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ON THE DERIVATIVE OF A RATIONAL POLYNOMIAL WITH PRESCRIBED POLES

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Abstract. Let $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n and $W(z)=\prod_{\nu=1}^{n}(z-a_i)$, where $|a_i|>1$, $j=1,2,\dots n$. If $r(z)=p(z)/W(z)$ be a rational function does not vanish in $|z|>1$. The aim of this paper is to obtained some generalization of an inequality due to Xin Li, R. N. Mohapatra and R.S. Rodriguez [Inequality (12), J. London Math. Soc. 51 (20), 1995, pp. 523–531] for the polynomial $r(z)$ having all its zeros in $|z|\leqslant k$ and other related results.

1. Introduction

Let $\mathbb{P}_n$ denote the class all complex of polynomial $p(z)=\sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n. For all $a_\nu, \nu=1,2,\dots,n$ with $|a_\nu|>1$, we write

$$W(z)=\prod_{\nu=1}^{n}(z-a_\nu), \qquad B(z)=\prod_{\nu=1}^{n}\left(\frac{1-\overline{a_\nu}z}{z-a_\nu}\right)$$

and

$$\mathbb{R}_n=\mathbb{R}_n(a_1,a_2,\dots,a_n)=\frac{p(z)}{W(z)}; \qquad p\in\mathbb{P}_n$$

Here $\mathbb{R}_n$ is the set of all rational function with pole $a_1,a_2,\dots,a_n$ at most and with finite limit at infinity. So, it is clear that $B(z)\in\mathbb{R}_n$. Assume that $T_k:=\{z;\ |z|=k, k>0\}$, D_{k+} denotes the region outside T_k and D_{k-} denotes region inside T_k.

For $p\in\mathbb{P}_n$;

THEOREM A. *If $p\in\mathbb{P}_n$, then*

$$\max_{z\in T_1}|p'(z)|\leqslant n\max_{z\in T_1}|p(z)|. \tag{1.1}$$

The result is best possible and equality holds for $p(z)=\lambda z^n$.

The above result is known as Bernstein inequality due to S. N. Bernstein [4] on the derivative of polynomial. By taking a restriction on the zeros of $p(z)$ the above result is sharpened. It was first conjectured by Prof. P. Erdös and later verify by P. D. Lax [4] by proving that

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THEOREM B. *If $p \in \mathbb{P}_n$ and having all it zeros in $T_1 \cup D_{1+}$, then*

$$\max_{z \in T_1} |p'(z)| \leqslant \frac{n}{2} \max_{z \in T_1} |p(z)|. \tag{1.2}$$

The result is sharp and equality holds for $p(z) = \lambda z^n$.

P. Turán [4] consider the class of polynomial $p(z)$ having all its zeros in $T_1 \cup D_{1-}$ and prove the following.

THEOREM C. *If $p \in \mathbb{P}_n$ and having all it zeros in $T_1 \cup D_{1+}$, then*

$$\max_{z \in T_1} |p'(z)| \geqslant \frac{n}{2} \max_{z \in T_1} |p(z)|. \tag{1.3}$$

The result is sharp and equality holds for $p(z) = \lambda z^n$.

Li, Mohapatra and Rodriguez [3] have proved the following results similar to Theorem A, Theorem B and Theorem C respectively for rational functions $r \in \mathbb{R}_n$ with prescribed poles where they replaced z^n by Blaschke product $B(z)$.

THEOREM D. *If $r \in \mathbb{R}_n$, then*

$$\sup_{z \in T_1} |r'(z)| \leqslant |B'(z)| \sup_{z \in T_1} |r(z)|. \tag{1.4}$$

Equality in above holds for $r(z) = \alpha B(z), \alpha \in T_1$.

THEOREM E. *Suppose $r \in \mathbb{R}_n$ and all the zeros of $r(a)$ lies in $T_1 \cup D_{1+}$, then*

$$\sup_{z \in T_1} |r'(z)| \leqslant \frac{|B'(z)|}{2} \sup_{z \in T_1} |r(z)|. \tag{1.5}$$

Equality in above holds for $r(z) = \alpha B(z) + \beta$ with $\alpha, \beta \in T_1$.

