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A REVERSE BLASCHKE–SANTALÓ INEQUALITY

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Abstract. Motivated by works of Lutwak, Yang and Zhang (in [17], [20] and [21]), a new reverse Blaschke-Santaló inequality that connects the volume of an origin-symmetric convex body K with the volume of the polar body Γ_2^*K of the L_2 centroid body of K is established.

1. Introduction

Throughout this paper a convex body K in Euclidean n-space $\mathbb{R}^n$ is a compact convex subset with non-empty interior. The support function $h_K : \mathbb{R}^n \to [0,\infty)$ of K is defined by

$$h_K(x) = \sup\{x\cdot y : y \in K,\ x \in \mathbb{R}^n\},$$

where $x\cdot y$ is the usual inner product of x and y in $\mathbb{R}^n$. The polar body K^* of K is defined by

$$K^* = \{x \in \mathbb{R}^n : x\cdot y \leqslant 1 \ for\ all\ y \in K\}.$$

The Blaschke-Santaló inequality is a well know affine isoperimetric inequality which connect the volume of a convex body with that of its polar body. The Blaschke-Santaló inequality states that

$$V(K)V(K^*) \leqslant \omega_n^2, \tag{1.1}$$

where $V(K)$ and ω_n denote the volume of K and the volume of the unit ball B^n in $\mathbb{R}^n$, respectively, and the equality holds if and only if K is an ellipsoid. A classical proof of this inequality deduced it from the classical affine isoperimetric inequality of affine differential geometry (cf. [28]).

For a compact star-shaped (about the origin) subset K in $\mathbb{R}^n$, and each p such that $1 \leqslant p \leqslant \infty$, the L_p centroid body $\Gamma_p K$ of K is defined by

$$h_{\Gamma_p K}(x)^p = \frac{1}{c_{n,p}V(K)} \int_K |x\cdot y|^p dy, \tag{1.2}$$

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where $c_{n,p} = \frac{\omega_{n+p}}{\omega_2\omega_n\omega_{p-1}}$. The normalization is made in such a way that $\Gamma_p B^n = B^n$. The bodies $\Gamma_2 K$ and $\Gamma_{-2} K$ are all ellipsoids, which are called Legendre ellipsoid and LYZ ellipsoid of K, respectively (cf. [17], [32]). For the case $p = \infty$, this definition is to be interpreted as the limit as $p \to \infty$, which exists, and in the case of an o-symmetric convex body K the polar body of $\Gamma_p K$ coincides with K^*.

Let $\Gamma_p^* K$ denote the polar body of $\Gamma_p K$, in [21] the following centro-affine inequality involving the volume of K and the volume of $\Gamma_2^* K$ was established:

If K is a star-shaped (about the origin) subset of $\mathbb{R}^n$, then for $1 \leqslant p \leqslant \infty$,

$$V(K)V(\Gamma_p^* K) \leqslant \omega_n^2, \tag{1.3}$$

with equality if and only if K is an ellipsoid centered at the origin. If K is an o-symmetric convex body, for $p = \infty$, inequality (1.3) reduces to the classical Blaschke-Santaló inequality (1.1).

For more information on volume inequalities see, e.g., [4]–[16], [19], [22]–[27], [29]–[32] and the references therein. The lower bound of the volume product in (1.3) is of considerable interest. In [21], Lutwak and Zhang provide the following conjecture.

Lutwak-Zhang Problem. *For $p \geqslant 1$, is there a constant $c_p > 0$, independent of n (and perhaps even independent of p), so that for each centered convex body K in $\mathbb{R}^n$,*

$$V(K)V(\Gamma_p^* K) \geqslant \omega_n^2 c_p^n? \tag{1.4}$$

For a star body K and $p = 2$, $\Gamma_2 K$ is the Legendre ellipsoid of K, and inequality (1.4) becomes

$$V(K)V(\Gamma_2^* K) \geqslant \omega_n^2 c_2^n.$$

This inequality is one of the equivalent forms of the slicing problem: Does there exist an absolute constant $c > 0$ such that each centered convex body of unit volume in $\mathbb{R}^n$, has an $(n-1)$-dimensional slice of $(n-1)$-dimensional volume greater than c?

