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APPROXIMATION PROPERTIES OF COMBINATION OF MULTIVARIATE AVERAGES ON TRIEBEL–LIZORKIN SPACES

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Abstract. The purpose of this paper is to establish the rate of approximation of the combination of some generalized multivariate average on Triebel-Lizorkin spaces and obtain its equivalent relation to the K-functionals. These results significantly generalize some known results in the literatures.

1. Introduction

Let $\gamma \in \mathbb{R}$ and I_γ be the Riesz potential of order γ defined on functions or distributions g via the Fourier transform

$$\widehat{I_\gamma(g)}(\xi) = |\xi|^{-\gamma}\hat{g}(\xi).$$

The Laplacian $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$ on the n-dimensional Euclidean space $\mathbb{R}^n$ satisfies $\Delta = -I_{-2}$.

Fix a Schwartz function ψ satisfying

$$\operatorname{supp}\, \hat{\psi} \subset \{\xi : \frac{1}{2} \leqslant |\xi| \leqslant 2\}.$$

Let $\psi_j(\cdot) = 2^{jn}\psi(2^j\cdot)$ and require that $\hat{\psi}$ satisfies

$$\sum_{j\in\mathbb{Z}} |\widehat{\psi}(2^{-j}\xi)|^2 = 1, \ \ \xi \in \mathbb{R}^n \setminus \{0\}.$$

Let $\mathscr{S}(\mathbb{R}^n)$ be the Schwartz space and $\mathscr{S}'(\mathbb{R}^n)$ be the space of tempered distributions. For $s \in \mathbb{R}$, $0 < p,q < \infty$, the Triebel-Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}^s_{p,q}(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n)/\mathscr{P}(\mathbb{R}^n) : \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = \left\| \Big(\sum_j (2^{sj}|\psi_j * f|)^q \Big)^{\frac{1}{q}} \right\|_{L^p(\mathbb{R}^n)} < \infty \right\},$$

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where $\mathscr{P}(\mathbb{R}^n)$ denotes the set of all polynomials on $\mathbb{R}^n$. It is well known that the function ψ in the above definition is flexible and any two different functions ψ give the equivalent norms. With the rapidly developing wavelet analysis, one important function space, the Triebel–Lizorkin space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ arises and is well studied (see [11, 12, 21]). The significance of the space $\dot{F}^s_{p,q}(\mathbb{R}^n)$ is that it provides a uniform setting of many important function spaces in analysis, such as Lebesgue spaces, Hardy spaces, Sobolev spaces, BMO spaces, Lipschitz spaces, etc. Particularly, we know that $\dot{F}^0_{p,2}(\mathbb{R}^n) \approx H^p(\mathbb{R}^n)$ if $0 < p \leqslant 1$ and $\dot{F}^0_{p,2}(\mathbb{R}^n) \approx L^p(\mathbb{R}^n)$ if $1 < p < \infty$. We recall that the Hardy spaces $H^p(\mathbb{R}^n), 0 < p < \infty$, is the space of all distributions f satisfying

$$\|f\|_{H^p(\mathbb{R}^n)} = \Big\| \sup_{t>0} |\varphi_t * f| \Big\|_{L^p(\mathbb{R}^n)} < \infty$$

for some $\varphi \in \mathscr{S}(\mathbb{R}^n)$ with $\int \varphi \neq 0$, where $\varphi_t(x) = \frac{1}{t^n}\varphi(\frac{x}{t})$. The symbol $\mathscr{S}(\mathbb{R}^n)$ denotes the set of Schwartz functions in $\mathbb{R}^n$.

Suppose that $t > 0$, $\gamma > 0$, $0 < p < \infty$, $0 < q < \infty$ and $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$. Let $K_\gamma(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ denote the γth order K-functional of f, that is

$$K_\gamma(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = \inf_{g \in \dot{F}^{s,\gamma}_{p,q}(\mathbb{R}^n)} \left\{ \|f-g\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^\gamma \|I_{-\gamma}(g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \right\}, \tag{1}$$

where

$$\dot{F}^{s,\gamma}_{p,q}(\mathbb{R}^n) = \{g \in \dot{F}^s_{p,q}(\mathbb{R}^n) : I_{-\gamma}(g) \in \dot{F}^s_{p,q}(\mathbb{R}^n)\}.$$

The K-functional of f, $K_\gamma(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ is used to measure the smoothness of f in $\dot{F}^s_{p,q}(\mathbb{R}^n)$. Similarly, for $t > 0$, $\gamma > 0$ and $0 < p < \infty$, we use the symbol $K_\gamma(f,t)_{H^p(\mathbb{R}^n)}$ to denote γth order K-functional of f

$$K_\gamma(f,t)_{H^p(\mathbb{R}^n)} = \inf_{g \in H^{p,\gamma}(\mathbb{R}^n)} \left\{ \|f-g\|_{H^p(\mathbb{R}^n)} + t^\gamma \|I_{-\gamma}(g)\|_{H^p(\mathbb{R}^n)} \right\},$$

where

$$H^{p,\gamma}(\mathbb{R}^n) = \{g \in H^p(\mathbb{R}^n) : I_{-\gamma}(g) \in H^p(\mathbb{R}^n)\}.$$

If $f \in L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, we can also use the symbol $K_\gamma(f,t)_{L^p(\mathbb{R}^n)}$ to denote γth order K-functional of f. In fact, it is convenient to use the K-functionals to deal with the approximation of operators (see [8, 9, 10]).

Belinsky, Dai and Ditzian in [2] considered the average on a sphere with radius t defined as

$$A_t(f)(x) = \int_{\mathbb{S}^{n-1}} f(x - ty') d\sigma(y'), n \geqslant 2,$$

where $\mathbb{S}^{n-1}$ is the unit sphere with the surface Lebesgue measure normalized by

$$\int_{\mathbb{S}^{n-1}} d\sigma(y') = 1,$$

and $y' = y/|y|$ is the unit vector for any $y \neq 0$. They obtained the following equivalent relation

$$\|A_t(f) - f\|_{L^p(\mathbb{R}^n)} \approx K_2(f,t)_{L^p(\mathbb{R}^n)}$$

for all $1 \leqslant p \leqslant \infty$.

Later, Dai and Ditzian in [7] studied the combination of multivariate averages

$$A_{l,t}(f)(x) = \frac{-2}{\binom{2l}{l}} \sum_{j=1}^{l} (-1)^j \binom{2l}{l-j} A_{jt}(f)(x),$$

where l and j are positive integers. And they proved the equivalent relation

$$\|A_{l,t}(f) - f\|_{L^p(\mathbb{R}^n)} \approx K_{2l}(f,t)_{L^p(\mathbb{R}^n)} \tag{2}$$

for $n \geqslant 2$ and $1 \leqslant p \leqslant \infty$.

