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NONLINEAR ELLIPTIC EQUATIONS WITH SMALL BMO COEFFICIENTS IN NONSMOOTH DOMAINS IN GENERALIZED MORREY SPACES

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Abstract. We obtain the generalized Sobolev-Morrey spaces $W^1_{p,\varphi}(\Omega)$ estimate for weak solutions of a boundary value problem for nonlinear elliptic equations with BMO coefficients in nonsmooth domains. We investigate regularity of the weak solutions in generalized Morrey spaces $M_{p,\varphi}(\Omega)$. The nonlinearity has sufficiently small BMO seminorm and the boundary of the domain is sufficiently flat.

1. Introduction

The classical Morrey spaces $L_{p,\lambda}$ is originally introduced in order to study the local behavior of the solutions to elliptic differential equations. The inclusion from the Morrey spaces into Hölder spaces permits to obtain regularity of the solution to elliptic boundary problems. For the properties and applications of the classical Morrey spaces we refer the readers to [27, 29]. In [8] Chiarenza and Frasca show boundedness of the Hardy-Littlewood maximal operator in $L_{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity of fractional and classical Calderon-Zygmund operators in these spaces and solvability boundary problem. The second author, Mizuhara and Nakai [15, 28, 30] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [16, 17, 35]). In [15, 17, 28, 30], the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}$ was also studied, see also [10, 18, 36].

Let φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. The space $M_{p,\varphi}(\mathbb{R}^n)$ is defined by the norm

$$\|f\|_{M_{p,\varphi}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(x,r)^{-1}\, |B(x,r)|^{-1/p}\, \|f\|_{L^p(B(x,r))}.$$

Here and everywhere in the sequel $B(x,r)$ is the ball in $\mathbb{R}^n$ of radius r centered at x and $|B(x,r)| = v_n r^n$ is its Lebesgue measure, where v_n is the volume of the unit ball in $\mathbb{R}^n$.

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The second author studies the continuity and boundedness properties of sublinear operators generated by various integral operators as maximal operator, Calderon-Zygmund operators and etc. in generalized Morrey space ([1, 15, 17, 18], see also [28, 30, 35, 36]). These results have many applications in the theory of differential equations ([8, 12, 13, 14, 19, 21, 22, 33, 37], see also [2, 4, 5, 6, 7, 9, 11, 20, 23, 25, 26]). Therefore, in the first part of the references we cited those works that are closely related to our work. In the second part of the references we cited those works that are close to this topic. Moreover, in [19] Dirichlet boundary value problems for the higher order linear uniformly elliptic equations in generalized Morrey spaces were considered.

In this paper we investigate the regularity of weak solutions in generalized Morrey spaces $M_{p,\varphi}(\Omega)$. Specifically, we find the conditions on the nonlinearity of equation and the most general geometric requirement to the boundary $\partial\Omega$ so that obtain of the following global $W^1_{p,\varphi}(\Omega)$ estimate

$$\|\nabla u\|_{M_{p,\varphi}(\Omega)} \leqslant C\big(\|f\|_{M_{p,\varphi}} + 1\big). \tag{1.1}$$

Our result holds for any value of $p \in [2,+\infty)$. Obviously, such a result requires regularity condition for coefficients of the equation and the geometric conditions on a bounded open domain Ω.

There have been many works with $W^1_p(\Omega)$ estimates in this direction. For the linear case Di Fazio in [11] obtained the global $W^1_p(\Omega)$ estimate (1.1) for each $1 < p < \infty$ provided the coefficients of equation are in VMO and the domain is in $C^{1,1}$. The result in [11] was extended to the case that Ω is in C^1 by Auscher and Qafsaoni [2]. The main argument in [2, 11] is based on explicit representation formulas involving singular integral operators and commutators. In [4] it is obtained that the same result as in [2, 11] under the condition that the coefficients have small BMO seminorms, which is weaker than VMO condition, and in the geometric setting that the domain is sufficiently flat in the Reifenberg sence. The approach in [4] relies on weak compactness, the Hardy-Littlewood maximal function, the Vitali covering lemma, good λ – inequalities, energy estimates.

We would like to remark that the general theory of singular integral operators and commutators has some limitation to Lipshitz domains and may seem to work only for the linear equation in this direction.

For the quasilinear elliptic quations of p-Laplacian type DiBenedetto and Manfredi in [9] obtained the interior $W^1_q(\Omega)$, $q \geqslant p$ estimates coefficients is the identity matrix by applying maximal function inequalities. When coefficients is uniformly elliptic, the interior and boundary $W^1_q(\Omega)$, $q \geqslant p$ estimates were obtained by Kinnunen and Zhou in [25, 26] under the assumptions that coefficients are of VMO class and Ω is of $C^{1,\alpha}$, $0 < \alpha \leqslant 1$. In [25, 26] it is found that a local version for the sharp maximal functions to prove the interior estimates and employed the flattening argument to obtain the boundary estimates. Their results were extended to the case that coefficients have the small BMO seminorms and Ω is Reifenberg flat in [6].

Our approach is based on the maximal function and the Caldron-Zygmund decomposition. The goal of this paper is to formulate and prove of the results in [6, 25, 26] for equations on nonvariational type in generalized Sobolev-Morrey type spaces when domain Ω is nonsmooth. We also want to find the regularity results for elliptic equations with discontinuous nonlinearity in nondivergence form in $W^2_{p,\varphi}(\Omega)$, $p > n$.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Boundedness of the maximal operator in generalized Morrey spaces

The Hardy-Littlewood maximal function Mf for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy.$$

If Ω is a bounded domain in $\mathbb{R}^n$ and $f \in L_1(\Omega)$, then $Mf = M\tilde{f}$, where $\tilde{f}$ is the zero extension of f in the whole space. It is well known that M is a bounded sub-linear operator from L_p into itself. Precisely, if $f \in L_p(\mathbb{R}^n), p \in (1,\infty)$, then

$$\int_{\mathbb{R}^n} |f(x)|^p\, dx \leqslant \int_{\mathbb{R}^n} |Mf(x)|^p\, dx \leqslant C_p \int_{\mathbb{R}^n} |f(x)|^p\, dx \tag{2.1}$$

for some positive constant $C_p = C(p,n)$. Moreover, the following weak type estimate holds

$$\left|\{x \in \mathbb{R}^n : Mf(x) > t\}\right| \leqslant \frac{C_p}{t^p} \int_{\mathbb{R}^n} |f(x)|^p\, dx \tag{2.2}$$

for any $1 \leqslant p < \infty$ and any $t > 0$.

Our standing assumption is that the domain $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ is a bounded domain with nonsmooth boundary $\partial\Omega$.