Motivated by Lutwak-Yang-Zhang [17], [20], [21], in this paper, for the case $p = 2$, we establish the following reverse Blaschke-Santaló inequality.

THEOREM 1.1. *Let K be an origin-symmetric convex body in $\mathbb{R}^n$ and Let $\Gamma_2^* K$ be the polar body of the L_2 centroid body $\Gamma_2 K$, then*

$$V(K)V(\Gamma_2^* K) \geqslant \frac{\omega_n (n+1)^{(n+1)/2}}{n! n^{n/2}}. \tag{1.5}$$

This paper is organized as follows. In Section 2, we collect some basic concepts and various facts of convex bodies. In Section 3, we present some results of isotropic measures which will be used. The main theorem is proved in Section 4.

2. Basic concepts of convex body

A convex body is a nonempty compact, convex subset of $\mathbb{R}^n$. The set of all convex bodies in $\mathbb{R}^n$ contain the origin of in its interior is denoted by $\mathscr{K}_o^n$. From the definition of K^* it follows that for $K \in \mathscr{K}_o^n$,

$$K^{**} = K. \tag{2.1}$$

For $\phi \in GL(n)$, we have

$$(\phi K)^* = \phi^{-t} K^*, \tag{2.2}$$

where ϕ^{-t} denotes the inverse of the transpose of ϕ.

A set K in $\mathbb{R}^n$ is star shaped with respect to a point $p \in K$ if the intersection of every line through p with K is a line segment. Let K be a compact star shaped set with respect to the origin. Its radial function is defined by The radial function $\rho_K : \mathbb{R}^n\{0\} \to (0,\infty)$ of K is defined by

$$\rho_K(x) = \max\{\lambda \geqslant 0 : \lambda x \in K\}.$$

If the radial function is continuous for u, then K is called a star body.

The support function is homogeneous of degree 1 while the radial function is homogeneous of degree -1. From the definitions of the support and radial functions and the definition of the polar body, for $K \in \mathscr{K}_o^n$, it follows that,

$$h_K = \frac{1}{\rho_{K^*}} \text{ and } \rho_K = \frac{1}{h_{K^*}}. \tag{2.3}$$

From the definitions (1.2) and (2.3), the radial function of $\Gamma_2^* K$ is given by

$$\begin{aligned} \rho_{\Gamma_2^* K}(x)^{-2} &= \frac{1}{c_{n,2} V(K)} \int_K |x \cdot y|^2 dy \\ &= \frac{n+2}{V(K)} \int_{S^{n-1}} \int_0^{\rho_K} |x \cdot u|^2 r^{n+1} dr du \\ &= \frac{1}{V(K)} \int_{S^{n-1}} |x \cdot u|^2 \rho_K^{n+2} du. \end{aligned} \tag{2.4}$$

For two star bodies K, L, and $\varepsilon > 0$, the L_2-harmonic radial combination $K \widetilde{+}_{-2} \varepsilon \cdot L$ is the star body defined by

$$\rho_{K \widetilde{+}_{-2} \varepsilon \cdot L}^{-2} = \rho_K^{-2} + \varepsilon \rho_L^{-2}. \tag{2.5}$$

The dual mixed volume $V_{-2}(K,L)$ of star bodies K, L can be defined by

$$\frac{n}{-2} V_{-2}(K,L) = \lim_{\varepsilon \to 0^+} \frac{V(K \widetilde{+}_{-2} \varepsilon \cdot L) - V(K)}{\varepsilon}. \tag{2.6}$$

From the definitions (2.5) and (2.6), it follows that for each star body K,

$$V(K) = V_{-2}(K,K). \tag{2.7}$$

Definitions (2.5) and (2.6) and the polar coordinate formula for volume give the following integral representation of the dual mixed volume $V_{-2}(K,L)$ of the star bodies K,L:

$$V_{-2}(K,L) = \frac{1}{n}\int_{S^{n-1}} \rho_K^{n+2}(u)\rho_L^{-2}(u)du. \tag{2.8}$$

Note that for $\phi \in \mathrm{GL(n)}$,

$$\Gamma_{-2}\phi K = \phi\Gamma_{-2}K, \quad \phi \in \mathrm{GL}(n). \tag{2.9}$$

The following lemma is due to Lutwak-Yang-Zhang in [20].