Recently, Fan and Zhao extended the result in [11] to consider the combination for a more general operator, i.e.

$$\Im^{\beta}_{l,t}(f)(x) = \frac{-2}{\binom{2l}{l}} \sum_{j=1}^{l} (-1)^j S^{\beta}_{jt}(f)(x),$$

where

$$S^{\beta}_t(f)(x) = \frac{\Gamma(\beta + \frac{n}{2})}{\pi^{\frac{n}{2}}\Gamma(\beta)} t^{-n} \int_{|y| \leqslant t} \left(1 - |\frac{y}{t}|^2\right)^{\beta-1} f(x-y)dy.$$

$\Gamma(z)$ denotes the usual gamma function with $z \in \mathbb{Z}$ and $Rez > 0$. This family of operators S^{β}_t has received extensive study in the history (see e.g. [1, 4, 17, 18, 19]). By taking Fourier transform, one can embed the operator A_t in an analytic family S^{β}_t with the complex parameter β, so we get

$$S^0_t(f)(x) = A_t(f)(x).$$

If taking $\beta = 0$, then we have

$$\Im^0_{l,t}(f)(x) = A_{l,t}(f)(x).$$

Moreover, Fan and Zhao [11] obtained the following theorem.

THEOREM A. ([11], Theorem 1.1, p. 79) *Let $l \in \mathbb{Z}^+$, $\beta \geqslant 0$, $n \geqslant 2$, $p \geqslant \frac{n-1}{n-1+\beta}$ and $t > 0$. Then for $f \in H^p(\mathbb{R}^n)$, we have*

$$\|\Im^{\beta}_{l,t}(f) - f\|_{H^p(\mathbb{R}^n)} \approx K_{2l}(f,t)_{H^p(\mathbb{R}^n)}.$$

Inspired by the above results, it is of interest to know whether we have a uniform equivalent relation

$$\|\Im^{\beta}_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \approx K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \tag{3}$$

holds for $n \geqslant 2$, $s \in \mathbb{R}$ and $0 < p,q < \infty$, where the notation of $K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$ is given by (1).

In this paper, our main purpose is to address this question. The first aim of this paper is to establish the following theorem for different p and q.

THEOREM 1. *Let $l \in \mathbb{Z}^+$, $\beta \geqslant 0$, $n \geqslant 2$ and $t > 0$. For any $1 < p,q < \infty$ or $0 < q \leqslant p < 1$, we have*

$$\|\Im^{\beta}_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \approx K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

if $p > \frac{n-1}{n-1+\beta}$. Besides, for any $0 < p \leqslant 1 < q < \infty$, we have

$$\|\Im^{\beta}_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \approx K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

if $p \geqslant \frac{n-1}{n-1+\beta}$.

Note that Theorem 1 returns to Theorem A when taking $0 < p < \infty, q = 2$, since $\dot{F}^0_{p,2}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$. Moreover, let $p \geqslant 1, q = 2$ and $\gamma = 0$, then the result is the same as (2) obtained in [7].

Let η be a radial C^∞ function satisfying $\eta(\xi) \equiv 1$ if $|\xi| \leqslant 1$ and $\operatorname{supp}(\eta) \subset \{\xi : |\xi| \leqslant 2\}$. And we define $\eta_t f$ by $\widehat{\eta_t f}(\xi) = \eta(t\xi)\hat{f}(\xi)$. To prove Theorem 1, we begin by proving the following auxiliary theorem.

THEOREM 2. *Let $l \in \mathbb{Z}^+, s \in \mathbb{R}$, $0 < p,q < \infty$ and $t > 0$. Suppose $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$, then we have*

$$\|\eta_t(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^{2l}\|\Delta^l \eta_t(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \approx K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)},$$

where $\Delta^l h = \Delta(\Delta^{l-1}h)$.

From Theorem 2, to show Theorem 1, it suffices to prove that

$$\|\Im^{\beta}_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \approx \|\eta_t(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^{2l}\|\Delta^l \eta_t(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

for different p and q. To this end, we will write the multiplier of $(\Im^{\beta}_{l,t}(f) - f)$ as a sum of two multipliers, where one multiplier is supported in a neighborhood of zero and the other is supported away from zero. We will check that the first multiplier is a Triebel-Lizorkin multiplier with the help of a well-known Triebel-Lizorkin multiplier theorem (see Theorem B below). For the second multiplier, we will reduce it to a sum of multipliers of wave operators, then invoke the boundedness of wave operator on Triebel-Lizorkin spaces obtained by Cao, Chen and Fan [3] achieve our target (see Theorem C in Section 4).

Finally, we also consider the iterates $(A_t)^N(f)$ on the Triebel-Lizorkin space and get the following theorem.

THEOREM 3. *Let $n \geqslant 2$ and $t > 0$, $s \in \mathbb{R}$. For any $1 < p \leqslant q \leqslant 2$ or $2 \leqslant q \leqslant p < \infty$, we have*

$$\|\Delta(A^N_t(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq t^{-2}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

if $|\frac{1}{2} - \frac{1}{p}| \leqslant \frac{N(\frac{n-1}{2})-2}{n-1}$. For any $1 < p,q < \infty$ or $0 < q \leqslant p \leqslant 1$, we have

$$\|\Delta(A^N_t(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq t^{-2}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

if $|\frac{1}{2}-\frac{1}{p}|<\frac{N(\frac{n-1}{2})-2}{n-1}$*; And for* $0<p\leqslant 1<q<\infty$*, we have*

$$\|\Delta(A_t^N(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\preceq t^{-2}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

if $|\frac{1}{2}-\frac{1}{p}|\leqslant\frac{N(\frac{n-1}{2})-2}{n-1}$.

This paper is organized as follows. In the next section, we will introduce some preliminary knowledge. Section 3 is devoted to the proof of Theorem 2. Finally, in section 4, we will give the proofs of Theorems 1 and 3.

Throughout this paper, the letter C stands for a positive constant which is independent of the essential variables, but whose value may vary from line to line. We use the notion $A\preceq B$ to mean that there exists a positive constant C independent of all essential variables such that $A\leqslant CB$. The notion $A\approx B$ means that there are two positive constant C_1 and C_2 independent of all essential variables such that $C_1A\leqslant B\leqslant C_2A$.

2. Preliminary knowledge

Let T_μ be a convolution operator and $\widehat{T_\mu(f)}(\xi)=\mu(\xi)\hat{f}(\xi)$, where μ is called the multiplier of T_μ. If T_μ is a bounded operator on $\dot{F}^s_{p,q}(\mathbb{R}^n)$, then we say that μ is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier and denote by $\|\mu(\cdot)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)\to\dot{F}^s_{p,q}(\mathbb{R}^n)}$ the operator norm of T_μ.

Let $0<p<\infty$. Denote by $[r]$ the largest integer less than or equal to the real number r. The following $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier theorem will be used repeatedly in the sequel.