DEFINITION 2.1. Let $\varphi : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ be a measurable function and $1 \leqslant p < \infty$. For any domain Ω the generalized Morrey space $M_{p,\varphi}(\Omega)$ (the weak generalized Morrey space $WM_{p,\varphi}(\Omega)$) consists of all $f \in L_p^{\mathrm{loc}}(\Omega)$

$$\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x\in\Omega, 0<r<d} \varphi^{-1}(x,r)\, r^{-n/p}\, \|f\|_{L_p(\Omega(x,r))} < \infty,$$

$$\left(\|f\|_{WM_{p,\varphi}(\Omega)} = \sup_{x\in\Omega, 0<r<d} \varphi^{-1}(x,r)\, r^{-n/p}\, \|f\|_{WL_p(\Omega(x,r))} < \infty\right)$$

where $d = \sup_{x,y\in\Omega}|x-y|$, $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$ and $\Omega(x,r) = \Omega \cap B(x,r)$.

In the case of $\varphi(x,r) = r^{\frac{\lambda-n}{p}}$ the generalized Morrey space $M_{p,\varphi}$ (the weak generalized Morrey space $WM_{p,\varphi}$) is the classical Morrey space $L_{p,\lambda}$ (classical weak Morrey space $WL_{p,\lambda}$). However, there exist examples of weight functions of more general form as $\varphi(r) = r\ln(r+2)$ or $\varphi(B(x,r)) = \left(\int_{B(x,r)} w(y)\, dy\right)^{\alpha}$, $0 < \alpha < 1$, where $w \in A_q$ is Muckenhoupt weight with $q \in \left(1, \frac{1}{\alpha}\right)$ (see [30]). One more example is the following.

In [32] it is shown that the function $f(x) = \chi_{[-1,1]}(x)\,|x|^{-1/2}$ belongs to $L_{1,\varphi}(\mathbb{R})$ with

$$\varphi(I) = \int_I |x|^\alpha\, dx, \quad -1 < \alpha \leqslant -\frac{1}{2},$$

where I is any interval in $\mathbb{R}$.

We denote by $L_{\infty,v}(0,\infty)$ the space of all functions $g(t)$, $t > 0$ with finite norm

$$\|g\|_{L_{\infty,v}(0,\infty)} = \sup_{t>0} v(t)g(t)$$

and $L_\infty(0,\infty) \equiv L_{\infty,1}(0,\infty)$. Let $\mathfrak{M}(0,\infty)$ be the set of all Lebesgue-measurable functions on $(0,\infty)$ and $\mathfrak{M}^+(0,\infty)$ its subset consisting of all nonnegative functions on $(0,\infty)$. We denote by $\mathfrak{M}^+(0,\infty;\uparrow)$ the cone of all functions in $\mathfrak{M}^+(0,\infty)$ which are non-decreasing on $(0,\infty)$ and

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(0,\infty;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $(0,\infty)$. We define the supremal operator $\overline{S}_u$ on $g \in \mathfrak{M}(0,\infty)$ by

$$(\overline{S}_u g)(t) := \|u\,g\|_{L_\infty(t,\infty)}, \;\; t \in (0,\infty).$$

We invoke the following theorem.

THEOREM 2.1. [3] *Let v_1, v_2 be non-negative measurable functions satisfying $0 < \|v_1\|_{L_\infty(t,\infty)} < \infty$ for any $t > 0$ and let u be a continuous non-negative function on $(0,\infty)$.*

Then the operator $\overline{S}_u$ is bounded from $L_{\infty,v_1}(0,\infty)$ to $L_{\infty,v_2}(0,\infty)$ on the cone $\mathbb{A}$ if and only if

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(\cdot,\infty)} \right) \right\|_{L_\infty(0,\infty)} < \infty. \tag{2.3}$$

The following lemmas were proved in [1].

LEMMA 2.1. Let $1 < p < \infty$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$ the inequality

$$\|Mf\|_{L_p(B(x,r))} \lesssim \|f\|_{L_p(B(x,2r))} + r^{\frac{n}{p}} \sup_{s>2r} s^{-n} \|f\|_{L_1(B(x,s))} \tag{2.4}$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|Mf\|_{WL_1(B(x,r))} \lesssim \|f\|_{L_1(B(x,2r))} + r^n \sup_{s>2r} s^{-n} \|f\|_{L_1(B(x,s))} \tag{2.5}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

LEMMA 2.2. Let $1 < p < \infty$. Then for any ball $B = B(x,r)$ in $\mathbb{R}^n$, the inequality

$$\|Mf\|_{L_p(B)} \lesssim r^{\frac{n}{p}} \sup_{s>2r} s^{-\frac{n}{p}} \|f\|_{L_p(B(x,s))} \tag{2.6}$$

holds for all $f \in L_p^{\mathrm{loc}}(\mathbb{R}^n)$.

Moreover, the inequality

$$\|Mf\|_{WL_1(B)} \lesssim r^n \sup_{s>2r} s^{-n} \|f\|_{L_1(B(x,s))} \tag{2.7}$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

We give the following boundedness result of maximal operators in generalized Morrey spaces which is obtained from Theorem 2.1 and Lemma 2.1.

THEOREM 2.2. [1] *Assume that* $1 \leqslant p < \infty$ *and there is a positive constant* C *such that for any fixed* $x \in \mathbb{R}^n$ *and* $r > 0$ *the following inequality holds*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\tau<\infty} \varphi_1(B(x,\tau))\tau^{\frac{n}{p}}}{s^{\frac{n}{p}}} \leqslant C\varphi_2(B(x,r)). \tag{2.8}$$

Then for $p > 1$, M *is bounded from* $M_{p,\varphi_1}(\mathbb{R}^n)$ *to* $M_{p,\varphi_2}(\mathbb{R}^n)$ *and for* $p = 1$, M *is bounded from* $M_{1,\varphi_1}(\mathbb{R}^n)$ *to* $WM_{1,\varphi_2}(\mathbb{R}^n)$.

COROLLARY 2.1. (Maximal inequality) *Assume that for* $1 < p < \infty$, *there is a positive constant* C *such that for any fixed* $x \in \mathbb{R}^n$ *and* $r > 0$ *the following inequality holds*

$$\sup_{r<s<\infty} \frac{\operatorname*{ess\,inf}_{s<\tau<\infty} \varphi(B(x,\tau))\tau^{\frac{n}{p}}}{s^{\frac{n}{p}}} \leqslant C\varphi(B(x,r)), \tag{2.9}$$

where C *does not depend on* x *and* r. *Then there is a constant* $C_p > 0$ *such that*

$$\|f\|_{M_{p,\varphi}(\mathbb{R}^n)} \leqslant \|Mf\|_{M_{p,\varphi}(\mathbb{R}^n)} \leqslant C_p \|f\|_{M_{p,\varphi}(\mathbb{R}^n)}, \quad f \in M_{p,\varphi}(\mathbb{R}^n).$$

Denote by $\mathscr{G}_p$ the set of all decreasing functions $\phi : (0,\infty) \to (0,\infty)$ such that $r \in (0,\infty) \mapsto r^{\frac{n}{q}}\phi(r) \in (0,\infty)$ is almost increasing.

From Theorem 2.2 we get the following statement, which were proved in [31, 34].

