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SOME RESULTS ON BLOCK KRONECKER AND BLOCK HADAMARD PRODUCT OF MATRICES

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Abstract. In this paper, we first construct a relationship between a block Kronecker product and a block Hadamard product of matrices. Then we give some power rules for the block Kronecker product of matrices under commutation assumptions. We extend some inequalities for powers of the block Hadamard product of a finite number of positive definite Hermitian matrices.

1. Introduction

In 1991, Horn, Mathias and Nakamura [3] defined a block Hadamard product and gave some useful results for singular values and norms of the block Hadamard product of two matrices. In 2012, Günther and Klotz [1] generalized Kronecker product for block matrices and extended their studies to the block Hadamard product of positive semidefinite matrices. Additionally, Mond and Pecaric [6] investigated several inequalities involving eigenvalues and powers of the Hadamard product of positive definite Hermitian matrices.

Firstly, we will construct a relationship between the block Kronecker and the block Hadamard product of matrices. Further, we will prove the property associated with the block Kronecker product of the powers of matrices. Finally, we shall give some inequalities related to the powers of the block Hadamard product of a finite number of positive definite Hermitian matrices.

Now, let us give the notation and terminology we will use throughout our study. Let $M_{m\times n}$ be the linear space of $m\times n$ matrices with complex entries and $\mathbb{M}_{p,q}(M_{m,n})$ be the space of $p\times q$ block matrices, and write $\mathbb{M}_{p,q} := \mathbb{M}_{p,q}(M_{n,n})$ and $\mathbb{M}_p := \mathbb{M}_{p,p}(M_{n,n})$. The identity matrix in $\mathbb{M}_p$ is denoted by $\mathbb{I}_p = diag(I_n,\dots,I_n)$ where $I_n \in M_{n,n}$.

A matrix $\mathbb{A}\in\mathbb{M}_p$ is Hermitian if $\mathbb{A}^* = \mathbb{A}$ where $\mathbb{A}^*$ is the conjugate transpose of $\mathbb{A}$. A Hermitian matrix $\mathbb{A}$ is said to be positive definite if $x^*\mathbb{A}x > 0$ for all nonzero $x\in\mathbb{C}$.

Let $A\in M_{m,l}$ and $\mathbb{B} = (B_{ij})\in\mathbb{M}_{s,t}(M_{l,n})$. Then the block Kronecker product of A and $\mathbb{B}$ is defined by $A\boxtimes\mathbb{B} = (AB_{ij})_{i=1,\dots,s}^{j=1,\dots,t}$, where AB_{ij} is the usual matrix product

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of A and B_{ij}. For $\mathbb{A}=(A_{ij})\in\mathbb{M}_{p,q}(M_{m,l})$, the block Kronecker product is given by $\mathbb{A}\boxtimes\mathbb{B}=(A_{ij}\boxtimes\mathbb{B})_{i=1,\dots,p}^{j=1,\dots,q}$. If $\mathbb{A}=(A_{ij})\in\mathbb{M}_{p,q}(M_{m,l})$ and $\mathbb{B}=(B_{ij})\in\mathbb{M}_{p,q}(M_{l,n})$ the block Hadamard product of $\mathbb{A}$ and $\mathbb{B}$ is defined by $\mathbb{A}\Box\mathbb{B}=(A_{ij}B_{ij})$. Two matrices $\mathbb{A}\in\mathbb{M}_{p,q}$ and $\mathbb{B}\in\mathbb{M}_{s,t}$ are called block commuting if every $n\times n$ block of $\mathbb{A}$ commutes with every $n\times n$ block of $\mathbb{B}$. It is denoted by $\mathbb{A}_{bc}\mathbb{B}$. If $\mathbb{A}$ and $\mathbb{B}$ are both positive definite and $\mathbb{A}_{bc}\mathbb{B}$ then $\mathbb{A}\boxtimes\mathbb{B}$ and $\mathbb{A}\Box\mathbb{B}$ are positive definite matrices[1].