LEMMA 2.1. (cf. [20]) *If ν is a finite positive Borel measure on S^{n-1}, and Q is a convex body that contains the origin in its interior, then*

$$\int_{S^{n-1}} (v\rho_Q(v),1)d\nu(v) \in r_0Q \times \{r_0\} \in \mathbb{R}^{n+1},$$

where $r_0 = \nu(S^{n-1})$.

3. Isotropic embedding

A finite nonnegative Borel measure ν on S^{n-1} is said to be isotropic if

$$\int_{S^{n-1}} |v\cdot u|^2 d\nu(u) = 1, \tag{3.1}$$

for all $v \in S^{n-1}$. Applying (3.1) to the vectors of an orthonormal basis and subsequent summation yields

$$\nu(S^{n-1}) = n. \tag{3.2}$$

For a finite Borel measure ν on S^{n-1}, the bilinear form $F = \int_{S^{n-1}} u\otimes u d\nu(u)$ is defined by

$$F(x,y) = \int_{S^{n-1}} (u\cdot x)(u\cdot y)d\nu(u),\ x,\ y \in \mathbb{R}^n.$$

The Ball-Barthe inequality for isotropic measure is:

Ball-Barth inequality. (cf. [18]) *If ν is an isotropic measure on S^{n-1}, then for each continuous $l : S^{n-1} \to (0,\infty)$ is continuous,*

$$\det\int_{S^{n-1}} l(u)u\otimes u d\nu(u) \geqslant \exp\left\{\int_{S^{n-1}} \log l(u)d\nu(u)\right\}, \tag{3.3}$$

with equality if and only if $l(u_1),\cdots,l(u_n)$ *is constants for linearly independent* $u_1,\cdots,u_n$ *in* $\mathrm{supp}(\nu)$.

The concept of an isotropic embedding is critical in establishing Theorem 1.1.

Isotropic embedding. *If* (S^{n-1},ν) *is a Borel measure space, then a continuous map* $v: S^{n-1}\to S^n$ *is said to be an isotropic embedding of the Borel measure space* (S^{n-1},ν) *into* S^n *if*

$$\int_{S^{n-1}}|\omega\cdot v(x)|^2d\nu(x)=1,\ \ for\ all\ \omega\in S^n. \tag{3.4}$$

Summing the equation (3.4) with $w=e_1,\ldots,e_{n+1}$ shows that if (S^{n-1},ν) is isotropically embeddable into S^n, then

$$\nu(S^{n-1})=n+1. \tag{3.5}$$

Note that, if $\mathrm{supp}\,\nu=\{u_1,\cdots,u_{n+1}\}$, then $v(u_1),\cdots,v(u_n)$ are orthogonal and $\nu(\{u_i\})=1$ for all i.

The Ball-Barthe inequality [1, 2, 3] for isotropic embedding is:

LEMMA 3.1. *If* $h: S^{n-1}\to S^n$ *is an isotropic embedding of the Borel measure space* (S^{n-1},ν) *into* S^n, *then for each continuous* $l: S^{n-1}\to(0,\infty)$

$$\det\int_{S^{n-1}}l(u)h(u)\otimes h(u)d\nu(u)\geqslant\exp\left\{\int_{S^{n-1}}\log l(u)d\nu(u)\right\}, \tag{3.6}$$

with equality if and only if $l(u_1),\cdots,l(u_{n+1})$ *is constants for* $u_1,\cdots,u_{n+1}$ *in* $\mathrm{supp}(\nu)$ *such that* $h(u_1),\cdots,h(u_{n+1})$ *are linearly independent.*