THEOREM B. ([6], Theorem 5.1, p. 851) *Let* $s\in\mathbb{R}$, $0<p<\infty$*, and* $0<q\leqslant\infty$*. If for all multi-indices* α *satisfying* $|\alpha|\leqslant[\max\{\frac{n}{p},\frac{n}{q}\}+\frac{n}{2}]+1$, $\mu(\xi)$ *satisfies the following condition*

$$\sup_{\xi\in\mathbb{R}^n}|\xi|^{|\alpha|}|D^\alpha\mu(\xi)|\leqslant A,$$

then

$$\|\mu(\cdot)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)\to\dot{F}^s_{p,q}(\mathbb{R}^n)}\leqslant C,$$

where C *is a positive constant depending only* α*, the dimension* n *and the constant* A*. Also, if* $A>1$*, then* C *is not larger than* $A^{N(\alpha,n)}$*, where* $N(\alpha,n)$ *is an integer depending only on* α *and* n*.*

By an easy scaling argument, it is easy to get the following lemma.

LEMMA 1. *Let* $0<p<\infty,0<q<\infty$ *and let* μ *be an* $\dot{F}^s_{p,q}(\mathbb{R}^n)$ *multiplier. Then for any* $t>0$, $\mu(t\cdot)$ *is also an* $\dot{F}^s_{p,q}(\mathbb{R}^n)$ *multiplier. Moreover,*

$$\|\mu(\cdot)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)\to\dot{F}^s_{p,q}(\mathbb{R}^n)}=\|\mu(t\cdot)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)\to\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

The Bochner-Riesz multiplier $B^{\delta}_{t,\gamma}$ is defined by

$$\widehat{B^{\delta}_{t,\gamma}(f)}(\xi) = (1-(t|\xi|)^{\gamma})^{\delta}_{+}\hat{f}(\xi), f \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n),$$

where t, γ and δ are positive.

In fact, if we set $\dot{F}^{s,\gamma}_{p,q,2}(\mathbb{R}^n) = \{g \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n) : I_{-\gamma}(g) \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)\}$, then the K-functional $K_{\gamma}(f,t)_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}$ has the following property:

$$K_{\gamma}(f,t)_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} = \inf_{g\in \dot{F}^{s,\gamma}_{p,q,2}(\mathbb{R}^n)} \left\{\|f-g\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + t^{\gamma}\|I_{-\gamma}(g)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}\right\}.$$

We only need to prove that for every $g \in \dot{F}^{s,\gamma}_{p,q}(\mathbb{R}^n)$ and any $\varepsilon > 0$, there exists a $g_1 \in \dot{F}^{s,\gamma}_{p,q,2}(\mathbb{R}^n)$ such that

$$\|g-g_1\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + \|I_{-\gamma}(g) - I_{-\gamma}(g_1)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} < \varepsilon. \tag{4}$$

Let $p_y(x) = C_n \frac{y}{(y^2+x^2)^{(n+1)/2}}$ be the Poisson kernel. It is easy to see that $p_y * g \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and

$$[\widehat{I^{-\gamma}(p_y * g)}](\xi) = |\xi|^{\gamma} e^{-2\pi y|\xi|}\hat{g}(\xi) = [\widehat{p_y * I_{-\gamma}(g)}](\xi).$$

Thus, $I_{-\gamma}(p_y * g) = p_y * I_{-\gamma}(g) \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. From the properties of the Poisson integral, we get

$$\|f - p_y * g\|_{\dot{F}^{s,\gamma}_{p,q}(\mathbb{R}^n)} + \|I_{-\gamma}(g) - p_y * I_{-\gamma}(g)\|_{\dot{F}^{s,\gamma}_{p,q}(\mathbb{R}^n)} \to 0 (y \to 0^{+}).$$

Hence, (4) is proved.

Next, we prove the following lemma. It is an analogue of Theorem 3.1 in [14] with a slight difference.

LEMMA 2. *Let* $0 < p < \infty$, $0 < q < \infty$, $\gamma > 0$ *and* $t > 0$. *If* $\delta > [\max\{\frac{n}{p},\frac{n}{q}\} + \frac{n}{2}] + 1$, *then for* $f \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)$, *we have*

$$\|B^{\delta}_{t,\gamma}(f) - f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \approx K_{\gamma}(f,t)_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}.$$

Proof. On one hand, let $m(\xi) = (1-|\xi|^{\gamma})^{\delta}_{+}$. For every $g \in \dot{F}^{s,\gamma}_{p,q,2}(\mathbb{R}^n)$, we have

$$[\widehat{B^{\delta}_{t,\gamma}(g) - g}](\xi) = (m(t\xi)-1)\hat{g}(\xi) = t^{\gamma}\frac{m(t\xi)-1}{|t\xi|^{\gamma}}\widehat{I^{-\gamma}g}(\xi).$$

Let $u(\xi) = \frac{(1-|\xi|^{\gamma})^{\delta}_{+}-1}{|\xi|^{\gamma}}$. If $|\xi| \leqslant 1$, for every multi-indices α satisfying $|\alpha| \leqslant [\max\{\frac{n}{p},\frac{n}{q}\} + \frac{n}{2}] + 1$ and $\xi \in \mathbb{R}^n\setminus\{0\}$, we have

$$\begin{aligned} D^{\alpha}u(\xi) &= D^{\alpha}(|\xi|^{-\gamma}(m(\xi)-1)) \\ &= \sum_{\beta\leqslant\alpha} C_{\beta}(D^{\alpha-\beta}|\xi|^{-\gamma})D^{\beta}(m(\xi)-1) \\ &= (m(\xi)-1)D^{\alpha}(|\xi|^{-\gamma}) + \sum_{0\neq\beta\leqslant\alpha} C_{\beta}(D^{\alpha-\beta}|\xi|^{-\gamma})D^{\beta}m(\xi) \end{aligned}$$

Since $|m(\xi)-1| \leqslant C|\xi|^{\gamma}$, there holds

$$\begin{aligned}
&|D^{\alpha}u(\xi)| \\
&\leqslant |\xi|^{\gamma}|\xi|^{-\gamma-|\alpha|} + \sum_{0\neq\beta\leqslant\alpha} C_{\beta}|\xi|^{-\gamma-|\alpha|+|\beta|}|\xi|^{-|\beta|+\gamma} \\
&\leqslant C|\xi|^{-|\alpha|}.
\end{aligned}$$

If $|\xi|>1$, then

$$|D^{\alpha}u(\xi)| \leqslant |\xi|^{-\gamma-|\alpha|}.$$

From Theorem B, we know that u is a multiplier on $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$. Similarly, we can show that m is also a multiplier on $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$ when $\delta > [\max\{\frac{n}{p},\frac{n}{q}\}+\frac{n}{2}]+1$. Hence for any $f \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)$ and $g \in \dot{F}^{s,\gamma}_{p,q,2}(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\|B^{\delta}_{t,\gamma}(f)-f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \\
&\leqslant C\big\{\|B^{\delta}_{t,\gamma}(f)-B^{\delta}_{t,\gamma}(g)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + \|B^{\delta}_{t,\gamma}(g)-g\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + \|f-g\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}\big\} \\
&\leqslant C\big\{\|B^{\delta}_{t,\gamma}(f)-B^{\delta}_{t,\gamma}(g)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + t^{\gamma}\|I^{-\gamma}g\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + \|f-g\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}\big\} \\
&\leqslant C\big\{\|f-g\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + t^{\gamma}\|I^{-\gamma}g\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}\big\} \\
&\leqslant CK_{\gamma}(f,t)_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}.
\end{aligned}$$

The last inequality follows from (4).