These block products are related to each other for $\mathbb{A},\mathbb{B}\in\mathbb{M}_p(M_{n,n})$ by a $p^2n\times pn$ selection matrix J such that

$$\mathbb{A}\Box\mathbb{B}=J^T(\mathbb{A}\boxtimes\mathbb{B})J \quad \text{and } J^TJ=I,$$

where $J^T=[E_{11}E_{22}\cdots E_{pp}]$ for which E_{ii} is the $p\times p$ block matrix of zero matrices except an identity matrix I_n in the ii th position.

2. Main results

Firstly, we shall give some properties of block commuting matrices which are useful to establish our results. Let $\mathbb{A}\in\mathbb{M}_{p,q}$, $\mathbb{B}\in\mathbb{M}_{s,t}$, $\mathbb{C}\in\mathbb{M}_{q,u}$, and $\mathbb{D}\in\mathbb{M}_{t,v}$ for $p,q,s,t,u,v\in\mathbb{N}$.

LEMMA 1. *(a) If $\mathbb{A}_{bc}\mathbb{B}$ and $\mathbb{C}_{bc}\mathbb{B}$ then $\mathbb{A}\mathbb{C}_{bc}\mathbb{B}$.*
(b) Let $\widetilde{\mathbb{D}}$ be a block diagonal matrix such that $\widetilde{\mathbb{D}}=diag(D_1,\cdots,D_r)$ with $D_i\in M_{n,n}$. If $\mathbb{A}_{bc}\widetilde{\mathbb{D}}$ then $\mathbb{A}_{bc}\widetilde{\mathbb{D}}^s$ for a real number s.
(c) If $\mathbb{A}$ and $\mathbb{B}$ are Hermitian matrices and $\mathbb{A}_{bc}\mathbb{B}$ then $\mathbb{A}_{bc}\mathbb{B}^*$.

Proof. (a) For all $i,j,l,m=1,\dots,k$, we have

$$A_{ij}C_{ij}B_{lm}=A_{ij}B_{lm}C_{ij}=B_{lm}A_{ij}C_{ij}.$$

(b) It follows from the fact that is given in [2] pg.30 a matrix A_{ij} commutes with a diagonal matrix D_k if and only if every entry a_{ij} of A_{ij} is zero whenever $d_{ii}\neq d_{ij}$.
(c) Clear. □

LEMMA 2. [1] *(a) $(\mathbb{A}\boxtimes\mathbb{B})^*=\mathbb{A}^*\boxtimes\mathbb{B}^*$ if and only if $\mathbb{A}_{bc}\mathbb{B}$.*
(b) If $\mathbb{B}_{bc}\mathbb{C}$ then $(\mathbb{A}\boxtimes\mathbb{B})(\mathbb{C}\boxtimes\mathbb{D})=\mathbb{A}\mathbb{C}\boxtimes\mathbb{B}\mathbb{D}$.
(c) $\mathbb{A}\boxtimes\mathbb{B}=(\mathbb{A}\boxtimes\mathbb{I}_s)(\mathbb{I}_q\boxtimes\mathbb{B})$.
(d) $\mathbb{A}\boxtimes\mathbb{B}=(\mathbb{I}_p\boxtimes\mathbb{B})(\mathbb{A}\boxtimes\mathbb{I}_t)$ if and only if $\mathbb{A}_{bc}\mathbb{B}$.

For a finite number of matrices $\mathbb{A}_i$, $\mathbb{B}_i\in\mathbb{M}_p$, $i=1,\dots,k$, we write the following results.