4. Proof of the main theorem

THEOREM 4.1. *Let* K *be an origin-symmetric convex body in* $\mathbb{R}^n$ *and Let* Γ_2^*K *be the polar body of the* L_2 *centroid body* Γ_2K, *then*

$$V(K)V(\Gamma_2^*K)\geqslant\frac{\omega_n(n+1)^{(n+1)/2}}{n!n^{n/2}}. \tag{4.1}$$

Proof. We set

$$h:=\frac{1}{\sqrt{n}\rho_K}=\rho_{\sqrt{n}K}^{-1}\quad\text{and}\quad\mu:=\frac{1}{V(K)}\rho_K^{n+2}du. \tag{4.2}$$

From (2.2) and (2.9), we assume that the body K has been $\mathrm{GL}(n)$-transformed such that $\Gamma_2^*K=B$. According to (2.4) and (3.1), we infer that

the measure μ is isotropic.

Since K is origin-symmetric, hence the measure μ is even, i.e.,

$$\int_{S^{n-1}} uh(u)d\mu(u) = 0, \tag{4.3}$$

and from (2.7), (2.8) and (4.2), we have

$$\int_{S^{n-1}} h(u)^2 d\mu(u) = 1. \tag{4.4}$$

Define $q : S^{n-1} \to \mathbb{R}^{n+1}$ by

$$q(u) = (u, h(u)) \tag{4.5}$$

for $u \in S^{n-1}$, and define $\bar{q} : S^{n-1} \to S^n$ by

$$\bar{q} = \frac{q}{|q|}. \tag{4.6}$$

Suppose $y = (z, r) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$. By (3.1), (4.3) and (4.4), we have

$$\begin{aligned}
&\int_{S^{n-1}} |y \cdot \bar{q}(u)|^2 |q(u)|^2 d\mu(u) \\
&\quad = \int_{S^{n-1}} |(z, r) \cdot (u, h(u))|^2 d\mu(u) \\
&\quad = \int_{S^{n-1}} |z \cdot u + rh(u))|^2 d\mu(u) \\
&\quad = \int_{S^{n-1}} |z \cdot u|^2 d\mu(u) + 2rz \cdot \int_{S^{n-1}} uh(u)d\mu(u) + r^2 \int_{S^{n-1}} h(u)^2 d\mu(u) \\
&\quad = |z|^2 + r^2 \\
&\quad = |y|^2.
\end{aligned}$$

This means that $\bar{q} : S^{n-1} \to S^n$ is an isotropic embedding of the measure space $(S^{n-1}, |q|^2 d\mu)$ into S^n. Therefore, there does not exist a non-zero $y \in \mathbb{R}^{n+1}$ that is orthogonal to every vector in $q(\text{supp}\,\mu)$.

Define the smooth, monotone, strictly increasing function $\tau : \mathbb{R} \to (0, \infty)$ by

$$\int_0^{\tau(t)} e^{-l} dl = \frac{1}{\sqrt{\pi}} \int_{-\infty}^t e^{-l^2} dl,$$

which satisfies

$$-t^2 = \log\sqrt{\pi} - \tau(t) + \log\tau'(t). \tag{4.7}$$

For $y \in \mathbb{R}^{n+1}$ and $u \in S^{n-1}$, (4.7) gives

$$-|y \cdot \bar{q}(u)|^2 = \log\sqrt{\pi} - \tau(y \cdot \bar{q}(u)) + \log\frac{\tau'(y \cdot \bar{q}(u))}{e_{n+1} \cdot \bar{q}(u)} + \log(e_{n+1} \cdot \bar{q}(u)). \tag{4.8}$$

We now integrate (4.8) over all $u \in S^{n-1}$ with respect to the measure $|q|^2 d\mu$. Since $\bar{q} : S^{n-1} \to S^n$ is an isotropic embedding of the measure space $(S^{n-1}, |q|^2 d\mu)$ into S^n, hence

$$- \int_{S^{n-1}} |y \cdot \bar{q}(u)|^2 |q(u)|^2 d\mu(u) = -|y|^2. \tag{4.9}$$

From (3.5), we have

$$\int_{S^{n-1}} \log \sqrt{\pi} |q(u)|^2 d\mu(u) = (n+1) \log \sqrt{\pi}. \tag{4.10}$$