One the other hand, it is easy to obtain that $(B^{\delta}_{t,\gamma}(f)-f) \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ for any $f \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Thus, when writing $v(\xi)=|\xi|^{\gamma}m(\xi)(1-m(\xi))^{-1}$, for $f \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}
\widehat{I^{-\gamma}(B^{\delta}_{t,\gamma}f)}(\xi) &= |\xi|^{\gamma}m(t\xi)\hat{f}(\xi) \\
&= t^{-\gamma}v(t\xi)(m(t\xi)-1)\hat{f}(\xi) \\
&= t^{-\gamma}v(t\xi)[\widehat{B^{\delta}_{t,\gamma}(f)-f}].
\end{aligned}$$

If we can prove v is a multiplier on $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$, then

$$\|I^{-\gamma}(B^{\delta}_{t,\gamma}f)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \leqslant Ct^{-\gamma}\|B^{\delta}_{t,\gamma}(f)-f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}.$$

Consequently,

$$\begin{aligned}
K_{\gamma}(f,t)_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} &\leqslant \|f-B^{\delta}_{t,\gamma}f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} + t^{\gamma}\|I^{-\gamma}(B^{\delta}_{t,\gamma}f)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \\
&\leqslant C\|B^{\delta}_{t,\gamma}(f)-f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}.
\end{aligned}$$

for all $f \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$ and therefore for all $f \in \dot{F}^{s}_{p,q}(\mathbb{R}^n)$.

Our remaining task is to show that v satisfies the condition in Theorem B. If $|\xi| > 1$, then $v(\xi) = 0$. When $|\xi| \leqslant 1$, some direct computations show that

$$\begin{aligned}
&|D^{\beta}(m(\xi)-1)^{-1}| \\
&\leqslant C_{\beta}\sum_{k=1}^{|\beta|}|m(\xi)-1|^{k-1}\sum_{\beta^1+\beta^2+\cdots+\beta^k}^{\beta^j\neq 0}|D^{\beta^1}m(\xi)|\cdots|D^{\beta^k}m(\xi)|
\end{aligned}$$

As $|D^{\tilde{\alpha}}m(\xi)| \leqslant C|\xi|^{\gamma-|\tilde{\alpha}|}$, if $|\tilde{\alpha}| \leqslant [\max\{\frac{n}{p},\frac{n}{q}\}+\frac{n}{2}]+1$ and $\delta > [\max\{\frac{n}{p},\frac{n}{q}\}+\frac{n}{2}]+1$. Which implies that

$$\begin{aligned}
|D^{\beta}(m(\xi)-1)^{-1}| &\leqslant C_{\beta}\sum_{k=1}^{|\beta|}|m(\xi)-1|^{k-1}|\xi|^{k\gamma-|\beta|} \\
&\leqslant C|\xi|^{\gamma-|\beta|}.
\end{aligned}$$

Combining the above estimates, for every multi-indices α satisfying $|\alpha| \leqslant [\max\{\frac{n}{p},\frac{n}{q}\}+\frac{n}{2}]+1$ and $\xi \in \mathbb{R}^n\setminus\{0\}$, we have

$$\begin{aligned}
&|D^{\alpha}v(\xi)| \\
&\leqslant |m(\xi)(m(\xi)-1)^{-1}|\cdot|D^{\alpha}|\xi|^{\gamma}| + \sum_{0<\beta\leqslant\alpha}C_{\beta}|D^{\alpha-\beta}|\xi|^{\gamma}|\cdot|D^{\beta}(m(\xi)-1)^{-1}| \\
&\leqslant C|\xi|^{-|\alpha|}.
\end{aligned}$$

This completes the proof the Lemma 2. □

We recall the following lifting property on the Triebel-Lizorkin spaces (see [3]).

LEMMA 3. *Let* $-\infty < s < \infty$, $-\infty < \gamma < \infty$, *and* $0 < p,q \leqslant \infty$. *The space* $\dot{F}^{s-\gamma}_{p,q}(\mathbb{R}^n)$ *has the lifting property*

$$\|I_{-\gamma}f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \approx \|f\|_{\dot{F}^{s+\gamma}_{p,q}(\mathbb{R}^n)}.$$

With the help of Lemma 3, we obtain the following Plancherel-Polya-Nikol'skij-type inequality on the Triebel-Lizorkin spaces.

LEMMA 4. *Let* $l \in \mathbb{Z}^+$, $0 < p < \infty$. *Suppose that* $f \in \dot{F}^{s}_{p,q}(R^n)$ *and supp* $\hat{f} \subset \{\xi : |\xi| \leqslant \frac{1}{t}\}$. *Then we obtain that* $\Delta^l(f)$ *is in* $\dot{F}^{s}_{p,q}(R^n)$ *and*

$$\|\Delta^l(f)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \preceq t^{-2l}\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}.$$

Similarly, for any positive integer γ, *there holds*

$$\|I_{-\gamma}(f)\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)} \preceq t^{-\gamma}\|f\|_{\dot{F}^{s}_{p,q}(\mathbb{R}^n)}.$$

Proof. By Lemma 3, we have that

$$\|\Delta^l(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \approx \|f\|_{\dot{F}^{s+2l}_{p,q}(\mathbb{R}^n)}.$$

Since $\mathrm{supp}\hat{f} \subset \{\xi : |\xi| \leqslant \frac{1}{t}\}$, and $\mathrm{supp}\hat{\psi}_j \subset \{\xi : 2^{j-1} \leqslant |\xi| \leqslant 2^{j+1}\}$, which yields

$$\begin{aligned}\|f\|_{\dot{F}^{s+2l}_{p,q}(\mathbb{R}^n)} &= \|(\sum_j 2^{j(s+2l)q}|\psi_j * f|^q)^{\frac{1}{q}}\|_p \\ &= \|(\sum_{2^j \leqslant \frac{1}{t}} 2^{j(s+2l)q}|\psi_j * f|^q)^{\frac{1}{q}}\|_p \\ &\leqslant t^{-2l}\|(\sum_j 2^{jsq}|\psi_j * f|^q)^{\frac{1}{q}}\|_p \\ &= t^{-2l}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

By the similar computations above, we can also get that

$$\|I_{-\gamma}(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq t^{-\gamma}\|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

Thus we complete the proof of Lemma 4. □

From Lemma 2, we can obtain the following result.