LEMMA 3. *Let $\mathbb{A}_i$ and $\mathbb{B}_i$ be matrices such that $\mathbb{B}_i$ block commutes with $\mathbb{A}_{i+j}$ for all $i=1,\dots,k-1$ and $j=1,\dots,k-i$. Then*

$$\left(\prod_{i=1}^{k}A_i\right)\boxtimes\left(\prod_{i=1}^{k}B_i\right)=\prod_{i=1}^{k}(A_i\boxtimes B_i). \tag{1}$$

Proof. Let $k=3$. For $i=1$, $\mathbb{B}_1$ commutes with $\mathbb{A}_2$ and $\mathbb{A}_3$. For $i=2$, $\mathbb{B}_2$ commutes with $\mathbb{A}_3$. Applying Lemma 1(a) and Lemma 2(b), we get

$$\begin{aligned}(\mathbb{A}_1\boxtimes\mathbb{B}_1)(\mathbb{A}_2\boxtimes\mathbb{B}_2)(\mathbb{A}_3\boxtimes\mathbb{B}_3) &= (\mathbb{A}_1\mathbb{A}_2\boxtimes\mathbb{B}_1\mathbb{B}_2)(\mathbb{A}_3\boxtimes\mathbb{B}_3)\\ &= (\mathbb{A}_1\mathbb{A}_2)\mathbb{A}_3\boxtimes(\mathbb{B}_1\mathbb{B}_2)\mathbb{B}_3.\end{aligned}$$

The proof is completed by induction. □

Let $\mathbb{A}\in\mathbb{M}_p$ be a positive definite $n\times n$ Hermitian matrix. There exits a matrix $\mathbb{U}$ such that

$$\mathbb{A}=\mathbb{U}^*[\lambda_1,\lambda_2,\ldots,\lambda_{pn}]\mathbb{U},\quad \mathbb{U}^*\mathbb{U}=\mathbb{I},$$

where $[\lambda_1,\lambda_2,\ldots,\lambda_n]$ is the diagonal matrix with λ_i eigenvalues of A [6]. Then for any real number s, A^s is defined by

$$\mathbb{A}^s=\mathbb{U}^*[\lambda_1^s,\lambda_2^s,\ldots,\lambda_n^s]\mathbb{U}.$$

LEMMA 4. *Let $\mathbb{A},\mathbb{B}\in\mathbb{M}_p$ be positive definite matrices such that $\mathbb{A}=\mathbb{U}^*\mathbb{D}\mathbb{U}$ and $\mathbb{B}=\mathbb{V}^*\mathbb{K}\mathbb{V}$ where $\mathbb{D},\mathbb{K}$ are diagonal matrices and s a nonzero real number. Suppose that $\mathbb{A}$ block commutes with $\mathbb{B}$, then*

$$\mathbb{A}^s\boxtimes\mathbb{B}^s=(\mathbb{A}\boxtimes\mathbb{B})^s. \tag{2}$$

Proof. First note that if $\mathbb{A}_{bc}\mathbb{B}$ then $\mathbb{U}_{bc}\mathbb{V}$, $\mathbb{U}_{bc}\mathbb{K}$, $\mathbb{U}_{bc}\mathbb{D}$, $\mathbb{V}_{bc}\mathbb{K}$, $\mathbb{V}_{bc}\mathbb{D}$ and $\mathbb{D}_{bc}\mathbb{K}$. By Lemma 1, we get

$$\begin{aligned}\mathbb{A}^s\boxtimes\mathbb{B}^s &= (\mathbb{U}^*\mathbb{D}^s\mathbb{U})\boxtimes(\mathbb{V}^*\mathbb{K}^s\mathbb{V})\\ &= ((\mathbb{U}^*\mathbb{D}^s)\mathbb{U})\boxtimes(\mathbb{V}^*(\mathbb{K}^s\mathbb{V}))\\ &= (\mathbb{U}^*\mathbb{D}^s\boxtimes\mathbb{V}^*)(\mathbb{U}\boxtimes\mathbb{K}^s\mathbb{V})\\ &= (\mathbb{U}^*\mathbb{D}^s\boxtimes\mathbb{V}^*\mathbb{I})(\mathbb{I}\mathbb{U}\boxtimes\mathbb{K}^s\mathbb{V})\\ &= (\mathbb{U}^*\boxtimes\mathbb{V}^*)(\mathbb{D}^s\boxtimes\mathbb{I})(\mathbb{I}\boxtimes\mathbb{K}^s)(\mathbb{U}\boxtimes\mathbb{V})\\ &= (\mathbb{U}\boxtimes\mathbb{V})^*(\mathbb{D}^s\boxtimes\mathbb{K}^s)(\mathbb{U}\boxtimes\mathbb{V})\\ &= (\mathbb{U}\boxtimes\mathbb{V})^*(\mathbb{D}\boxtimes\mathbb{K})^s(\mathbb{U}\boxtimes\mathbb{V})\\ &= (\mathbb{A}\boxtimes\mathbb{B})^s.\end{aligned}$$