We now estimate the integral of the last term on the right-hand side of (4.8):

$$I_4 := \int_{S^{n-1}} \log(e_{n+1} \cdot \bar{q}(u)) |q(u)|^2 d\mu(u).$$

By (3.5), the measure $\frac{1}{n+1}|q|^2 d\mu$ is a probability measure and since, on a probability space, the L_0-mean of a function never exceeds its L_2-mean,

$$\exp\left(\frac{1}{n+1} I_4\right) \leqslant \left(\frac{1}{n+1} \int_{S^{n-1}} |e_{n+1} \cdot \bar{q}(u)|^2 |q(u)|^2 d\mu(u)\right)^{\frac{1}{2}}. \tag{4.11}$$

Since $\bar{q} : S^{n-1} \to S^n$ is an isotropic embedding of the measure space $(S^{n-1}, |q|^2 d\mu)$ into S^n, we have

$$\int_{S^{n-1}} |e_{n+1} \cdot \bar{q}(u)|^2 |q(u)|^2 d\mu(u) = \int_{\mathbb{R}^n} h(u)^2 d\mu(u) = 1$$

and hence

$$I_4 \leqslant -\log(n+1)^{\frac{n+1}{2}}. \tag{4.12}$$

From (4.9), (4.10) and (4.12), we see that

$$\begin{aligned} -|y|^2 \leqslant \log\left(\tfrac{\pi}{n+1}\right)^{\frac{n+1}{2}} - e_{n+1} \cdot \int_{S^{n-1}} \bar{q}(u) \frac{\tau(y \cdot \bar{q}(u))}{e_{n+1} \cdot \bar{q}(u)} |q(u)|^2 d\mu(u) \\ + \int_{S^{n-1}} \log \frac{\tau'(y \cdot \bar{q}(u))}{e_{n+1} \cdot \bar{q}(u)} |q(u)|^2 d\mu(u). \end{aligned} \tag{4.13}$$

Define $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ by

$$Ty = \int_{S^{n-1}} \bar{q}(u) \frac{\tau(y \cdot \bar{q}(u))}{e_{n+1} \cdot \bar{q}(u)} |q(u)|^2 d\mu(u), \tag{4.14}$$

for $y \in \mathbb{R}^{n+1}$. Hence,

$$dTy = \int_{S^{n-1}} \bar{q}(u) \otimes \bar{q}(u) \frac{\tau'(y \cdot \bar{q}(u))}{e_{n+1} \cdot \bar{q}(u)} |q(u)|^2 d\mu(u). \tag{4.15}$$

It is clear that for $z \in \mathbb{R}^{n+1}$

$$z \cdot dTyz = \int_{S^{n-1}} |z \cdot \bar{q}(u)|^2 \tau'(y \cdot \bar{q}(u)) \frac{\sqrt{1+h(u)^2}}{h(u)} d\mu(u).$$

Since there exists no nonzero $z \in \mathbb{R}^{n+1}$ such that $z \cdot \bar{q}(u) = 0$ for every $u \in \operatorname{supp} \mu$. This together with the fact that $\tau'(y \cdot \bar{q}(u))\sqrt{1+h(u)^2}/h(u) > 0$ shows that $z \cdot dTyz > 0$ for all $z \neq 0$. Therefore, the mean value theorem shows that $T : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is globally injective.

From Lemma 3.1 and (4.15), we infer that

$$|dTy| \geqslant \exp\left\{ \int_{S^{n-1}} \log \frac{\tau'(y \cdot \bar{q}(u))}{e_{n+1} \cdot \bar{q}(u)} |q(u)|^2 d\mu(u) \right\}. \tag{4.16}$$

Moreover, substituting (4.14) and (4.16) into (4.13), we have

$$e^{-|y|^2} \leqslant \left(\frac{\pi}{n+1} \right)^{\frac{n+1}{2}} e^{-e_{n+1} \cdot Ty} |dTy|. \tag{4.17}$$