LEMMA 5. *Let* $l \in \mathbb{Z}^+$, $s \in \mathbb{R}, 0 < p,q < \infty$ *and* $t > 0$. *If* $\delta > [\max\{\frac{n}{p}, \frac{n}{q}\} + \frac{n}{2}] + 1$, *then for* $f \in \dot{F}^s_{p,q}(\mathbb{R}^n)$, *we have*

$$\|B^\delta_{t,2l}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^{2l}\|\Delta^l B^\delta_{t,2l}(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \approx K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

Proof. By Lemma 2, we have

$$K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq \|B^\delta_{t,2l}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^{2l}\|\Delta^l B^\delta_{t,2l}(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

To show the reverse inequality, by Lemma 2 again, it only needs to prove

$$t^{2l}\|\Delta^l B^\delta_{t,2l}(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

In fact, by the definition of $B^\delta_{t,2l}(f)$, it is easy to check that

$$t^{2l}\Delta^l B^\delta_{t,2l}(f) \approx B^\delta_{t,2l}(f) - B^{\delta+1}_{t,2l}(f)$$

Hence,

$$\begin{aligned}\|B^\delta_{t,2l}(f) - B^{\delta+1}_{t,2l}(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} &\preceq \|B^{\delta+1}_{t,2l}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + \|B^\delta_{t,2l}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \\ &\preceq K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

This finishes the proof of Lemma 5. □

We can proceed similarly to the proof of Lemma 5, and formulate the general result.

LEMMA 6. *Let* $0<\gamma<\infty$*,* $s\in\mathbb{R}$*,* $0<p,q<\infty$ *and* $t>0$*. If* $\delta>[\max\{\frac{n}{p},\frac{n}{q}\}+\frac{n}{2}]+1$*, then for* $f\in\dot{F}^s_{p,q}(\mathbb{R}^n)$*, we have*

$$\|B^{\delta}_{t,\gamma}(f)-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}+t^{\gamma}\|I^{-\gamma}B^{\delta}_{t,\gamma}(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\approx K_{\gamma}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

3. Proof of Theorem 2

The inequality

$$K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\leqslant\|\eta_t f-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}+t^{2l}\|\Delta^l\eta_t f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

follows from the definition directly. To show the reverse inequality, by taking Fourier transform, we get

$$\eta_t(B^{\delta}_{t,\gamma}(f))(x)=B^{\delta}_{t,\gamma}(f)(x),x\in\mathbb{R}^n.$$

From Lemma 2, we have

$$\begin{aligned}\|\eta_t f-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}&=\|B^{\delta}_{t,\gamma}(f)-f+\eta_t(f)-\eta_t(B^{\delta}_{t,\gamma}(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\\&\preceq\|B^{\delta}_{t,\gamma}(f)-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}+\|\eta_t(f-B^{\delta}_{t,\gamma}(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\\&\preceq\|B^{\delta}_{t,\gamma}(f)-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\\&\preceq K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

It remains to prove

$$t^{2l}\|\Delta^l\eta_t f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\preceq K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

By Lemma 5, we have that

$$\begin{aligned}t^{2l}\|\Delta^l\eta_t f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}&\preceq t^{2l}\|\Delta^l B^{\delta}_{t,2l}(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}+t^{2l}\|\Delta^l(B^{\delta}_{t,2l}(f)-\eta_t(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\\&\preceq K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}+t^{2l}\|\Delta^l(B^{\delta}_{t,2l}(f)-\eta_t(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\end{aligned}$$

The second part can be estimated as the following

$$\begin{aligned}t^{2l}\|\Delta^l(B^{\delta}_{t,2l}(f)-\eta_t(f))\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}&\preceq\|B^{\delta}_{t,2l}(f)-\eta_t(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\\&\preceq\|B^{\delta}_{t,2l}(f)-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}+\|\eta_t(f)-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\\&\preceq K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\end{aligned}$$

Using the same argument, we can obtain the general inequality

$$\|\eta_t(f)-f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}+t^{\gamma}\|I_{-\gamma}\eta_t(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\approx K_{\gamma}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

Thus we have completed the proof.

4. Proof of Theorems 1 and 3

For convenience, we let $V_\sigma(u) = \frac{J_\sigma(u)}{|u|^\sigma}$, where J_σ is the Bessel function of order σ, $\sigma > -\frac{1}{2}$ and $u \in \mathbb{C}$. Recall that in [7, 19]

$$\widehat{A_t(f)(\xi)} = m(2\pi t|\xi|)\hat{f}(\xi),$$

where $m(u) = 2^{(n-2)/2}\Gamma(n/2)V_{\frac{n-2}{2}}(u)$. By checking the Fourier transform of the operator S_t^β defined in , we get that

$$\widehat{S_t^\beta f}(\xi) = \mu^\beta(2\pi t|\xi|)\hat{f}(\xi).$$

By using the formula in ([20], pp. 153-154) and the identity in ([13], p. 427), we have

$$\mu^\beta(u) = C_{n,\beta}(2\pi)^{\frac{n}{2}}2^{\beta-1}\Gamma(\beta)V_{\frac{n-2}{2}+\beta}(u). \tag{5}$$

Hence, the multiplier of $\mathfrak{I}_{l,t}^\beta$ is

$$\mu_l^\beta(2\pi t|\xi|) = \frac{-2}{\binom{2l}{l}}\sum_{j=1}^{l}(-1)^j\binom{2l}{l-j}\mu^\beta(2\pi tj|\xi|),$$

namely,

$$\widehat{\mathfrak{I}_{l,t}^\beta(f)}(\xi) = \mu_l^\beta(2\pi t|\xi|)\hat{f}(\xi).$$

By a similar calculation of (9) and Lemma 3.2 in [7] , we get

$$\begin{aligned} 1-\mu_l^\beta(u) = &\frac{2\Gamma(\beta+n/2)}{\Gamma(1/2)\Gamma(\beta+n/2-1/2)} \times \frac{4^l}{\binom{2l}{l}} \\ &\times \int_0^1 (1-s^2)^{\frac{n+\beta-3}{2}}\left(\sin\left(\frac{|u|s}{2}\right)\right)^{2l} ds. \end{aligned} \tag{6}$$

At first, we prove the following conclusion about the multiplier μ^β defined in (5) for different p and q.

LEMMA 7. *Let $\beta \geqslant 0$, $s \in \mathbb{R}$. For any $1 < p,q < \infty$, μ^β is a multiplier of $\dot{F}_{p,q}^s(\mathbb{R}^n)$; For any $0 < q \leqslant p \leqslant 1$, μ^β is a multiplier of $\dot{F}_{p,q}^s(\mathbb{R}^n)$ if $p > \frac{n-1}{\beta+n-1}$; For any $0 < p \leqslant 1 < q < \infty$, μ^β is a multiplier of $\dot{F}_{p,q}^s(\mathbb{R}^n)$ if $p \geqslant \frac{n-1}{\beta+n-1}$.*

Proof. Let $\eta(\xi)$ be the same as the above, and set $\psi(\xi) = 1-\eta(\xi)$. Since μ^β is a radial function, we can write

$$\begin{aligned} \mu^\beta(|\xi|) &= C_{n,\beta}(2\pi)^{\frac{n}{2}}2^{\beta-1}\Gamma(\beta)V_{\frac{n-2}{2}+\beta}(|\xi|) \\ &= C_{n,\beta}(2\pi)^{\frac{n}{2}}2^{\beta-1}\Gamma(\beta)(V_{\frac{n-2}{2}+\beta}(|\xi|)\eta(\xi) + V_{\frac{n-2}{2}+\beta}(|\xi|)\Psi(\xi)) \end{aligned}$$