Since $\mathbb{V}_{bc}^*\mathbb{U}$, we have

$$\begin{aligned}(\mathbb{U}\boxtimes\mathbb{V})^*(\mathbb{U}\boxtimes\mathbb{V}) &= (\mathbb{U}^*\boxtimes\mathbb{V}^*)(\mathbb{U}\boxtimes\mathbb{V})\\ &= \mathbb{U}^*\mathbb{U}\boxtimes\mathbb{V}^*\mathbb{V}\\ &= \mathbb{I}_{pn}\boxtimes\mathbb{I}_{pn}=\mathbb{I}_{p^2n}.\end{aligned}$$ □

The block Hadamard and the block Kronecker product of matrices $\mathbb{A}_i$, $i=1,\ldots,k$ will be denoted by $\square_{i=1}^k A_i$ and $\boxtimes_{i=1}^k A_i$, respectively.

For a finite number of positive definite Hermitian matrices $\mathbb{A}_i\in\mathbb{M}_p$, we can extend the equation (2) as follows.

THEOREM 1. *Let* $\mathbb{A}_i = \mathbb{U}_i^* \mathbb{D}_i \mathbb{U}_i$ *be a positive definite Hermitian matrix with a diagonal matrix* $\mathbb{D}_i$ *and a matrix* $\mathbb{U}_i$ *such that* $\mathbb{U}_i^* \mathbb{U}_i = \mathbb{I}$ *for* $i = 1, \dots, k$. *Assume that* $\mathbb{A}_i$ *block commutes with* $\mathbb{A}_j$ *for all* $i \neq j$. *Then we have*

$$\boxtimes_{i=1}^k \mathbb{A}_i^s = \left(\boxtimes_{i=1}^k \mathbb{A}_i \right)^s.$$

Proof. First we prove this for $k = 3$. By considering assumptions and (2), we obtain

$$\begin{aligned}(\mathbb{A}_1^s \boxtimes \mathbb{A}_2^s) \boxtimes \mathbb{A}_3^s &= (\mathbb{A}_1 \boxtimes \mathbb{A}_2)^s \boxtimes \mathbb{A}_3^s \\ &= (\mathbb{A}_1 \boxtimes \mathbb{A}_2 \boxtimes \mathbb{A}_3)^s.\end{aligned}$$

This can be extended from m to $m+1$ similarly. Thus, the proof is accomplished by induction. □

We shall give a result that we use mainly in the proofs of Theorems 2 and 3.

LEMMA 5. *Let* $\mathbb{A}_i \in \mathbb{M}_p$. *Then there exists a* $p^k n \times pn$ *selection matrix* $\mathbb{J}$ *such that* $\mathbb{J}^T \mathbb{J} = \mathbb{I}$ *and*

$$\square_{i=1}^k A_i = \mathbb{J}^T \left(\boxtimes_{i=1}^k \mathbb{A}_i \right) \mathbb{J}.$$

Proof. We prove for three block matrices. The extension from m to $m+1$ is similar. Using the fact that $\mathbb{J}^T$ block commutes with $\mathbb{A}$, we get