COROLLARY 2.2. [31, 34] *Let* $1 \leqslant p < \infty$ *and* $\phi \in \mathscr{G}_p$. *Then for* $p > 1$, M *is bounded on* $M_{p,\varphi}(\mathbb{R}^n)$ *and for* $p = 1$, M *is bounded from* $M_{1,\varphi}(\mathbb{R}^n)$ *to* $WM_{1,\varphi_2}(\mathbb{R}^n)$.

Impose in addition a kind of monotonicity condition on φ, precisely

$$\varphi(B(y,r)) \leqslant \varphi(B(z,s)) \quad \text{for all} \quad B(y,r) \subset B(z,s). \tag{2.10}$$

This implies that for a given bounded domain $\Omega \subset \mathbb{R}^n$, the following inequality holds

$$\sup_{\substack{y\in\Omega \\ r>0}} \frac{|\Omega(y,r)|}{\varphi(B(y,r))} \leqslant k_1, \tag{2.11}$$

with a positive constant k_1 depending on n, φ and Ω. In fact, since Ω is bounded domain there exists $d > 0$ such that $\Omega \subset B(0,d)$. Then, if $r \geqslant 2d$ for any $y \in \Omega$ we have

$$\frac{|\Omega(y,r)|}{\varphi(B(y,r))} \leqslant \frac{|\Omega|}{\varphi(B(0,d))}.$$

On the other hand, if $0 < r < 2d$, then we see from (2.9) that

$$k_2\,\frac{\varphi(B(y,r))}{r^n} \geqslant \frac{\varphi(B(0,d))}{n(2d)^n}.$$

It implies that for some positive constant $c = c(n)$ the following inequalities hold

$$\frac{|\Omega(y,r)|}{\varphi(B(y,r))} \leqslant \frac{c\,r^n}{\varphi(B(y,r))} \leqslant \frac{c\,k_2\,n(2d)^n}{\varphi(B(0,d))}.$$

3. Definition and statement of the problem

Assume that Ω is bounded. We now introduce some geometric notation. $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$ is an open ball in $\mathbb{R}^n$ with center 0 and radius $\rho > 0$, $B_\rho(y) = B_\rho + y$, $B_\rho^+ = B_\rho \cap \{x : x_n > 0\}$, $B_\rho^+(y) = B_\rho^+ + y$, $T_\rho = B_\rho \cap \{x : x_n = 0\}$ and $T_\rho(y) = T_\rho + y$. $\Omega_\rho = \Omega \cap B_\rho$ and $\Omega_\rho(y) = \Omega \cap B_\rho(y)$. $\partial\Omega$ is the boundary of Ω, $\partial\Omega_\rho = \partial\Omega \cap B_\rho$ is the wiggled part of $\partial\Omega_\rho$.

The generalized Sobolev-Morrey space $W^1_{p,\varphi}(\Omega)$ consists of all functions $u \in W^1_p(\Omega)$ with distributional derivatives $D^s u \in M_{p,\varphi}(\Omega)$, endowed with the norm

$$\|u\|_{W^1_{p,\varphi}(\Omega)} = \sum_{0 \leqslant |s| \leqslant 1} \|D^s u\|_{M_{p,\varphi}(\Omega)}.$$

The space $W^1_{p,\varphi}(\Omega) \cap \mathring{W}^1_p(\Omega)$ consists of all functions $u \in \mathring{W}^1_p(\Omega)$ with $D^s u \in M_{p,\varphi}(\Omega)$, $0 \leqslant |s| \leqslant 1$ and is endowed by the same norm. Recall that $\mathring{W}^1_p(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm in $W^1_p(\Omega)$.

We need the following version of the Vitali covering result.

LEMMA 3.3. [4] Let A and D be measurable sets with $A \subset D \subset \Omega$. Assume that Ω has nonsmooth boundary and there exists a small $\varepsilon > 0$ such that $|A| < \varepsilon |B_1|$ and $|A \cap B_r(x)| \geqslant \varepsilon |B_r(x)|$, $B_r(x) \cap \Omega \subset D$ with $x \in \Omega$, $r \in (0,1]$. Then we have

$$|A| < [10/(1-\delta)]^n \varepsilon |D|.$$

The next result follows from the standard measure theory (see, [22]).

LEMMA 3.4. Let $f \in L^1(U)$ be a nonnegative function on a bounded domain $U \subset \mathbb{R}^n$, φ be a weight satisfying (2.9) and (2.10), $p \in (1,\infty)$ and $\theta > 0, \lambda > 1$ be constants. Then $f \in M_{p,\varphi}(U)$ if and only if

$$S = \sup_{y \in U, r>0} \sum_{k \geqslant 1} \lambda^{kp} \frac{|\{x \in U \cap B(y,r) : f(x) > \theta\lambda^k\}|}{\varphi(B(y,r))^p\, r^n} < \infty$$

and

$$\frac{1}{C} S \leqslant \|f\|^p_{M_{p,\varphi}(U)} \leqslant C(|U| + S),$$

where $C>0$ is a constant depending only on θ, λ, p and φ.

Proof. Choose $y\in U$ and take a ball $B(y,r)$, then

$$\begin{aligned}\frac{r^{-n}}{\varphi(B(y,r))^p}\int_{B(y,r)} f(x)^p\,dx &= \frac{r^{-n}}{\varphi(B(y,r))^p}\int_{\{x\in B(y,r):f(x)\leqslant\theta\lambda\}} f(x)^p\,dx\\ &\quad+\sum_{k\geqslant 1}\frac{r^{-n}}{\varphi(B(y,r))^p}\int_{\{x\in B(y,r):\theta\lambda^k<f(x)\leqslant\theta\lambda^{k+1}\}} f(x)^p\,dx\\ &\leqslant(\theta\lambda)^p\frac{|B(y,r)|}{\varphi(B(y,r))^p r^n}+\sum_{k\geqslant 1}\frac{(\theta\lambda^{k+1})^p}{\varphi(B(y,r))^p r^n}\big|\{x\in B(y,r):f(x)>\theta\lambda^k\}\big|\\ &=(\theta\lambda)^p\left(\frac{|B(y,r)|}{\varphi(B(y,r))^p r^n}+\sum_{k\geqslant 1}\frac{\lambda^{kp}\big|\{x\in B(y,r):f(x)>\theta\lambda^k\}\big|}{\varphi(B(y,r))^p r^n}\right).\end{aligned}$$