By Leibniz' rule of differentiation, for any multi-indices α, $\partial^{\alpha}\big(V_{\frac{n-2}{2}+\beta}(|\xi|)\eta(\xi)\big)$ can be written as a finite linear combination of the following functions:

$$\partial^{\alpha'}V_{\frac{n-2}{2}+\beta}(|\xi|)\partial^{\beta'}\eta(\xi)$$

where α' and β' are multi-indices with $\alpha'+\beta'=\alpha$. Using the derivative formula for Bessel function

$$\begin{aligned}&\frac{dV_{\gamma}(t)}{dt}=-tV_{\gamma+1}(t),\\&V_{\gamma}(|\xi|)=O(1)\ \text{ if }\ |\xi|\leqslant 2,\end{aligned}\tag{7}$$

and noting that $\operatorname{supp}\eta\subset\{\xi:|\xi|\leqslant 2\}$, we have

$$\begin{aligned}|\partial^{\alpha}V_{\frac{n-2}{2}+\beta}(|\xi|)\eta(\xi)|&=|\sum_{\alpha=\alpha'+\beta'}\partial^{\alpha'}V_{\frac{n-2}{2}+\beta}(|\xi|)\partial^{\beta'}\eta(\xi)|\\&\leqslant C\sum_{|\beta'|\leqslant|\alpha|}|\partial^{\beta'}(|\xi|)|.\end{aligned}$$

Since η is a smooth function, we get that

$$\sup_{\xi\in\mathbb{R}^n}|\xi|^{|\alpha|}\big|\partial^{\alpha}\big(V_{\frac{n-2}{2}+\beta}(|\xi|)\eta(\xi)\big)\big|\leqslant A$$

for any multi-indices α. By Theorem B , we know that $V_{\frac{n-2}{2}+\beta}(|\xi|)\eta(\xi)$ is a $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$ multiplier for any $0<p,q<\infty$. By Proposition 5.1 in [11], we can write the second multiplier

$$\begin{aligned}&V_{\frac{n-2}{2}+\beta}(|\xi|)\Psi(\xi)\\&=\sum_{j=0}^{L}a_j\Psi(\xi)e^{i\xi}|\xi|^{-\frac{n-1}{2}-\beta-j}+\sum_{0}^{L}b_j\Psi(\xi)e^{-i\xi}|\xi|^{-\frac{n-1}{2}-\beta-j}+E(\xi)\Psi(\xi),\end{aligned}$$

where $E(\xi)$ is a C^{∞} function satisfying

$$|\partial^{\alpha}E(\xi)|\preceq|\xi|^{-\frac{n-1}{2}-\beta-j},|\xi|>1$$

for any multi-index α. Noting $\Psi(\xi)=0$ if $|\xi|\leqslant 1$, we may choose a suitably large L such that $E(\xi)\Psi(\xi)$ is a $\dot{F}^{s}_{p,q}(\mathbb{R}^n)$ for different p,q. Moreover, we know that , for each j,

$$m_j^{+}(\xi)=\Psi(\xi)e^{i\xi}|\xi|^{-\frac{n-1}{2}-\beta-j}\ \text{ or }\ m_j^{-}(\xi)=\Psi(\xi)e^{i\xi}|\xi|^{-\frac{n-1}{2}-\beta-j}$$

is the multiplier of the wave operator W_{ν} with $\nu=-\frac{n-1}{2}-\beta-j$ (see [3, 5, 15, 16]).

For the terms m_j^{+} and m_j^{-}, we need the following result about oscillating multipliers on Triebel-Lizorkin spaces in [3].

THEOREM C. *Suppose that* W_ν *is the wave operator with Fourier multiplier* $\Psi(\xi)e^{ic|\xi|}|\xi|^{-\nu}$, *where* $\Psi(\xi) \in C^\infty(R^n)$ *and equals to zero near zero and one for* $|\xi| \geqslant 2$, c *is a non-zero real number and* $\nu > 0$. *Let* $s \in \mathbb{R}$. *For any* $1 < p \leqslant q \leqslant 2$ *or* $2 \leqslant q \leqslant p < \infty$, *the operator* W_ν *is bounded on the space* $\dot{F}^s_{p,q}(\mathbb{R}^n)$ *if* $\nu \geqslant (n-1)|\frac{1}{2} - \frac{1}{p}|$;

For any $1 < p,q < \infty$ *or* $0 < q \leqslant p \leqslant 1$, *the operator* W_ν *is bounded on the space* $\dot{F}^s_{p,q}(\mathbb{R}^n)$ *if* $\nu > (n-1)|\frac{1}{2} - \frac{1}{p}|$;

For any $0 < p \leqslant 1 < q < \infty$, *the operator* W_ν *is bounded on the space* $\dot{F}^s_{p,q}(\mathbb{R}^n)$ *if* $\nu \geqslant (n-1)|\frac{1}{2} - \frac{1}{p}|$.

Taking $\nu = \frac{n-1}{2} + \beta$ in Theorem C, for any $1 < p,q < \infty$, we have

$$-\frac{1}{2} < \frac{1}{p} - \frac{1}{2} < \frac{1}{2} + \frac{\beta}{n-1} = \frac{\frac{n-1}{2} + \beta}{n-1},$$

which satisfies the conditions of Theorem C.

For any $0 < q \leqslant p \leqslant 1$, from Theorem C, we know that $\Psi(\xi)e^{\mp i\xi}|\xi|^{-\nu}$ is a multiplier of $\dot{F}^s_{p,q}(\mathbb{R}^n)$ if

$$\Big|\frac{1}{p} - \frac{1}{2}\Big| < \frac{\frac{n-1}{2} + \beta}{n-1}.$$

This inequality is equivalent to $p \in (\frac{n-1}{\beta+n-1}, 1]$. Similarly, we can get that $\Psi(\xi)e^{\mp i\xi}|\xi|^{-\nu}$ is a multiplier of $\dot{F}^s_{p,q}(\mathbb{R}^n)$ if $p \geqslant \frac{n-1}{\beta+n-1}$ for any $0 < p \leqslant 1 < q < \infty$. It is an easy fact that if $\Psi(\xi)e^{\mp i\xi}|\xi|^{-\nu}$ is a H^p multiplier, then $\Psi(\xi)e^{\mp i\xi}|\xi|^{-\nu-\varepsilon}$ is, for any positive ε. Hence, Lemma 7 is proved. □

We now are ready to show Theorem 3.