Integrating (4.17) over all $y \in \mathbb{R}^{n+1}$ gives

$$(n+1)^{\frac{n+1}{2}} \leqslant \int_{\mathbb{R}^{n+1}} e^{-e_{n+1} \cdot Ty} |dTy| dy = \int_{T(\mathbb{R}^{n+1})} e^{-e_{n+1} \cdot z} dz. \tag{4.18}$$

From the definitions of Ty and q, we have

$$\begin{aligned} Ty &= \int_{S^{n-1}} \bar{q}(u) \frac{\tau(y \cdot \bar{q}(u))}{e_{n+1} \cdot \bar{q}(u)} |q(u)|^2 d\mu(u) \\ &= \int_{S^{n-1}} q(u) \frac{\tau(y \cdot \bar{q}(u))}{e_{n+1} \cdot q(u)} |q(u)|^2 d\mu(u) \\ &= \int_{S^{n-1}} (\rho_{\sqrt{n}K}(u)u, 1) \tau(y \cdot \bar{q}(u)) |q(u)|^2 d\mu(u), \end{aligned}$$

hence Lemma 2.1 shows that

$$Ty \in \bigcup_{r>0} r\sqrt{n}K \times \{r\} =: C \subseteq \mathbb{R}^n \times \mathbb{R}.$$

Take $z = (x, r) \in \mathbb{R}^n \times \mathbb{R}$, we have

$$\begin{aligned} (n+1)^{\frac{n+1}{2}} &\leqslant \int_{T(\mathbb{R}^{n+1})} e^{-e_{n+1} \cdot z} dz \\ &\leqslant \int_C e^{-e_{n+1} \cdot z} dz \\ &= \int_0^\infty \int_{r\sqrt{n}K} e^{-r} dx dr \\ &= \int_0^\infty |r\sqrt{n}K| e^{-r} dr \\ &= n! n^{n/2} V(K). \end{aligned}$$

We establish inequality (4.1) for $\Gamma_2^* K = B$. Inequality (4.1) for origin-symmetric bodies can be obtained by (2.2) and (2.9). □

REMARK 4.1. Inequality (4.1) is strict for each origin-symmetric convex bodies in $\mathbb{R}^n$. In fact, the equality of inequality (4.11) in the proof holds if and only if $e_{n+1} \cdot \bar{q}(u)$ is a constant for $u \in \text{supp}\,\mu$, that is, $h(u)$ is a constant for $u \in \text{supp}\,\mu$. From the definition of $h(u)$ and μ, we have $\rho_K(u)$ is a constant for $u \in \text{supp}\,\mu$, and $\rho_K(u) = 0$ for $u \in S^{n-1}/\text{supp}\,\mu$. Then followed by the continuity of ρ_K, this means that

$$\text{supp}\,\mu = S^{n-1}.$$

Therefore, if the equality in inequality (4.1) holds, K must be a ball. But since inequality (4.1) is strict for the ball, then inequality (4.1) is strict for each origin-symmetric convex bodies.