Taking the supremum over $y\in U$, and $r>0$, and making use of (2.11), we get

$$\|f\|^p_{M_{p,\varphi}(U)}\lesssim |U|+S$$

with a constant depending on p, n, φ, λ and θ. On the other hand

$$\begin{aligned}\frac{r^{-n}}{\varphi(B(y,r))^q}\int_{B(y,r)} f(x)^p\,dx &= \frac{pr^{-n}}{\varphi(B(y,r))^p}\int_{B(y,r)}\left(\int_0^{f(x)}\xi^{p-1}d\xi\right)dx\\ &=\frac{pr^{-n}}{\varphi(B(y,r))^p}\int_0^\infty\big|\{x\in B(y,r):f(x)>\xi\}\big|\xi^{p-1}d\xi\\ &\geqslant\frac{pr^{-n}}{\varphi(B(y,r))^p}\sum_{k\geqslant 1}\big|\{x\in B(y,r):f(x)>\theta\lambda^k\}\big|\int_{\theta\lambda^{k-1}}^{\theta\lambda^k}\xi^{p-1}d\xi\\ &=\theta^q(1-\lambda^{-p})\frac{r^{-n}}{\varphi(B(y,r))^p}\sum_{k\geqslant 1}\lambda^{kp}\big|\{x\in B(y,r):f(x)>\theta\lambda^k\}\big|.\end{aligned}$$

Taking again the supremum over $y\in U$ and $r>0$ we get

$$\|f\|^p_{M_{p,\varphi}(\Omega)}\geqslant\frac{1}{C}S$$

with a positive constant $C=C(\theta,\lambda,p)$. $\square$

Now we give the statement of problem. We are interested in the well-posedness in generalized Sobolev-Morrey space $W^1_{p,\varphi}(\Omega)$ of the following nonlinear boundary value problem:

$$\begin{cases}\operatorname{div} a(x,u,\nabla u)=f & \text{in}\quad\Omega,\\ u=0 & \text{on}\quad\partial\Omega,\end{cases}\tag{3.1}$$

where $f\in M_{p,\varphi}(\Omega;\mathbb{R}^n)$ is a given vector-valued function for some $2\leqslant p<\infty$, a is the vector field $a=a(x,\xi,\xi_x):\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ which is measurable in x for almost every ξ, and continuous in ξ for each x. For simplicity we consider $a(x,\xi,\xi_x)\equiv$

$a(x,\xi)$. The unknown is $u(\cdot):\Omega\to\mathbb{R}$, where Ω is a bounded, open subset of $\mathbb{R}^n$ with very nonsmooth boundary. We say that $u\in\mathring{W}_p^1(\Omega)$ is a weak solution of (3.1) if it satisfies

$$\int_\Omega a(x,\nabla u)\nabla\varphi dx=\int_\Omega f\varphi dx$$

for all $\varphi\in\mathring{W}_p^1(\Omega)$.

If there exists a function $L:\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ such that $a(x,\xi)$ is the gradient of $L(x,\xi)$ with respect to $\xi\in\mathbb{R}^n$

$$a(x,\xi)=\Delta_\xi L(x,\xi),$$

then (3.1) is the Euler-Lagrange equation corresponding to the integral

$$I(u)=\int_\Omega(L(x,\nabla u)-fu)dx.$$

However, if the problem (3.1) is not of variational type i.e., there is no such a functional L and the variational methods do not apply to our problem for the existence of a weak solution. We will use the method of Browder and Minty. We suppose that there exists a positive constant c_0 such that

$$(a(x,\xi)-a(x,\eta))(\xi-\eta)\geqslant c_0|\xi-\eta|^p \tag{3.2}$$

for all $\xi,\eta\in\mathbb{R}^n$, almost every $x\in\Omega$ and

$$|a(x,\xi)|\leqslant c_1(1+|\xi|)^{p-1}. \tag{3.3}$$

One can show that there exists a unique weak solution of (3.1) in generalized Morrey spaces. We also suppose that $a(x,\xi)$ is uniformly Lipschitz continuous with respect to the variable ξ and

$$|\nabla_\xi a(x,\xi)|\leqslant c_2. \tag{3.4}$$

We will impose in the nonlinearity a. For $\rho>0$ and $y\in\mathbb{R}^n$ in order to measure the oscillation of $a(x,\xi)$ in the variable x over $B_\rho(y)$, we define the function $\beta:\Omega\to\mathbb{R}$ by

$$\beta[a,B_\rho(y)](x)=\sup_{\xi\in\mathbb{R}^n}\frac{|a(x,\xi)-\bar{a}_{B_\rho(y)}(\xi)|}{1+|\xi|}, \tag{3.5}$$

where we denote $\bar{a}_{B_\rho(y)}$ by the integral average of a on $B_\rho(y)$

$$\bar{a}_{B_\rho(y)}(\xi)=\fint_{B_\rho(y)}a(x,\xi)dx=\frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}a(x,\xi)dx$$

is the integral average of $a(x,\xi)$ for each fixed ξ over $B_\rho(y)$.

In the linear case, if $a(x,\xi)=A(x)\cdot\xi$ for each $\xi\in\mathbb{R}^n$ and for almost every $x\in\mathbb{R}^n$

$$\beta[a,B_\rho(y)](x)\leqslant|A(x)-\bar{A}_{B_\rho(y)}|,$$

where $\bar{A}_{B_\rho(y)}$ is the integral average of A on $B_\rho(y)$ and so it seems natural to consider $\beta[a,B_\rho(y)]$ to be a version of the function of mean oscillation over $B_\rho(y)$ uniformly in the variable ξ for the nonlinear case.

We use the following assumption on the nonlinearity coefficients $a(x,\xi)$ (see, also [4]).

DEFINITION 3.2. We say that the vector field $a(x,\xi)$ satisfies the (δ,R)-*BMO* condition if

$$\sup_{0<\rho\leqslant R}\sup_{y\in\mathbb{R}^n}\fint_{B_\rho(y)}|\beta[a,B_\rho(y)]|^2dx\leqslant\delta^2. \tag{3.6}$$

This definition is called also small *BMO* condition. Condition (3.6) is a good replacement of small *BMO* condition used in [4]. The small *BMO* condition has been extensively studied as an appropriate substitute for *VMO* condition (see [4]). Also we use the following regularity condition on the boundary $\partial\Omega$.

DEFINITION 3.3. We say that Ω is the (δ,R)- Reifenberg flat if every $x\in\partial\Omega$ and every $r\in(0,R]$, there exists a coordinate system $\{y_1,y_2,\dots,y_n\}$, which can depend on r and x so that $x=0$ in this coordinate system and that

$$B_r\cap\{y_n>\delta r\}\subset B_r(0)\cap\Omega\subset B_r(0)\cap\{y_n>-\delta r\}. \tag{3.7}$$

Reifenberg domains arise naturally in minimal surface theory and free boundary problems. The boundary of a Reifenberg domain is very rough and could be a fractal. This is a geometric condition exhibiting a very low level of regularity, prescribing that all scales the boundary can be trapped between two hyperplanes depending on the scale chosen, see, for example [24]. Also we remark that a Lipschitz domain is a Reifenberg flat domain provided its Lipschitz constant is sufficiently small (see [38, 39]). On the other hand an inner neighborhood of the δ Reifenberg domain is a Lipschitz domain for small δ (see [5]).

Our problem (3.1) is invariant under a normalization and a scaling though the problem considered here is highly nonlinear.