Proof of Theorem 3. With loss of generality, we will only prove $0 < p \leqslant 1 < q < \infty$. The other cases of p and q are similar but easier. By an easy scaling argument, it suffices to show that

$$F\Big(N, \frac{n-2}{2}, |\xi|\Big) = |\xi|^2 V^N_{\frac{n-2}{2}}(|\xi|)$$

is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier if

$$\Big|\frac{1}{2} - \frac{1}{p}\Big| \leqslant \frac{N(\frac{n-1}{2}) - 2}{n-1}.$$

As we did in the proof of Lemma 7, we may write

$$\begin{aligned} F\Big(N, \frac{n-2}{2}, |\xi|\Big) &= |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|)\eta(\xi) + V_{\frac{n-2}{2}}(|\xi|)\Psi(\xi))^N \\ &= |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|)\eta(\xi))^N + |\xi|^2 (V_{\frac{n-2}{2}}(|\xi|)\Psi(\xi))^N \end{aligned}$$

Applying the same argument in the proof of Lemma 7, the first part is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier for any $0 < p,q < \infty$. Hence, we consider the multiplier $|\xi|^2 (V_{\frac{n-2}{2}}(|\xi|)\Psi(\xi))^N$.

Using the Proposition 5.1 in [11] again, we write

$$\begin{aligned}
&F\left(N,\frac{n-2}{2},|\xi|\right)\\
&=|\xi|^2\left(e^{i|\xi|}\sum_{j=0}^{j=L}a_j\psi(\xi)|\xi|^{-\frac{n-1}{2}-j}+e^{-i|\xi|}\sum_{j=0}^{L}b_j\psi_\xi|\xi|^{-\frac{n-1}{2}-j}+E(\xi)\psi(\xi)\right)^N\\
&=a_0^Ne^{iN|\xi|}\psi^N(\xi)|\xi|^{-\frac{N(n-1)}{2}+2}+b_0^Ne^{-iN|\xi|}\psi^N(\xi)|\xi|^{-\frac{N(n-1)}{2}+2}+\varepsilon(\xi),
\end{aligned}$$

where $\varepsilon(\xi)$ is a function satisfying that if $a_0^Ne^{iN|\xi|}\psi^N(\xi)|\xi|^{-\frac{N(n-1)}{2}+2}$ and $b_0^Ne^{-iN|\xi|}\psi^N(\xi)$ $\times|\xi|^{-\frac{N(n-1)}{2}+2}$ are $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier for $0<p\leqslant 1<q<\infty$. From Theorem C, we know that $a_0^Ne^{iN|\xi|}\psi^N(\xi)|\xi|^{-\frac{N(n-1)}{2}+2}$ and $b_0^Ne^{-iN|\xi|}\psi^N(\xi)$ are $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier for $0<p\leqslant 1<q<\infty$ if

$$\left|\frac{1}{2}-\frac{1}{p}\right|\leqslant\frac{N(\frac{n-1}{2})-2}{n-1}.$$

This concludes the proof of Theorem 3. □

Finally, we end this paper with the

Proof of Theorem 1. First we shall prove the following result.

LEMMA 8. *Let $l\in\mathbb{Z}$, $\beta\geqslant 0$ and $t>0$. If $\gamma\leqslant 2l$ and $I_{-\gamma}(g)\in\dot{F}^s_{p,q}(\mathbb{R}^n)$, for any $0<q\leqslant p\leqslant 1$ or $1<p,q<\infty$, we have*

$$\|t^{-\gamma}(\mathfrak{I}^\beta_{l,t}(g)-g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\preceq\|I_{-\gamma}(g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

if $p>\frac{n-1}{n-1+\beta+\gamma}$. Besides, for any $0<p\leqslant 1<q<\infty$, we have

$$\|t^{-\gamma}(\mathfrak{I}^\beta_{l,t}(g)-g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}\preceq\|I_{-\gamma}(g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$$

if $p\geqslant\frac{n-1}{n-1+\beta+\gamma}$.

Proof. Note that $t^{-\gamma}(\mathfrak{I}^\beta_{l,t}(g)-g)$ has the Fourier transform

$$\frac{\mu^\beta_l(t\xi)-1}{|t\xi|^\gamma}\hat{g}(\xi)=\frac{\mu^\beta_l(t\xi)-1}{|t\xi|^\gamma}\widehat{I_{-\gamma}(g)}(\xi).$$

By Lemma 1, to prove Lemma 8, we only need to show that $\frac{\mu^\beta_l(\xi)-1}{|\xi|^\gamma}$ is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier. Write

$$\frac{\mu^\beta_l(\xi)-1}{|\xi|^\gamma}=\frac{(\mu^\beta_l(\xi)-1)\eta(\xi)}{|\xi|^\gamma}+\frac{(\mu^\beta_l(\xi)-1)\Psi(\xi)}{|\xi|^\gamma}.$$

Since supp $\eta \subset \{\xi \in \mathbb{R}^n : |\xi| < 2\}$, then we have

$$\sin^{2l}\left(\frac{s|\xi|}{2}\right) = \left(\frac{s|\xi|}{2}\right)^{2l} + \Phi(s|\xi|), \text{for } |\xi| < 2,$$

where $\Phi(u)$ is a C^∞ function satisfying $|\frac{d^k}{du^k}\Phi(u)| \preceq \min\{1, u^{2l+2-k}\}$ for all $0 < u < 2$ and all integers k. Using (6) and Theorem B, it is easy to show that the first part is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier for any $0 < p,q < \infty$.

Next we turn to study the multiplier $\frac{(\mu_l^\beta(\xi)-1)\Psi(\xi)}{|\xi|^\gamma}$. As

$$1 - \mu_l^\beta(2\pi t|\xi|) = 1 + \frac{2}{\binom{2l}{l}} \sum_{j=1}^{l} (-1)^j \binom{2l}{l-j} \mu^\beta(2\pi t j|\xi|),$$

which implies that we only need to consider the multiplier $\mu^\beta(2\pi t j|\xi|)\Psi(\xi)|\xi|^{-\gamma}$. But it can be done by following the same argument as Lemma 7. Hence, Lemma 8 is proved. $\square$

Now we start to prove Theorem 1. Without loss of generality, we will only consider $0 < p \leqslant 1 < q < \infty$. The other cases are similar and easier. First, from the definition of the γth order K-functional $K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}$, there exists a function g such that

$$\|f - g\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^{2l}\|\Delta^l(g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \leqslant 2K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

By Lemma 7, we have

$$\begin{aligned} \|\Im^\beta_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} &\preceq \|\Im^\beta_{l,t}(f-g) + (g-f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + \|\Im^\beta_{l,t}(g) - g\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \\ &\preceq \|g - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + \|\Im^\beta_{l,t}(g) - g\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \end{aligned}$$

From Lemma 8 , we get

$$\|\Im^\beta_{l,t}(g) - g\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} = t^{2l}\|t^{-2l}(\Im^\beta_{l,t}(g) - g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq t^{2l}\|\Delta^l(g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)},$$

which implies that

$$\begin{aligned} \|\Im^\beta_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} &\preceq \|g - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^{2l}\|\Delta^l(g)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \\ &\preceq 2K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)}. \end{aligned}$$

It remains to show

$$K_{2l}(f,t)_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq \|\Im^\beta_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

To this end, by Theorem 2, it suffices to prove

$$\|\eta_t(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} + t^{2l}\|\Delta^l\eta_t(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} \preceq \|\Im^\beta_{l,t}(f) - f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.$$

By Lemma 1, we need to show that the following two multipliers

$$\frac{1-\eta(\xi)}{1-\mu_\beta^l(\xi)},\quad \frac{|\xi|^{2l}\eta(\xi)}{1-\mu_l^\beta(\xi)}$$

are $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multipliers for $p \geqslant \frac{n-1}{n-1+\beta}$.