$$\begin{aligned}\mathbb{A} \square \mathbb{B} \square \mathbb{C} &= \mathbb{A} \square (\mathbb{J}^T (\mathbb{B} \boxtimes \mathbb{C}) \mathbb{J}) \\ &= \mathbb{J}^T (\mathbb{A} \boxtimes (\mathbb{J}^T (\mathbb{B} \boxtimes \mathbb{C}) \mathbb{J})) \mathbb{J} \\ &= \mathbb{J}^T \big((\mathbb{I} \mathbb{A} \mathbb{I}) \boxtimes (\mathbb{J}^T (\mathbb{B} \boxtimes \mathbb{C}) \mathbb{J}) \big) \mathbb{J} \\ &= \mathbb{J}^T \big((\mathbb{I} \boxtimes \mathbb{J}^T)(\mathbb{A} \boxtimes \mathbb{B} \boxtimes \mathbb{C})(\mathbb{I} \boxtimes \mathbb{J}) \big) \mathbb{J} \\ &= \tilde{\mathbb{J}}^T (\mathbb{A} \boxtimes \mathbb{B} \boxtimes \mathbb{C}) \tilde{\mathbb{J}}\end{aligned}$$

where $\tilde{\mathbb{J}} = (\mathbb{I} \boxtimes \mathbb{J}) \mathbb{J} \in \mathbb{M}_{p^3, p}$ with

$$\tilde{\mathbb{J}}^T \tilde{\mathbb{J}} = \mathbb{J}^T (\mathbb{I} \boxtimes \mathbb{J}^T)(\mathbb{I} \boxtimes \mathbb{J}) \mathbb{J} = \mathbb{J}^T (\mathbb{I} \boxtimes \mathbb{I}) \mathbb{J} = \mathbb{I}. \quad \square$$

LEMMA 6. [4] *Let* $A \in M_n$ *be a positive definite Hermitian matrix and* $V \in M_{n,p}$ *matrix such that* $V^* V = I$. *Then for all real* r *and* s, $r < s$,

$$(V^* A^s V)^{1/s} \geqslant (V^* A^r V)^{1/r}$$

where $r \notin (-1,1)$ *and* $s \notin (-1,1)$ *or* $s \geqslant 1 \geqslant r \geqslant \frac{1}{2}$ *or* $r \leqslant -1 \leqslant s \leqslant -\frac{1}{2}$.

THEOREM 2. *Let* $\mathbb{A}_i$ *and* $\mathbb{A}_j$ *be block commuting matrices for* $i \neq j$*. Suppose that* $\mathbb{A}_i$ *are positive definite matrices and* r *and* s *be real numbers such that* $r < s$*. Then*

$$\left(\square_{i=1}^{k}\mathbb{A}_i^s\right)^{1/s} \geqslant \left(\square_{i=1}^{k}\mathbb{A}_i^r\right)^{1/r} \tag{3}$$

for $r \notin (-1,1)$ *and* $s \notin (-1,1)$ *or* $s \geqslant 1 \geqslant r \geqslant \frac{1}{2}$ *or* $r \leqslant -1 \leqslant s \leqslant -\frac{1}{2}$.

Proof. By using the selection matrix $\mathbb{J}$ instead of V in Lemma 6, we have

$$\begin{aligned}\left(\square_{i=1}^{k}\mathbb{A}_i^s\right)^{1/s} &= \left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i^s\right)\mathbb{J}\right)^{1/s}\\ &= \left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i\right)^s\mathbb{J}\right)^{1/s}\\ &\geqslant \left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i\right)^r\mathbb{J}\right)^{1/r}\\ &= \left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i^r\right)\mathbb{J}\right)^{1/r}\\ &= \left(\square_{i=1}^{k}\mathbb{A}_i^r\right)^{1/r}. \quad \square\end{aligned}$$

Similarly using (3) we obtain the following corollary.

