S(r,f). \tag{8}$$

From the first fundamental theorem of Nevanlinna theory, we have

$$m\left(r,\tfrac{\Psi(z)-b}{\Delta_c(\Psi(z)-b)}\right)\leqslant m\left(r,\tfrac{\Delta_c(\Psi(z)-b)}{\Psi(z)-b}\right)+N\left(r,\tfrac{\Delta_c(\Psi(z)-b)}{\Psi(z)-b}\right)+O(1). \tag{9}$$

It follows from Lemma 3 that

$$m\left(r,\tfrac{\Delta_c(\Psi(z)-b)}{\Psi(z)-b}\right)=S(r,\Psi(z)). \tag{10}$$

It follows from Lemma 1 that

$$N\left(r,\tfrac{\Delta_c(\Psi(z)-b)}{\Psi(z)-b}\right)\leqslant N\left(r,\tfrac{1}{\Psi(z)-b}\right)+4nN(r,f)+S(r,f). \tag{11}$$

From (4), (8)–(11) and Lemma 4, we have

$$\begin{aligned} nT(r,f)&=T\left(r,\tfrac{1}{f^n}\right)+S(r,f)\\ &\leqslant nN\left(r,\tfrac{1}{f}\right)+4nN(r,f)+N\left(r,\tfrac{1}{\Psi(z)-b}\right)+S(r,f). \end{aligned} \tag{12}$$

2.2. Proof of Theorem 2

Since $\delta(0,f)>0$, then $\Psi(z)$ is transcendental. If $\Psi(z)$ is not a transcendental meromorphic function. Then there is a rational function $Q(z)$ such that $Q(z)\Psi(z)\equiv 1$, i.e.

$$\frac{1}{f^n}\equiv Q(z)\frac{\Psi(z)}{f^n}.$$

Apply Lemma 2 and note that $f(z)$ is transcendental, we can get

$$m\left(r,\tfrac{1}{f^n}\right)\leqslant m(r,Q(z))+m\left(r,\tfrac{\Psi(z)}{f^n}\right)=S(r,f).$$

Therefore

$$\begin{aligned} m\left(r,\tfrac{1}{f^n}\right)+N\left(r,\tfrac{1}{f^n}\right)&\leqslant N\left(r,\tfrac{1}{f^n}\right)+S(r,f)\\ &\leqslant nN\left(r,\tfrac{1}{f}\right)+S(r,f). \end{aligned}$$

Apply Lemma 4 and the first fundamental theorem of Nevanlinna theory, we can get

$$nT(r,f)\leqslant nN\left(r,\tfrac{1}{f}\right)+S(r,f).$$

This contradicts with $\delta(0,f)>0$. Thus $\Psi(z)$ is a function transcendental and meromorphic function of finite order.

(i) Since $\delta(0,f) > 0$ and $\delta(\infty,f) = 1$, there is a positive number $\theta < 1$ such that

$$N\left(r,\frac{1}{f}\right) < \theta T(r,f), \tag{13}$$

$$N(r,f) = o(1)T(r,f). \tag{14}$$

By Theorem 1, we have

$$T(r,f) \leqslant 4N(r,f) + N\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{\Psi(z)-b}\right) + S(r,f). \tag{15}$$

Combining (13)–(15) we can get

$$(1-o(1)-\theta)T(r,f) \leqslant N\left(r,\frac{1}{\Psi(z)-b}\right), r \notin E, r \to \infty, \tag{16}$$

where E is a possible exceptional set with finite logarithmic measure. Noticing f is transcendental, applying Lemma 5 and (16), we can get that $\Psi(z)$ assumes every non-zero value b infinitely often and $\lambda(b,\Psi(z)) = \sigma(f)$.

(ii) Since $\delta(0,f) = 1$ and $\delta(\infty,f) = 1$,

$$N\left(r,\frac{1}{f}\right) = S(r,f), \tag{17}$$

$$N(r,f) = S(r,f). \tag{18}$$

From (17), (18) and Theorem 1, we have

$$\begin{aligned} nT(r,f) &\leqslant N\left(r,\tfrac{1}{\Psi(z)-b}\right) + S(r,f) \\ &\leqslant T\left(r,\Psi(z)\right) + S(r,f) \end{aligned} \tag{19}$$

From Lemma 1 and Lemma 4, we have

$$\begin{aligned} T(r,\Psi(z)) &= m(r,\Psi(z)) + N(r,\Psi(z)) \\ &= m\left(r,bf^n\tfrac{\Psi(z)}{bf^n}\right) + N(r,\Psi(z)) \\ &\leqslant m(r,bf^n) + m\left(r,\tfrac{\Psi(z)}{bf^n}\right) + N(r,\Psi(z)) \\ &\leqslant T(r,bf^n) + m\left(r,\tfrac{\Psi(z)}{bf^n}\right) + 4nN(r,f) + S(r,f) \\ &\leqslant nT(r,f) + m\left(r,\tfrac{\Psi(z)}{bf^n}\right) + 4nN(r,f) + S(r,f) \end{aligned} \tag{20}$$

It follows from (6), (18)–(20) that

$$\begin{aligned} nT(r,f) &\leqslant N\left(r,\tfrac{1}{\Psi(z)-b}\right) + S(r,f) \\ &\leqslant T\left(r,\Psi(z)\right) + S(r,f) \\ &\leqslant nT(r,f) + S(r,f). \end{aligned} \tag{21}$$

Since f is transcendental, (21) means that $\Psi(z)$ assumes every non-zero value b infinitely often and

$$T(r,\Psi) \sim nT(r,f) \sim N\left(r, \frac{1}{\Psi - b}\right)$$

as $r \notin E$, $r \to \infty$, where E is a possible exception set of r with finite logarithmic measure.

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