THEOREM F. *Suppose $r \in \mathbb{R}_n$ and all its zeros lies in $T_1 \cup D_{1-}$, then*

$$\sup_{z \in T_1} |r'(z)| \geqslant \frac{|B'(z)|}{2} \sup_{z \in T_1} |r(z)|. \tag{1.6}$$

Equality in above holds for $r(z) = \alpha B(z) + \beta$ with $\alpha, \beta \in T_1$.

Aziz and Zargar [1] proved the following generalization of Theorem E by taking all the zeros of $r(z)$ in $T_k \cup D_{k+}$, $k \geqslant 1$.

THEOREM G. *Suppose $r \in \mathbb{R}_n$ and having all its zeros in $T_k \cup D_{k+}$, $k \geqslant 1$, then*

$$\sup_{z \in T_1} |r'(z)| \leqslant \frac{1}{2} \left\{ |B'(z)| - \frac{n(k-1)}{k+1} . \frac{|r(z)|^2}{||r||^2} \right\} ||r||, \tag{1.7}$$

where $||r|| = \sup_{z \in T_1} |r(z)|$. Equality in above holds for $p(z) = (z+k)^n/(z-a)^n$, $a > 1$, $k \geqslant 1$.

Recently, Xin Li [2] proved another generalization of Theorem D by sating that

THEOREM H. *Let* $r,s \in \mathbb{R}_n$ *and assume* $s(z)$ *has all its zeros in* $T_1 \cup D_{1-}$ *and* $|r(z)| \leqslant |s(z)|$ *for* $z \in T_1$, *then for any* ρ *with* $|\rho| \leqslant 1/2$

$$\left|r'(z) + \rho B'(z) r(z)\right| \leqslant \left|s'(z) + \rho B'(z) s(z)\right| \tag{1.8}$$

Equality in above holds for $r(z) = s(z)$.

Some other extension and generalizations of above results was also proved by Tripathi, Hans, Mogbademu and Tyagi [5]. Here, we first prove the following generalization of Theorem F for $r \in \mathbb{R}_n$ and having all its zeros in $T_k \cup D_{k-}$, $k \leqslant 1$. More precisely, we prove

THEOREM 1. *Suppose* $r \in \mathbb{R}_n$ *and having all its zeros in* $T_k \cup D_{k-}$, $k \leqslant 1$, *then*

$$\sup_{z \in T_1} |r'(z)| \geqslant \frac{1}{2}\left\{|B'(z)| - \frac{n(1+k) - 2m}{1+k}\right\} \sup_{z \in T_1} |r(z)|, \tag{1.9}$$

where m *is number of zeros and* n *is number of poles of* $r(z)$. *Equality in above holds for* $r(z) = \alpha + \beta B(z)$, *where* $\alpha, \beta \in T_1$ *and* $k = 1, m = n$.

REMARK 1.1. The right hand side of inequality (1.9) may be negative for small m.

If $r \in \mathbb{R}_n$ has exactly n zeros, i.e. $m = n$, then we have following result.

COROLLARY 1.1. *Suppose* $r \in \mathbb{R}_n$ *and having all its zeros in* $T_k \cup D_{k-}$, $k \leqslant 1$, *then*

$$\sup_{z \in T_1} |r'(z)| \geqslant \frac{1}{2}\left\{|B'(z)| + \frac{n(1-k)}{1+k}\right\} \sup_{z \in T_1} |r(z)|, \tag{1.10}$$

Equality in above holds for $r(z) = \alpha + \beta B(z)$, *where* $\alpha, \beta \in T_1$ *and* $k = 1$.

REMARK 1.2. On taking $k = 1$ in inequality (1.10), Theorem F has been obtained.

Next, we prove the following generalization of Theorem H.

THEOREM 2. *Let* $r,s \in \mathbb{R}_n$ *and assume* $s(z)$ *has all its zeros in* $T_k \cup D_{k-}$, $k \leqslant 1$ *and* $|r(z)| \leqslant |s(z)|$ *for* $z \in T_1$, *then for any* β *with* $|\beta| \leqslant 1/2$

$$\left|zr'(z) + \beta\left\{|B'(z)| + \frac{n(1-k)}{1+k}\right\} r(z)\right| \leqslant \left|zs'(z) + \beta\left\{|B'(z)| + \frac{n(1-k)}{1+k}\right\} s(z)\right|. \tag{1.11}$$

Equality in above holds for $r(z) = s(z)$.