LEMMA 3.5. [4] Assume that $a(x,\xi)$ satisfies (3.2), (3.3), (3.4) and the (δ,R)-*BMO* condition. Suppose further $u\in\mathring{W}_p^1(\Omega)$ is the weak solution of the Dirichlet problem (3.1). For fixed $\lambda\geqslant 1$ let $u_\lambda=\frac{u}{\lambda}$, $f_\lambda=\frac{f}{\lambda}$. Define

$$a_\lambda(x,\xi)=\frac{a(x,\lambda\xi)}{\lambda},\quad x,\xi\in\mathbb{R}^n. \tag{3.8}$$

Then $a_\lambda(x,\xi)$ satisfies (3.2)–(3.4) and the (δ,R)-*BMO* condition with the same constants c_0,c_1,c_2, δ and R. Furthemore $u_\lambda\in\mathring{W}_p^1(\Omega)$ is the weak solution of

$$\begin{cases}\operatorname{div}a_\lambda(x,u_\lambda,\nabla u_\lambda)=f_\lambda & \text{in}\quad\Omega,\\ u_\lambda=0 \quad\text{on}\quad\partial\Omega.\end{cases} \tag{3.9}$$

LEMMA 3.6. [4] Under the same conditions as in Lemma 3.3 we define $a_r(x,\xi)=a(rx,\xi)$, $u_r(x)=u(rx)$, $f_r(x)=f(rx)$ and $\Omega_r=\{x/r:x\in\Omega\}$, $r>0$. Then

1. $a_r(x,\xi)$ satisfies (3.2)–(3.4) with the same constants.
2. $a_r(x,\xi)$ satisfies the $(\delta,R/r)$- BMO condition.

3. Ω_r is the $(\delta, R/r)$- Reifenberg flat.

4. $u_r \in W_0^1(\Omega_r)$ is the weak solution of $div\, a_r(x, \nabla u_r) = f_r$ in Ω_r and $u_r = 0$ in $\partial\Omega_r$.

4. Main result

The main theorem is stated as follows.

THEOREM 4.3. *Let φ satisfy the condition (2.9) and $2 \leqslant p < \infty$. Assume there exists $\delta > 0$, which dependent at c_0, c_1, c_2, p, n, such that if $a(x,\xi)$ satisfies (3.2), (3.3), (3.4) and the (δ, R)- BMO condition, Ω is the (δ, R)-Reifenberg flat and $f \in M_{p,\varphi}(\Omega)$. Then the weak solution $u \in \mathring{W}_p^1(\Omega)$ of the problem (3.1) belongs to $\mathring{W}_{p,\varphi}^1(\Omega)$ with estimate (1.1).*

We mean δ to be a small positive constant, which is determined later in the proof of Theorem 3.1. Before we will obtain interior $W_p^1(\Omega)$, $2 \leqslant p < \infty$ estimates for the elliptic equation

$$\operatorname{div} a(x, \nabla u) = f \quad \text{in} \quad \Omega \tag{4.1}$$

assuming that $a(x,\xi)$ satisfies the (δ, R)-BMO condition. By a scaling we take $R = 8$ and that $B_8 \subset \Omega$. We consider the following problem

$$\operatorname{div} \bar{a}_{B_6}(\nabla v) = 0 \quad \text{in} \quad B_6. \tag{4.2}$$

We say that $v \in W_p^1(B_6)$ is a weak solution to (4.2), if

$$\int_{B_6} \bar{a}_{B_6}(\nabla v)(x) \cdot \nabla \varphi(x) dx = 0 \tag{4.3}$$

for each $\varphi \in \mathring{W}_p^1(B_6)$.

Our sufficient regularity for (4.2) is the following interior W_∞^1-estimate.

LEMMA 4.7. [4] Suppose that a satisfies the condition (3.4). Then for any weak solution $v \in W_p^1(B_6)$ of the problem (4.2) we have

$$\|\nabla v\|_{L_\infty(B_3)}^p \leqslant C \fint_{B_5} |\nabla v(x)|^p dx$$

for some constant $C > 0$.

We consider the L_p-average of f instead of its BMO seminorm.

LEMMA 4.8. Suppose that a satisfy the condition (3.4). Given $\varepsilon > 0$, there exists a small $\delta = \delta(\varepsilon) > 0$ so that for any weak solution $u \in W_p^1(\Omega)$ of (4.1) with the following normalization conditions

$$\fint_{B_6} |\nabla u(x)|^p dx \leqslant 1$$

and

$$\fint_{B_6} (|\beta[a,B_6](x)|^p + |f(x)|^p)dx \leqslant \delta^p.$$

Then there exists a weak solution $v \in W_p^1(B_6)$ of the problem (4.2) such that

$$\int_{B_6} |(u - \bar{u}_{B_6})(x) - v(x)|^p dx \leqslant \varepsilon^p.$$

For the proof we use argue by contradiction. If not, there would exist $\varepsilon_0 > 0$, $\{a_k\}_{k=1}^{\infty}$, $\{u_k\}_{k=1}^{\infty}$ and $\{f_k\}_{k=1}^{\infty}$ such that $u_k \in W_p^1(\Omega)$ is a weak solution of

$$\operatorname{div} a_k(x, \nabla u_k) = \operatorname{div} f_k \quad \text{in} \quad \Omega$$

with

$$\fint_{B_6} |\nabla u_k(x)|^p dx \leqslant 1 \tag{4.4}$$

and

$$\fint_{B_6} \left(|\beta[a_k,B_6](x)|^p + |f_k(x)|^p\right) dx \leqslant \frac{1}{k^2},$$

but

$$\int_{B_6} \left|(u_k - \bar{u}_{k,B_6})(x) - v_k(x)\right|^p dx > \varepsilon_0^p$$

for any weak solution $v_k \in W_p^1(B_6)$ of

$$\operatorname{div} \bar{a}_{k,B_6}(\nabla v_k) = 0 \quad \text{in} \quad B_6.$$

Using Poincare's inequality and (4.4) we see that $\left\{u_k - \bar{u}_{k,B_6}\right\}_{k=1}^{\infty}$ is uniformly bounded in $W_p^1(B_6)$. Then there exist a subsequence $\{u_{k_j}\}$ and $u_0 \in W_p^1(B_6)$ such that

$$u_{k_j} \to u_0 \quad \text{in} \quad W_p^1(B_6).$$

For each fixed bounded domain U in $\mathbb{R}^n$, $\bar{a}_{k,B_6}(\xi)$ is uniformly bounded and equicontinious. Indeed, if $\xi \in U$, then the growth condition implies that

$$\left|\bar{a}_{k,B_6}(\xi)\right| \leqslant \fint_{B_6} \left|a_k(x,\xi)\right| dx \lesssim \fint_{B_6} (1+|\xi|)^{p-1} dx = (1+|\xi|)^{p-1} \approx 1 + |U| < \infty.$$

Similarly, by using (3.4) and the Lipschitz continuity of $a(x,\xi)$ in ξ for almost every x, we find

$$\left|\nabla_\xi \bar{a}_{k,B_6}(\xi)\right| \leqslant \fint_{B_6} \left|\nabla_\xi a_k(x,\xi)\right| dx \lesssim \fint_{B_6} dx \lesssim 1.$$

If we apply the Arzela-Ascoli compactness criterion, then we obtain a subsequence $\left\{\bar{a}_{k,B_6}(\xi)\right\}$ and a vector field $a_0(\xi)$ such that $\bar{a}_{k,B_6}(\xi) \to a_0(\xi)$ locally uniformly in $\mathbb{R}^n$. In fact, we use the Arzela-Ascoli compactness criterion on each $\bar{B}_i(0)$, $i = 1,2,\dots$ and obtain $\bar{a}_{k,B_6}(\xi)$ uniformly convergence to $a_0(\xi)$ in $\bar{B}_i(0)$ and as usual, the diagonal subsequence technique yields the locally uniform convergence.