Note that $|1-\mu_l^\beta(\xi)| \geqslant C|\xi|^{2l}$ if $|\xi| \leqslant 2$. So, with the same idea as we did in Lemma 8, we can use Theorem B to check that $\frac{|\xi|^{2l}\eta(\xi)}{1-\mu_l^\beta(\xi)}$ is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multipliers for any $0<p,q<\infty$. Let $\phi(\xi)$ be a C^∞ radial function supported in $\{|\xi|<2M\}$ and $\phi(\xi)=1$ on the set $\{|\xi|<M\}$, where M is a sufficiently large number to be determined. Let $\psi = 1-\varphi$. We can write

$$\frac{1-\eta(\xi)}{1-\mu_l^\beta(\xi)} = \frac{(1-\eta(\xi))\phi(\xi)}{1-\mu_l^\beta(\xi)} + \frac{(1-\eta(\xi))\psi(\xi)}{1-\mu_l^\beta(\xi)}.$$

With the help of

$$1-\mu_l^\beta(\xi) \geqslant c \ \text{ for } \ |\xi| \geqslant 1,$$

where c is positive constant depending only n,β and l (see [11], Lemma 4.1, p. 90). Since $\frac{(1-\eta(\xi))\phi(\xi)}{1-\mu_l^\beta(\xi)}$ is supported in $\{1 \leqslant |\xi| \leqslant 2M\}$, by Theorem B, it is easy to verify that $\frac{(1-\eta(\xi))\phi(\xi)}{1-\mu_l^\beta(\xi)}$ is a $\dot{F}^s_{p,q}(\mathbb{R}^n)$ multiplier for any $0<p,q<\infty$. Recall that

$$\mu_l^\beta(2\pi t|\xi|) = C_{n,\beta}(2\pi)^{\frac{n}{2}}2^{\beta-1}\Gamma(\beta)\frac{-2}{\binom{2l}{l}}\sum_{j=1}^{l}(-1)^j\binom{2l}{l-j}V_{\frac{n-2}{2}+\beta}(j|\xi|),$$

and we have the following asymptotic development,

$$V_{\frac{n-2}{2}+\beta}(j|\xi|) = O\Big(\frac{1}{(2\pi j|\xi|)^{\frac{n-1}{2}+\beta}}\Big), \text{ as } |\xi|\to\infty.$$

We can choose a sufficiently large $M>0$ and an integer N such that

$$|\mu_l^\beta(|\xi|)| \leqslant \frac{1}{4}.$$

Applying the estimates in (7), for all multi-indices α satisfying $|\alpha| \leqslant [\max\{\frac{n}{p},\frac{n}{q}\}+\frac{n}{2}]+1$, we obtain

$$|\partial_\xi^\alpha(2\mu_l^\beta(\xi))^k| \leqslant C(2k)^{|\alpha|}|\xi|^{-|\alpha|-1} \tag{8}$$

whenever $|\xi|>M$ and $k \geqslant N$, where C is independent of ξ and k.

Since $|\mu_l^\beta(|\xi|)| \leqslant \frac{1}{4}$, we can write

$$\frac{(1-\eta(\xi))\psi(\xi)}{1-\mu_l^\beta(\xi)} = (1-\eta(\xi))\psi(\xi)\sum_{k=0}^{\infty}2^{-k}(2\mu_l^\beta(\xi))^k.$$

We define T by $\widehat{Tf}(\xi) = \frac{(1-\eta(\xi))\psi(\xi)}{1-\mu_l^\beta(\xi)}\hat{f}(\xi)$, which can be decomposed as

$$T = \sum_{k=0}^{N-1} 2^{-k}T_k + \sum_{k=N}^{\infty} 2^{-k}T_k,$$

where T_k is associated to the multiplier $(2\mu_l^\beta(\xi))^k(1-\eta(\xi))\psi(\xi)$. By Lemma 7, the finite sum $\sum_{k=0}^{N-1} 2^{-k}T_k$ is bounded operator on $\dot{F}^s_{p,q}(\mathbb{R}^n)$ for $p \geqslant \frac{n-1}{n-1+\beta}$. For the infinite sum, by the lifting property, it it suffices to show its boundedness on $\dot{F}^0_{p,q}(\mathbb{R}^n)$. More precisely, once we prove

$$\|\sum_{k=N}^{\infty} 2^{-k}T_k(f)\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \preceq \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)}, \tag{9}$$

then

$$\begin{aligned}\|\sum_{k=N}^{\infty} 2^{-k}T_k(f)\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)} &\approx \|I_{-s}\big(\sum_{k=N}^{\infty} 2^{-k}T_k(f)\big)\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} = \|\sum_{k=N}^{\infty} 2^{-k}T_k(I_{-s}f)\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \\ &\preceq \|I_{-s}f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \approx \|f\|_{\dot{F}^s_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

To finish the proof of Theorem 1, it suffices to verify (9). If $0 < p \leqslant 1 < q < \infty$, using Minkowski inequality, we have

$$\begin{aligned}\|\sum_{k=N}^{\infty} 2^{-k}T_kf\|^p_{\dot{F}^0_{p,q}(\mathbb{R}^n)} &= \int \{\sum_j |\psi_j * \sum_{k=N}^{\infty} 2^{-k}T_kf(x)|^q\}^{\frac{p}{q}}dx \\ &\leqslant \int \{\sum_{k=N}^{\infty} 2^{-k}\big(\sum_j |\psi_j * T_kf(x)|^q\big)^{\frac{1}{q}}\}^p dx \\ &\leqslant \int \sum_{k=N}^{\infty} 2^{-kp}\{\sum_j |\psi_j * T_kf(x)|^q\}^{\frac{p}{q}}dx \\ &\leqslant \sum_{k=N}^{\infty} 2^{-kp}\|T_kf\|^p_{\dot{F}^0_{p,q}(\mathbb{R}^n)}.\end{aligned}$$

Applying the estimate of (8) and Theorem B, we have

$$\begin{aligned}\|\sum_{k=N}^{\infty} 2^{-k}T_kf\|^p_{\dot{F}^0_{p,q}(\mathbb{R}^n)} &\leqslant \sum_{k=N}^{\infty} 2^{-kp}\|T_kf\|^p_{\dot{F}^0_{p,q}(\mathbb{R}^n)} \\ \preceq \sum_{k=N}^{\infty} 2^{-kp}k^{N(\alpha,n)}\|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)} &\preceq \|f\|_{\dot{F}^0_{p,q}(\mathbb{R}^n)},\end{aligned}$$

This confirms (9) and hence Theorem 1 follows. $\square$

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