REMARK 2.1. Theorem H can be followed by taking $k = 1$ in inequlity (1.11).

If we take $s(z) = B(z) \sup_{z \in T_k} |r(z)|$ and using (3.5), following auxiliary result has been obtained.

COROLLARY 2.1. *If $r \in \mathbb{R}_n$, then for β with $|\beta| \leqslant 1/2$ and $z \in T_1$*

$$\left| r'(z) + \beta \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} r(z) \right| \leqslant \left\{ |1+\beta||B'(z)| + |\beta| \frac{n(1-k)}{1+k} \right\} \sup_{z \in T_k} |r(z)|, \tag{1.12}$$

for $k \leqslant 1$.

On taking $\beta = -\frac{1}{2}$, we obtained the following result.

COROLLARY 2.2. *If $r \in \mathbb{R}_n$, then for $z \in T_1$*

$$\left| r'(z) - \frac{1}{2} \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} r(z) \right| \leqslant \frac{1}{2} \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} \sup_{z \in T_k} |r(z)|, \tag{1.13}$$

for $k \leqslant 1$.

Finally, we prove following result concerning minimum modulus of polynomial.

THEOREM 3. *If $r \in R_n$ has n zeros all in D_{k-}, then for every β with $|\beta| \leqslant 1/2$*

$$\left| zr'(z) + \beta \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} r(z) \right| \geqslant \left| 1 + \beta \left\{ 1 + \frac{n(1-k)}{(1+k)|B'(z)|} \right\} \right| |B'(z)| m, \tag{1.14}$$

where $m = \inf_{z \in T_1} |r(z)|$. The equality in (1.14) holds for $r(z) = \lambda B(z)$, $\lambda > 0$ and $k = 1$.

By taking $\beta = 0$ and $k = 1$ in inequality (1.14), we get the following result.

COROLLARY 3.1. *If $r \in R_n$ has n zeros all in D_{1-}, then*

$$\inf_{z \in T_1} |r'(z)| \geqslant |B'(z)| \inf_{z \in T_1} |r(z)|. \tag{1.15}$$

The equality in (1.15) holds for $r(z) = \lambda B(z)$, $\lambda > 0$.

From inequality (1.10) and with suitable choice of β, we get for $z \in T_1$

$$\left| zr'(z) + \beta \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} r(z) \right| = |zr'(z)| - |\beta| \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} |r(z)|$$

Now combining inequality (1.14) with above inequality, we have for $z \in T_1$

$$\begin{aligned} |zr'(z)| - |\beta| \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} |r(z)| &= \left| zr'(z) + \beta \left\{ |B'(z)| + \frac{n(1-k)}{1+k} \right\} r(z) \right| \\ &\geqslant \left| 1 + \beta \left\{ 1 + \frac{n(1-k)}{(1+k)|B'(z)|} \right\} \right| |B'(z)| m \\ &\geqslant \left\{ 1 - |\beta| \left\{ 1 + \frac{n(1-k)}{(1+k)|B'(z)|} \right\} \right\} |B'(z)| m \end{aligned}$$

and by letting $\beta \to 1/2$, we obtained the following.

COROLLARY 3.2. *If $r \in R_n$ has n zeros all in D_{k-}, then*

$$\sup_{z\in T_1}|r'(z)| \geqslant \left\{|B'(z)| + \frac{n(1-k)}{1+k}\right\}\sup_{z\in T_1}|r(z)| + \left\{|B'(z)| - \frac{n(1-k)}{(1+k)}\right\}\inf_{z\in T_1}|r(z)|. \tag{1.16}$$

The equality in above holds for $r(z) = \lambda B(z)$ and $k = 1$.

The following result has been obtained by taking $k = 1$ in Corollary 3.2.

COROLLARY 3.2. *If $r \in R_n$ has n zeros all in D_{1-}, then*

$$\sup_{z\in T_1}|r'(z)| \geqslant \frac{|B'(z)|}{2}\left\{\sup_{z\in T_1}|r(z)| + \inf_{z\in T_1}|r(z)|\right\}. \tag{1.17}$$

The equality in above holds for $r(z) = \lambda B(z)$.

2. Lemma

For the proof our results, we require following Lemma due to Xin Li [2].