Now taking k large enough we reach a contradiction.

COROLLARY 4.3. *Under the same conditions as in Lemma 4.8 we have*

$$\fint_{B_2} |\nabla(u-v)(x)|^p dx \leqslant \varepsilon^p. \tag{4.5}$$

Corollary 4.3 follows from Lemmas 4.7 and 4.8.

LEMMA 4.9. For given $\varepsilon > 0$ there exists a small $\delta = \delta(\varepsilon) > 0$ such that for any weak solution $u \in W_p^1(\Omega)$ of (4.1) with

$$\fint_{B_r} |\beta[a, B_r](x)|^p dx \leqslant \delta_p \tag{4.6}$$

and

$$B_1 \cap \{x \in \Omega : M(|\nabla u|^p)(x) \leqslant 1\} \cap \{x \in \Omega : M(|f|^p)(x) \leqslant \delta^p\} \neq \emptyset, \tag{4.7}$$

then there is a constant $c_3 > 0$ which dependent only on c_0, c_1, c_2 and n, we have

$$\left|\{x \in \Omega : M(|\nabla u|^p)(x) > c_3^p\} \cap B_1\right| < \varepsilon |B_1|.$$

Proof. From the condition (4.7), we see that there is a point $x_0 \in B_1$ such that for each $\rho > 0$ we have

$$\fint_{B_\rho(x_0)} |\nabla u(x)|^p dx \leqslant 1, \quad \fint_{B_\rho(x_0)} |f(x)|^p dx \leqslant \delta^p.$$

Then, since $B_6 \subset B_7(x_0) \subset B_8 \subset \Omega$, we find

$$\fint_{B_6} |\nabla u(x)|^p dx \leqslant \left(\frac{7}{6}\right)^n \fint_{B_7(x_0)} |\nabla u(x)|^p dx \leqslant \left(\frac{7}{6}\right)^n.$$

By the same reason, we have

$$\fint_{B_6} |f(x)|^p dx \leqslant \left(\frac{7}{6}\right)^n \delta^p.$$

We fix any $\eta \in (0,1)$ and set $\lambda = \sqrt{(7/6)^n}$. Normalizing a, u and f to a_λ, u_λ and f_λ, respectively, as in Lemma 3.3 we have under the conditions of Lemma 4.8, which gives us there exists a weak solution $v_\lambda \in W_p^1(B_6)$ of

$$\operatorname{div} \bar{a}_{\lambda, B_6}(\nabla v_\lambda) = 0 \quad \text{in} \quad B_6,$$

such that

$$\fint_{B_2} |\nabla(u_\lambda - v_\lambda)(x)|^p dx \leqslant \eta^p$$

for some small $\delta = \delta(\eta) > 0$ satisfying normalization conditions. Continuing, we use inequality

$$\fint_{B_4} |\nabla v_\lambda(x)|^p dx \leqslant 2 \fint_{B_4} (|\nabla(v_\lambda - u_\lambda)(x)|^p + |\nabla u_\lambda(x)|^p) dx \lesssim \eta^p + 1 \lesssim 1.$$

By Lemma 4.7, we get

$$\|v_\lambda\|_{L_\infty(B_3)} \leqslant n_0$$

for some constant $n_0 > 0$. Note that for $n_1 = \max\{2n_0, \sqrt{(8/5)^n}\}$ the following embedding

$$\{x \in B_1 : M(|\nabla u_\lambda|^p)(x) > n_1^p\} \subset \{x \in B_1 : M_{B_6}(|\nabla(u_\lambda - v_\lambda)|^p)(x) > n_0^p\} \tag{4.8}$$

is valid.

Then using (4.8) and the weak type inequality (2.2), we obtain the following estimates

$$\begin{aligned}
&\left|\left\{x \in B_1 : M(|\nabla u_\lambda|^p)(x) > (\lambda n_1)^p\right\}\right| \\
&= \left|\left\{x \in B_1 : M(|\nabla u_\lambda|^p)(x) > n_1^p\right\}\right| \\
&\leqslant \left|\left\{x \in B_1 : M_{B_6}(|\nabla(u_\lambda - v_\lambda)|^p)(x) > n_0^p\right\}\right| \\
&\lesssim \int_{B_2} |\nabla(u_\lambda - v_\lambda)(x)|^p dx \lesssim \eta^p.
\end{aligned}$$

Now we select $\eta > 0$, thereby $\delta = \delta(\eta) > 0$ satisfying

$$\left|\{x \in B_1 : M(|\nabla u|^p)(x) \geqslant (\lambda n_1)^p\}\right| \lesssim \eta^p < \varepsilon |B_1|. \quad \square$$

Now let's fix ε, δ, c_3 given in Lemma 4.9. Then the lemma follows as the scaling invariant form of Lemma 4.9 using the Lemma 3.4.

LEMMA 4.10. Let $y \in \Omega$ and $r > 0$ be small with $B_{8r}(y) \subset \Omega$. Suppose that a satisfies (3.4) and the $(\delta, 8r)$-*BMO* condition. Then for any weak solution $u \in W_p^1(\Omega)$ of the problem (4.1) satisfying

$$B_r(y) \cap \{x \in \Omega : M\big(|\nabla u|^p\big)(x) \leqslant 1\} \cap \{x \in \Omega : M\big(|\nabla u|^p\big)(x)\} \neq \emptyset,$$

we have

$$|\{x \in \Omega : M\big(|\nabla u|^p\big)(x) > c_3^p\} \cap B_r(y)| < \varepsilon |B_r(y)|.$$

THEOREM 4.4. *Let* $\left|\left\{x \in \Omega : M\big(|\nabla u|^p\big)(x) > c_3^p\right\} \cap B_r(y)\right| > \varepsilon \left|B_r(y)\right|$. *Suppose that a satisfies* (3.4) *and the* $(\delta, 8r)$-*BMO condition. Then for any weak solution* $u \in W_p^1(\Omega)$ *of the problem* (4.1) *and for any small ball* $B_r(y)$ *with* $B_{8r} \subset \Omega$ *we have*

$$B_r(y) \subset \{x \in \Omega : M\big(|\nabla u|^p\big)(x) > 1\} \cup \{x \in \Omega : M(|f|^p)(x) > \delta^p\}.$$

