

NEW GENERALIZED RIEMANN—LIOUVILLE FRACTIONAL INTEGRAL INEQUALITIES FOR CONVEX FUNCTIONS

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Abstract. In the literature, the right-side of Hermite–Hadamard’s inequality is called trapezoid type inequality. In this paper, we obtain new integral inequalities of trapezoid type for convex functions involving generalized Riemann–Liouville fractional integrals (ψ -Riemann–Liouville fractional integrals). Our obtained inequalities generalize some recent classical integral inequalities and Riemann–Liouville fractional integral inequalities which are established in earlier works.

1. Introduction

A function $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be convex on the interval \mathcal{I} , if the inequality

$$g(\eta x + (1 - \eta)y) \leq \eta g(x) + (1 - \eta)g(y) \quad (1)$$

holds for all $x, y \in \mathcal{I}$ and $\eta \in [0, 1]$. We say that g is concave if $-g$ is convex.

For convex functions (1), many equalities and inequalities have been established by many authors; such as Ostrowski type inequality [7], Hardy type inequality [2], Olsen type inequality [8], Gagliardo-Nirenberg type inequality [28], and midpoint and trapezoid type inequalities [21, 22]. But the most important inequality is the Hermite–Hadamard type inequality [3], which is defined by:

$$\begin{aligned} g\left(\frac{u+v}{2}\right) &\leq \frac{1}{v-u} \int_u^v g(x)dx \\ &\leq \frac{g(u) + g(v)}{2}, \end{aligned} \quad (2)$$

where $g : \mathcal{I} \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be a convex function on \mathcal{I} with $u, v \in \mathcal{I}$ and $u < v$.

A number of mathematicians in the field of applied and pure mathematics have dedicated their efforts to generalize, refine, counterpart, and extend the Hermite–Hadamard inequality (2) for different classes of convex functions and mappings. For more recent results obtained in view of inequality (2); we refer our readers to [3, 32, 11, 4, 19].

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In [13], Sarikaya et al. obtained the following Hermite–Hadamard’s inequalities in fractional integral form:

$$\begin{aligned} g\left(\frac{u+v}{2}\right) &\leq \frac{\Gamma(\mu+2)}{2(v-u)^\mu} [I_{u^+}^\mu g(v) + I_{v^-}^\mu g(u)] \\ &\leq \frac{g(u) + g(v)}{2}, \end{aligned} \quad (3)$$

where $g : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is assumed to be a positive convex function on $[u, v]$, $g \in L_1[u, v]$ with $u < v$, and $I_{u^+}^\mu$ and $I_{v^-}^\mu$ are the left-sided and right-sided Riemann–Liouville fractional integrals of order $\mu > 0$, these are respectively defined as [10]:

$$\begin{aligned} I_{u^+}^\mu g(x) &= \frac{1}{\Gamma(\mu)} \int_u^x (x-t)^{\mu-1} g(t) dt, \quad x > u, \\ I_{v^-}^\mu g(x) &= \frac{1}{\Gamma(\mu)} \int_x^v (t-x)^{\mu-1} g(t) dt, \quad x < v. \end{aligned} \quad (4)$$

If we take $\mu = 1$ in (3) we obtain (2), so it is clear that inequality (3) is a generalization of Hermite–Hadamard inequality (2).

In view of inequality (3), many inequalities have been established for convex [27, 6], h -convex [13], MT -convex [26, 14], m -convex [31], (s, m) -convex [31], (α, m) -convex [24, 29], F -convex [21], invex and preinvex [9, 32, 12] functions. Furthermore, there are many related inequalities for other kinds of convex functions and fractional operators; so it is important that the readers of this article visit [11, 4, 1, 17, 18, 20, 23].

Let us now state the definition of generalized Riemann–Liouville fractional integrals:

DEFINITION 1. Let $(u, v) \subseteq (-\infty, \infty)$ be a finite or infinite interval of the real-axis \mathbb{R} and $\mu > 0$. Let $\psi(x)$ be an increasing and positive monotone function on the interval $(u, v]$ with a continuous derivative $\psi'(x)$ on the interval (u, v) . Then the left and right-sided ψ -Riemann–Liouville fractional integrals of a function g with respect to another function $\psi(x)$ on $[u, v]$ are defined by [30]:

$$\begin{aligned} I_{u^+}^{\mu;\psi} g(x) &= \frac{1}{\Gamma(\mu)} \int_u^x \psi'(t)(\psi(x) - \psi(t))^{\mu-1} g(t) dt, \\ I_{v^-}^{\mu;\psi} g(x) &= \frac{1}{\Gamma(\mu)} \int_x^v \psi'(t)(\psi(t) - \psi(x))^{\mu-1} g(t) dt. \end{aligned} \quad (5)$$

It is important to be noted that if we set $\psi(x) = x$ in (5), then ψ -Riemann–Liouville fractional integral reduces to Riemann–Liouville fractional integral (4).