COROLLARY 1. *Let* $\mathbb{A}_i$ *and* $\mathbb{A}_j$ *be block commuting matrices for* $i \neq j$*. Then*
(a) $\left(\square_{i=1}^{k}\mathbb{A}_i\right) \geqslant \left(\square_{i=1}^{k}\mathbb{A}_i^{-1}\right)^{-1}$ *or* $\left(\square_{i=1}^{k}\mathbb{A}_i^{-1}\right) \geqslant \left(\square_{i=1}^{k}\mathbb{A}_i\right)^{-1}$.
(b) $\left(\square_{i=1}^{k}\mathbb{A}_i^r\right)^{1/r} \geqslant \left(\square_{i=1}^{k}\mathbb{A}_i\right)$ *or* $\left(\square_{i=1}^{k}\mathbb{A}_i\right)^{1/r} \geqslant \left(\square_{i=1}^{k}\mathbb{A}_i^{1/r}\right)$ *for* $r > 1$.
(c) $\left(\square_{i=1}^{k}\mathbb{A}_i^2\right)^{1/2} \geqslant \left(\square_{i=1}^{k}\mathbb{A}_i\right)$ *or* $\left(\square_{i=1}^{k}\mathbb{A}_i\right)^{1/2} \geqslant \left(\square_{i=1}^{k}\mathbb{A}_i^{1/2}\right)$.

LEMMA 7. [5] *Let* $A \in M_n$ *be a positive definite Hermitian matrix with eigenvalues in* $[m,M]$, $m > 0$*. Let* $V \in M_{n,p}$ *such that* $V^*V = I$*. If* $r < s$ *real numbers such that either* $r \notin (-1,1)$ *or* $s \notin (-1,1)$*, then*

$$(V^*A^sV)^{1/s} \leqslant \Delta(V^*A^rV)^{1/r}, \tag{4}$$

where

$$\Delta = \left\{\frac{r(\alpha^s-\alpha^r)}{(s-r)(\alpha^r-1)}\right\}^{1/s}\left\{\frac{s(\alpha^r-\alpha^s)}{(r-s)(\alpha^s-1)}\right\}^{-1/r} \tag{5}$$

$\alpha = M/m$*, and* M *and* m *are the largest and smallest eigenvalues of* A*, respectively.*

THEOREM 3. *Let* $\mathbb{A}_i$ *and* $\mathbb{A}_j$ *be block commuting matrices for* $i \neq j$*. Suppose that* $\mathbb{A}_i$ *are positive definite matrices and* r *and* s *be nonzero real numbers such that* $r \notin (-1,1)$ *or* $s \notin (-1,1)$ *and* $r < s$*. Then*

$$\left(\square_{i=1}^{k}\mathbb{A}_i^s\right)^{1/s} \leqslant \Delta\left(\square_{i=1}^{k}\mathbb{A}_i^r\right)^{1/r}, \tag{6}$$

where Δ *is given by (5),* M_i *and* m_i *are the largest and smallest eigenvalues of* $\boxtimes_{i=1}^{k}\mathbb{A}_i$*, respectively.*

Proof. By Lemma 3 and Lemma 7, we obtain

$$\begin{aligned}\left(\square_{i=1}^{k}\mathbb{A}_i^s\right)^{1/s} &= \left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i^s\right)\mathbb{J}\right)^{1/s}\\ &= \left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i\right)^s\mathbb{J}\right)^{1/s}\\ &\leqslant \Delta\left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i\right)^r\mathbb{J}\right)^{1/r}\\ &= \Delta\left(\mathbb{J}^T\left(\boxtimes_{i=1}^{k}\mathbb{A}_i^r\right)\mathbb{J}\right)^{1/r}\\ &= \Delta\left(\square_{i=1}^{k}\mathbb{A}_i^r\right)^{1/r}. \quad \square\end{aligned}$$

In (6) setting $s=2$, $r=1$, and $s=1$, $r=-1$, respectively, we get the following corollary.

COROLLARY 2. *Let* $\mathbb{A}_i$ *and* $\mathbb{A}_j$ *be block commuting matrices for* $i\neq j$*. Then*

(a) $\left(\square_{i=1}^{k}\mathbb{A}_i^2\right)^{1/2}\leqslant\dfrac{M+m}{2\sqrt{Mm}}\left(\square_{i=1}^{k}\mathbb{A}_i\right)$.

(b) $\left(\square_{i=1}^{k}\mathbb{A}_i\right)\leqslant\dfrac{(M+m)^2}{4Mm}\left(\square_{i=1}^{k}\mathbb{A}_i^{-1}\right)^{-1}$.

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