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REFERENCES

[1] K. BALL, *Volume ratios and a reverse isoperimetric inequality*, J. London Math. Soc. (2) 44 (1991), 351–359.
[2] F. BARTHE, *On a reverse form of the Brascamp-Lieb inequality*, Invent. Math. 134 (1998), 335–361.
[3] F. BARTHE, *A continuous version of the Brascamp-Lieb inequalities*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1850, Springer, Berlin, (2004), 53–63.
[4] J. BOURGAIN, V. D. MILMAN, *New volume ratio properties for convex symmet-ric bodies in R_n*, Invent. Math. 88 (1987), 319–340.
[5] S. CAMPI, P. GRONCHI, *The L_p-Busemann-Petty centroid inequality*, Adv. Math. 167 (2002), 128–141.
[6] S. CAMPI, P. GRONCHI, *On the reverse L_p-Busemann-Petty centroid inequality*, Mathematika 49 (2002), 1–11.
[7] S. CAMPI, P. GRONCHI, *Extremal convex sets for Sylvester-Busemann type functionals*, Appl. Anal. 85 (2006), 129–141.
[8] S. CAMPI, P. GRONCHI, *On volume product inequalities for convex sets*, Proc. Amer. Math. Soc. 134 (2006) 2393–2402.
[9] S. CAMPI, P. GRONCHI, *Volume inequalities for L_p-zonotopes*, Mathematika 53 (2006), 71–80.
[10] R. J. GARDNER, *Geometric tomography*, Cambridge Univ. Press, Cambridge, 1995.
[11] R. J. GARDNER, *The Brunn-Minkowski inequality*, Bull. Amer. Math. Soc. (N. S.) 39 (2002), 355–405.
[12] A. A. GIANNOPOULOS, V. D. MILMAN, *Extremal problems and isotropic positions of convex bodies*, Israel J. Math. 117 (2000), 29–60.
[13] A. A. GIANNOPOULOS, V. D. MILMAN, M. RUDELSON, *Convex bodies with minimal mean width*, Geometric aspects of functional analysis, Lecture Notes in Math., vol. 1745, Springer, Berlin, (2000), 81–93.
[14] A. A. GIANNOPOULOS, M. PAPADIMITRAKIS, *Isotropic surface area measures*, Mathematika 46 (1999), 1–13.
[15] Y. GORDON, M. MEYER, S. REISNER, *Zonoids with minimal volume-product – a new proof*, Proc. Amer. Math. Soc. 104 (1988), 273–276.
[16] E. LUTWAK, D. YANG, G. ZHANG, *L_p affne isoperimetric inequalities*, J. Differential Geom. 56 (2000), 111–132.

[17] E. LUTWAK, D. YANG, G. ZHANG, *A new ellipsoid associated with convex bodies*, Duke Math. J. 104 (2000), 375–390.

[18] E. LUTWAK, D. YANG, G. ZHANG, *Volume inequalities for subspaces of L_p*, J. Differential Geom. 68 (2004), 159–184.

[19] E. LUTWAK, D. YANG, G. ZHANG, *Volume inequalities for isotropic measures*, Amer. J. Math. 129 (2007), 1711–1723.

[20] E. LUTWAK, D. YANG, G. ZHANG, *A volume inequality for polar bodies*, J. Differential Geom., 84 (2010), 163–178.

[21] E. LUTWAK, G. ZHANG, *Blaschke-Santaló inequalities*, J. Differential Geom., 45 (1997), 1–16.

[22] M. MEYER, *Convex bodies with minimal volume product in $\mathbb{R}^2$*, Monatsh. Math. 112 (1991), 297–301.

[23] M. MEYER, S. REISNER, *Shadow systems and volumes of polar convex bodies*, Mathematika 53 (2006), 129–148.

[24] M. MEYER, E. WERNER, *On the p-affine surface area*, Adv. Math. 152 (2000), 288–313.

[25] V. D. MILMAN, A. PAJOR, *Isotropic position and inertia ellipsoids and zonoids of the unit ball of a normed H-dimensional space*, Geometric Aspects of Functional Analysis (J. Lindenstrauss and V. D. Milman, eds.), vol. 1376, Springer Lecture Notes in Math., 1989, 64–104.

[26] S. REISNER, *Random polytopes and the volume-product of symmetric convex bodies*, Math. Scand. 57 (1985), 386–392.

[27] S. REISNER, *Zonoids with minimal volume-product*, Math. Z. 192 (1986), 339–346.

[28] R. SCHNEIDER, *Convex bodies: the Brunn-Minkowski theory*, Cambridge University Press, 2014.

[29] B. ZHU, H. HONG, D. YE, *The Orlicz-Petty bodies*, Interna. Math. Res. Notices, 14 (2018): 4356–4403.

[30] B. ZHU, J. ZHOU, W. XU, *Dual Orlicz-Brunn-Minkowski theory*, Adv. Math., 264 (2014): 700–725.

[31] D. ZOU, G. XIONG, *Orlicz-John ellipsoids*, Adv. Math., 265 (2014): 132–168.

[32] D. ZOU, G. XIONG, *Orlicz-Legendre ellipsoids*, J. Geom. Anal., 26 (2016): 2474–2502.

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