Now we extend this results for the interior estimates to study the well-posedness in $\mathring{W}_p^1(\Omega)$, $2 \leqslant p < \infty$, of the Dirichlet problem (4.1) with $f \in M_{p,\varphi}(\Omega)$ in the bounded Reifenberg flat domain $\Omega \subset \mathbb{R}^n$. Let $a(x,\xi)$ has a small *BMO* seminorm. We consider the situation that $\partial\Omega$ is the (δ, N)- Reifenberg flat, where N is a sufficiently big number. We are under the geometric setting

$$B_\rho^+ \subset \Omega_\rho \subset B_\rho \cap \{x_n > -2N\delta\}, \quad 1 \leqslant \rho \leqslant N. \tag{4.9}$$

Consider Dirichlet problem

$$\begin{cases} \operatorname{div} \bar{a}_{B_6^+}(\nabla v) = 0 \quad \text{in} \quad B_6^+, \\ v = 0 \quad \text{on} \quad \partial B_6^+. \end{cases} \tag{4.10}$$

The idea is that we can find local estimates in B_6^+ of the weak solution $u \in \mathring{W}_p^1(\Omega)$ of (4.1) by studying comparison with solutions of the problem (4.10) by studying the deviation of the nonlinearity $a(x,\xi)$ of (4.1) from the coefficients of $\bar{a}_{B_6^+}(\xi)$ and by measuring the deviation of $\partial\Omega$ from being a flat boundary at scale 6 and at the origin.

We remark that under conditions (3.2)–(3.4), the matrix $\nabla_\xi \bar{a}_{B_6^+}(\nabla v(x))$ is uniformly elliptic and bounded

$$\nabla_\xi \bar{a}_{B_6^+}(\nabla v(x)\xi)\xi \geqslant c_0|\xi|^p \quad \text{and} \quad \nabla_\xi \bar{a}_{B_6^+}(\nabla v(x)) \leqslant c_2 \tag{4.11}$$

for all $\xi \in R^n$ and almost every $x \in B_6^+$.

The sufficient regularity of (4.10) for our boundary estimates is the following interior regularity.

LEMMA 4.11. [4] Let $v \in W_p^1(B_6^+)$ be a weak solution of the problem (4.10). Then

$$\|\nabla v\|_{L_\infty(B_3^+)} \leqslant C \fint_{B_5^+} |\nabla v(x)|^p dx$$

for some constant $C > 0$.

LEMMA 4.12. [4] Let $a(x,\xi)$ satisfy (3.4). Then, for given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ so that for any weak solution $u \in \mathring{W}_p^1(\Omega)$ to the problem (3.1) with the following normalization conditions

$$B_6^+ \subset \Omega_6 \subset B_6 \cap \{x_n > -12\delta\}, \tag{4.12}$$

$$\fint_{\Omega_6} |\nabla u(x)|^p dx \leqslant 1 \tag{4.13}$$

and

$$\fint_{\Omega_6} (|\beta[a,\Omega_6](x)|^p + |f(x)|^p) dx \leqslant \delta^p. \tag{4.14}$$

Then there exists a weak solution $v \in W_p^1(B_6^+)$ of the problem (3.10) such that

$$\int_{B_6^+} |(u - \bar{u}_{B_6^+})(x) - v(x)|^p dx \leqslant \varepsilon^p. \tag{4.15}$$

We want to extend v to Ω_6. We know that $v = 0$ on ∂B_6^+ in the trace sences, it is natural to use the zero extension. Under the same conditions as in Lemma 4.12, we have

$$\fint_{\Omega_2} |\nabla(u - v)(x)|^p dx \leqslant \varepsilon^p. \tag{4.16}$$

LEMMA 4.13. Assume that $u \in \mathring{W}^1_p(\Omega)$ is the weak solution of the problem (3.1) and $a(x,\xi)$ satisfies (3.4). Then there is a positive constant $c_3(c_0,c_1,c_2,n)$ such that for given $\varepsilon > 0$ there exists a small $\delta(\varepsilon) > 0$, if

$$\fint_{\Omega_r} |\beta[a,\Omega_r](x)|^p dx \leqslant \delta^p \quad (1 \leqslant r \leqslant N), \tag{4.17}$$

$$B_r^+ \subset \Omega_r \subset B_r \cap \{x_n > -2r\delta\} \tag{4.18}$$

and

$$B_r \subset \{x \in \Omega : M\big(|\nabla u(x)|^p\big)(x) \leqslant 1\} \cup \{x \in \Omega : M(|f|^p)(x) \leqslant \delta^p\} \neq \emptyset. \tag{4.19}$$

Moreover

$$\big|\{x \in \Omega : M\big(|\nabla u|^p\big)(x) > c_3^p\} \cup B_r\big| < \varepsilon\big|B_r\big|.$$

Proof. By the definition of the maximal function and from (4.19), there exists a point $x_0 \in \Omega_1$ such that for $\rho > 0$

$$\fint_{\Omega_\rho(x_0)} |\nabla u(x)|^p dx \leqslant 1, \quad \fint_{\Omega_\rho(x_0)} |f(x)|^p dx \leqslant \delta^p. \tag{4.20}$$

We note that $x_0 \in \Omega_1$, $\Omega_r \subset \Omega_{2r} \subset \Omega_{4r}$. Then from (4.20)

$$\fint_{\Omega_r} |\nabla u(x)|^p dx \leqslant \frac{|\Omega_{2r}|}{|\Omega_r|} \fint_{\Omega_{2r}} |\nabla u(x)|^p dx \leqslant \frac{|B_{2r}|}{|B_r^+|} \lesssim r^n$$

and

$$\fint_{\Omega_r} |f(x)|^p dx \lesssim r^n \delta^p.$$

By Lemma 4.12 and by normalizing Lemma 3.5 with $\lambda = \sqrt{2r^n}$, for any $\eta > 0$, there exists a small $\delta\eta$ and a weak solution $v_\lambda \in W^1_p(\Omega_r)$ of

$$\begin{cases} \operatorname{div} \bar{a}_{\lambda,B_6^+}(\nabla v_\lambda) = 0 & \text{in} \quad B_6^+, \\ v_\lambda = 0 \quad \text{on} & \partial B_6^+. \end{cases}$$

We have

$$\int_{\Omega_2} |\nabla(u_\lambda - v_\lambda)(x)|^p dx \leqslant \eta^p, \tag{4.21}$$

where v_λ is extended by zero to Ω_r. We choose n_1 and n_0, so that

$$\{x \in \Omega_1 : M(|\nabla u_\lambda|^p)(x) > n_1^p\} \subset \{x \in \Omega_1 : M(|\nabla(u_\lambda - v_\lambda|^p)(x) > n_0^p)\}. \tag{4.22}$$