LEMMA 1. *Let x and y be two complex number.*

1. *If $|x| \geqslant |y|$ and $y \neq 0$, then $x \neq \delta y$ for all complex number δ satisfying $|\delta| < 1$.*
2. *Conversely, if $x \neq \delta y$ for all complex number δ with $|\delta| < 1$, then $|x| \geqslant |y|$.*

3. Proof of Results

Proof of Theorem 1. Since $r(z) = p(z)/W(z) \in \mathbb{R}_n$. Let b_j, $j = 1,2,\ldots,m$ are the zeros of $p(z)$, then $|b_j| \leqslant k \leqslant 1$. Therefore, we have

$$\frac{zr'(z)}{r(z)} = \frac{zp'(z)}{p(z)} - \frac{zW'(z)}{W(z)}. \tag{3.1}$$

Since $p(z) = \prod_{j=1}^{m}(z-b_j)$, therefore

$$\frac{zp'(z)}{p(z)} = \sum_{j=1}^{m}\frac{z}{z-b_j}.$$

So, equation (3.1) become

$$\frac{zr'(z)}{r(z)} = \sum_{j=1}^{m}\frac{z}{z-b_j} - \frac{zW'(z)}{W(z)}.$$

and for $z \in T_1$

$$Re\left(\frac{zr'(z)}{r(z)}\right) = Re\sum_{j=1}^{m}\frac{z}{z-b_j} - Re\left(\frac{zW'(z)}{W(z)}\right). \tag{3.2}$$

Now it is easy to verify that for $z \in T_1$, $|b_j| \leqslant k \leqslant 1$

$$Re\left(\frac{z}{z-b_j}\right) \geqslant \frac{1}{1+k} \tag{3.3}$$

and $W(z) = \prod_{j=1}^{n}(z-a_j)$. So, $W^*(z) = z^n\overline{W(1/\overline{z})}$ and for $z \in T_1, \overline{z} = 1/z$, we get

$$Re\left(\frac{zW^{*'}(z)}{W*(z)}\right) + Re\left(\frac{zW'(z)}{W(z)}\right) = n \tag{3.4}$$

Also we get from

$$B(z) = \prod_{\nu=1}^{n}\left(\frac{1-\overline{a_\nu}z}{z-a_\nu}\right),$$

which gives

$$\frac{zB'(z)}{B(z)} = \frac{zW^{*'}(z)}{W^*(z)} - \frac{zW'(z)}{W(z)}$$

i.e. for $z \in T_1$,

$$|B'(z)| = Re\left(\frac{zW^{*'}(z)}{W^*(z)}\right) - Re\left(\frac{zW'(z)}{W(z)}\right). \tag{3.5}$$

From equation (3.4) and (3.5), we have

$$Re\left(\frac{zW'(z)}{W(z)}\right) = \frac{n-|B'(z)|}{2} \tag{3.6}$$

On using (3.3) and (3.6) in (3.2), we get

$$Re\left(\frac{zr'(z)}{r(z)}\right) \geqslant \frac{m}{1+k} - \frac{n-|B'(z)|}{2} \tag{3.7}$$

Hence for $z \in T_1$ and $r \neq 0$, we have

$$\left|\frac{zr'(z)}{r(z)}\right| \geqslant Re\left(\frac{zr'(z)}{r(z)}\right) \geqslant \frac{m}{1+k} - \frac{n-|B'(z)|}{2}. \tag{3.8}$$

Furthermore, if $r(z) = 0$, then equality is trivially satisfied.Thus the result holds for all $z \in T_1$. Proof of Theorem 1 is completed. □

Proof of Theorem 2. Consider no zeros of $s(z)$ lies on T_k, therefore all the zeros of $s(z)$ lies in D_{k-} and $|r(z)| \leqslant |s(z)|$ for $z \in T_1$. Let α be an arbitrary number with $|\alpha| < 1$, therefore by direct application of Rouche's Theorem, $\alpha r(z) + s(z)$ has as many zeros in D_{k-} as $s(z)$. Thus all the zeros of $\alpha r(z)+s(z)$ lies in D_{k-}. By inequality (3.8), we get