By the definition of the maximal function

$$M_{\Omega_r}(|\nabla v_\lambda|^p) = M(\chi_{\Omega_r}|\nabla v_\lambda|^p),$$
$$M_{\Omega_r}(|\nabla(u_\lambda - v_\lambda)|^p) = M(\chi_{\Omega_r}|\nabla(u_\lambda - v_\lambda)|^p).$$

We choice $M_{\Omega_r}(|\nabla(u_\lambda - v_\lambda)|^p) \leqslant n_0^2$ such that for $\rho > 0$ and $y \in \Omega_1$ we have

$$\fint_{\Omega_\rho(y)} \chi_{\Omega_r}(x)\,|\nabla(u_\lambda(x) - v_\lambda(x)|^p dx \leqslant n_0^p.$$

If $0<\rho\leqslant 3$, then $\Omega_\rho(y)\subset\Omega_3$ and so

$$\fint_{\Omega_\rho(y)}|\nabla u_\lambda(x)|^p dx\leqslant C\fint_{\Omega_\rho(y)}\chi_{\Omega_r}(x)\left(|\nabla(u_\lambda-v_\lambda)(x)|^p+|\nabla v_\lambda(x)|^p\right)dx\lesssim n_0^2.$$

In case $\rho>3$, then $\Omega_\rho(y)\subset\Omega_{2\rho}(x_0)$,

$$\begin{aligned}\fint_{\Omega_\rho(y)}|\nabla u_\lambda(x)|^p dx&\leqslant\frac{|\Omega_{2\rho}|}{|\Omega_\rho|}\fint_{\Omega_{2\rho}(x_0)}|\nabla u_\lambda(x)|^p dx\\&\leqslant\frac{1}{\lambda^2}\frac{|B_{2\rho}|}{B_\rho^+}\fint_{\Omega_\rho(y)}|\nabla u(x)|^p dx\lesssim 1,\end{aligned}$$

by (4.20) and (4.18). Thus $y\in\Omega_1$ such that $M(|\nabla u_\lambda|^p)\leqslant n_1^2$. So we get $c_0\eta^2<\varepsilon|B_1|$. Thus lemma is proved. $\square$

THEOREM 4.5. *Assume $u\in\mathring{W}_p^1(\Omega)$ is the weak solution of the problem* (3.1) *and $a(x,\xi)$ satisfies* (3.2)–(3.4) *and the (δ,N)-BMO condition. Let N be a sufficiently big number, Ω be the (δ,N)-Reifenberg flat domain and let*

$$|\{x\in\Omega: M(|\nabla u|^p)(x)>c_3^p\}\cap B_r(y)|\geqslant\varepsilon|B_2(y)|. \tag{4.23}$$

Then for each $y\in\Omega$ and small $r>0$ we get

$$\Omega\cap B_r(y)\subset\{x\in\Omega: M(|\nabla u|^p)(x)>1\}\cup\{x\in\Omega: M(|f|^p)(x)>\delta^p\}. \tag{4.24}$$

Proof. Let fix $y\in\Omega$ and $0<r<1$. From Theorem 4.4 we get $B_{8r}(y)\subset\Omega$. Now consider the case that there is a boundary point $y^0\in\partial\Omega$ such that $y^0\in B_{8r}(y)$. We assume contradiction. Then there exists a $B_r(y)$ in which conditions (4.23) and (4.24) does not satisfy. Then for a point $x_0\in\Omega\cap B_r(y)$ and for all $\rho>0$ we obtain

$$\fint_{\Omega\cap B_\rho(x_0)}|\nabla u(x)|^p dx\leqslant 1,\quad \fint_{\Omega\cap B_\rho(x_0)}|f(x)|^p dx\leqslant\delta^p. \tag{4.25}$$

Then for the point $x_0\in\Omega\cap B_{9r}(y^0)$ with $y^0\in\partial\Omega$ we have

$$x_0\in\Omega\cap B_r(y)\subset\Omega\cap B_{9r}(y^0). \tag{4.26}$$

Recall that Ω is the (δ,N)- Reifenberg flat and $y^0\in\partial\Omega$. Then there exists a coordinate system $\{z_1,\dots,z_n\}$ such that

$$y^0+\delta r z_n=0,\quad y=\bar z,\quad x_0=z_0,\quad \Omega\cap B_{9r}(\bar z)\subset\Omega\cap B_{10r}(0) \tag{4.27}$$

and

$$B_{N_r}^+(0)\subset B_{N_r}\cap\Omega\subset B_{N_r}(0)\cap\{z_n>-2N\delta r\}. \tag{4.28}$$

By (4.25)–(4.28) we conclude

$$\left|\{z\in\Omega: M(|\nabla u|^p)(x)>c_3^p\}\cap B_{10r}(0)\right|<(\varepsilon/10^n)\left|B_{10r}\right|=\varepsilon|B_r|.$$

But (4.27) implies

$$\left|\{z\in\Omega: M(|\nabla u|^p)(x)>c_p^3\}\cap B_r(\bar z)\right|<\varepsilon\left|B_r\right|.$$

This is contradiction to (4.23). Thus, the proof is complete. □

Now we ready to prove Theorem 4.3.

The proof Theorem 4.3. By a scaling Lemma 3.4 we take $R = N$ and take λ large so that

$$\left|\left\{x \in \Omega : M(|\nabla u_\lambda|^p)(x) > c_p^3\right\}\right| < \varepsilon\left|B_1\right|. \tag{4.29}$$

From Theorem 4.5 by induction we have for all positive integer k.

$$\begin{aligned}|\{x \in \Omega : M(|\nabla u|^p)(x) > c_3^{pk}\}| < &\sum_{i=1}^{k} \varepsilon_1^i |\{x \in \Omega : M(|f|^p)(x) > \delta^p C_3^{2(k-i)}\}| \\ &+ \varepsilon_1^k |\{x \in \Omega : M(|\nabla u|^p)(x) > 1\}|.\end{aligned} \tag{4.30}$$

From (4.30) we have

$$\begin{aligned}\sum_{k=1}^{\infty} c_3^{pk} |x \in \Omega : M(|\nabla u_\lambda|^p)(x) > c_3^{pk}| &\lesssim \|f_\lambda\|_{M_{p,\varphi}(\Omega)} + |\Omega| \sum_{k=1}^{\infty} (c_3^p \varepsilon_1)^k \\ &\lesssim 1 + \|f_\lambda\|_{M_{p,\varphi}(\Omega)} \sum_{k=1}^{\infty} (c_3^p \varepsilon_1)^k\end{aligned}$$

by Lemma 3.4. We choose δ small. Then $c_3^p \varepsilon_1 < 1$ and the series above is summable. Then by Lemma 3.4 and Corollary 2.1 we have

$$\nabla u \in M_{p,\varphi}(\Omega)$$

with the estimate

$$\|\nabla u\|_{M_{p,\varphi}} \leqslant C(1 + \|f\|_{M_{p,\varphi}(\Omega)}),$$

where $C = C(C_0, C_1, C_2, p, n, |\Omega|)$.

Thus, the proof is complete. □

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