$$|\alpha z r'(z)+zs'(z)| \geqslant A_k|\alpha r(z)+s(z)|, \tag{3.9}$$

where

$$A_k=\frac{1}{2}\left\{|B'(z)|+\frac{n(1-k)}{1+k}\right\}. \tag{3.10}$$

It can be easily verify that $B'(z)\neq 0$, then R.H.S of inequality (3.9) is nonzero. Now, using 1 of Lemma 1, we have for any δ with $|\delta|<1$

$$\alpha z r'(z)+zs'(z)\neq \delta A_k(\alpha r(z)+s(z)) \quad \text{for} \quad z\in T_1,$$

or, equivalently for $|\alpha|<1$

$$\alpha\left\{zr'(z)-\delta A_k r(z)\right\}\neq -\left\{zs'(z)-\delta A_k s(z)\right\}.$$

On using 2 of Lemma 1, we have

$$\left|zs'(z)-\delta A_k s(z)\right| \geqslant \left|zr'(z)-\delta A_k r(z)\right|. \tag{3.11}$$

On taking $\beta=-\delta/2$, we get $|\beta|<1/2$. Therefor using (3.10) in inequality (3.11), we have

$$\left|zs'(z)+\beta\left\{|B'(z)|+\frac{n(1-k)}{1+k}\right\}s(z)\right| \geqslant \left|zr'(z)+\beta\left\{|B'(z)|+\frac{n(1-k)}{1+k}\right\}r(z)\right|. \tag{3.12}$$

Using the continuity in zeros and β with $|\beta|<1/2$, we can obtain the inequality when some zeros of $s(z)$ lie on the unit circle. Which proof the Theorem 2. □

Proof of Theorem 3. If $r(z)$ has a zero in T_k, then Theorem e is trivial. So, we consider that $r(z)$ has all its zeros in D_{k-}. If $m=\inf_{z\in T_1}|r(z)|$, then $m>0$ and $|r(z)|\geqslant m$ for $z\in T_1$. Therefore, for any δ with $|\delta|<1$, then the function $F(z)=r(z)-\delta m B(z)$ of degree n, has all its zeros in D_{k-}. From inequality (3.8)

$$|zF'(z)|\geqslant A_k|F(z)| \tag{3.13}$$

where A_k is defined in (3.10),i.e.

$$|zr'(z)-\delta m z B'(z)|\geqslant A_k|r(z)-\delta m B(z)| \tag{3.14}$$

Since $F(z)=r(z)-\delta m B(z)\neq 0$ in $T_k\cup D_{k+}$, then for any complex number ρ with $|\rho|<1$, we have from (i) of Lemma 1

$$T(z)=z\{r'(z)-\delta m B'(z)\}+\beta A_k\{r(z)-\delta m B(z)\}\neq 0 \tag{3.15}$$

i.e.

$$T(z)=\left\{zr'(z)+\rho A_k r(z)\right\}-\delta\left\{zB'(z)+\rho A_k B(z)\right\}m\neq 0$$

in $T_1 \cup D_{1+}$.Now from (ii) of Lemma 1, we obtained for $|\delta| < 1$

$$\left|zr'(z) + \rho A_k r(z)\right| \geqslant \left|zB'(z) + \rho A_k B(z)\right| m, \tag{3.16}$$

i.e.

$$\left|zr'(z) + \rho A_k r(z)\right| \geqslant \left|zB'(z) + \rho A_k B(z)\right| m, \tag{3.17}$$

Now for $z \in T_1$,

$$|B'(z)| = \frac{zB'(z)}{B(z)} = \sum_{j=1}^{n} \frac{(|a_j|^2 - 1)}{|z - a_j|^2},$$

i.e. $|B'(z)|B(z) = zB'(z)$ for $z \in T_1$ and $\frac{\rho}{2} = \beta$ with $|\beta| \leqslant 1/2$, therefore inequality (3.17) becomes

$$\left|zr'(z) + \beta\left\{|B'(z)| + \frac{n(1-k)}{1+k}\right\} r(z)\right| \geqslant \left|1 + \beta\left\{1 + \frac{n(1-k)}{(1+k)|B'(z)|}\right\}\right| |B'(z)| m \tag{3.18}$$

When a zero of $r(z)$ lie on T_1 and $|\beta| \leqslant 1/2$, the case can follow by using argument of continuity. Which complete Theorem 3. □

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