Based on this definition, we obtain several inequalities of trapezoid type for convex functions in the present paper.

2. The main results

Our main results depend on the following lemma:

LEMMA 1. Let $g : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function and $g'' \in L_1[u, v]$ with $0 \leq u < v$. If $\psi(x)$ is an increasing and positive monotone function on $(u, v]$ and its derivative $\psi'(x)$ is continuous on (u, v) , then for $t_1, t_2, \mu \in (0, 1)$ we have

$$\begin{aligned} \sigma_{\mu, \psi}(g; u, v) &= \frac{1}{2(\mu + 1)(v - u)^\mu} \\ &\times \left(\int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \left[(v - u)^\mu (v - \psi(t_1)) - (v - \psi(t_1))^{\mu+1} \right] \psi'(t_1) (g'' \circ \psi)(t_1) dt_1 \right. \\ &\left. + \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \left[(v - u)^\mu (\psi(t_2) - u) - (\psi(t_2) - u)^{\mu+1} \right] \psi'(t_2) (g'' \circ \psi)(t_2) dt_2 \right), \quad (6) \end{aligned}$$

where

$$\begin{aligned} \sigma_{\mu, \psi}(g; u, v) &= \frac{g(u) + g(v)}{2} \\ &- \frac{\Gamma(\mu + 1)}{2(v - u)^\mu} \left[I_{u^+}^{\mu; \psi}(g \circ \psi)(\psi^{-1}(v)) + I_{v^-}^{\mu; \psi}(g \circ \psi)(\psi^{-1}(u)) \right]. \end{aligned}$$

Proof. From Definition 1 we have

$$\begin{aligned} \bar{h}_1 &:= \frac{\Gamma(\mu + 1)}{2(v - u)^\mu} I_{u^+}^{\mu; \psi}(g \circ \psi)(\psi^{-1}(v)) \\ &= \frac{\mu}{2(v - u)^\mu} \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \psi'(t_1) (v - \psi(t_1))^{\mu-1} (g \circ \psi)(t_1) dt_1 \\ &= -\frac{1}{2(v - u)^\mu} \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} (g \circ \psi)(t_1) d(v - \psi(t_1))^\mu. \end{aligned}$$

Integrating by parts twice, we have

$$\begin{aligned} \bar{h}_1 &= \frac{1}{2}g(u) + \frac{1}{2(\mu + 1)(v - u)^\mu} \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \psi'(t_1) (v - \psi(t_1))^\mu (g' \circ \psi)(t_1) dt_1 \\ &= \frac{1}{2}g(u) + \frac{v - u}{2(\mu + 1)} g'(u) \\ &\quad + \frac{1}{2(\mu + 1)(v - u)^\mu} \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \psi'(t_1) (v - \psi(t_1))^{\mu+1} (g'' \circ \psi)(t_1) dt_1. \quad (7) \end{aligned}$$

Similarly,

$$\begin{aligned} \bar{h}_2 &:= \frac{\Gamma(\mu+1)}{2(v-u)^\mu} I_{v^-}^{\mu;\psi}(g \circ \psi)(\psi^{-1}(u)) \\ &= \frac{1}{2}g(v) - \frac{v-u}{2(\mu+1)}g'(v) \\ &\quad + \frac{1}{2(\mu+1)(v-u)^\mu} \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \psi'(t_2)(\psi(t_2)-u)^{\mu+1}(g'' \circ \psi)(t_2)dt_2. \end{aligned} \quad (8)$$

Integrating by parts, we have

$$\begin{aligned} \bar{h}_3 &:= \frac{1}{2(\mu+1)} \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \psi'(t_1)(v-\psi(t_1))(g'' \circ \psi)(t_1)dt_1 \\ &= -\frac{v-u}{2(\mu+1)}g'(u) + \frac{g(v)-g(u)}{2(\mu+1)}, \end{aligned} \quad (9)$$

and

$$\begin{aligned} \bar{h}_4 &:= \frac{1}{2(\mu+1)} \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \psi'(t_2)(\psi(t_2)-u)(g'' \circ \psi)(t_2)dt_2 \\ &= \frac{v-u}{2(\mu+1)}g'(v) - \frac{g(v)-g(u)}{2(\mu+1)}. \end{aligned} \quad (10)$$

It follows from (7)–(10) that

$$\begin{aligned} &\frac{g(u)+g(v)}{2} - (\bar{h}_1 + \bar{h}_2 + \bar{h}_3 + \bar{h}_4) \\ &= \frac{1}{2(\mu+1)(v-u)^\mu} \left(\int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \left[(v-u)^\mu(v-\psi(t_1)) - (v-\psi(t_1))^{\mu+1} \right] \psi'(t_1)(g'' \circ \psi)(t_1)dt_1 \right. \\ &\quad \left. + \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \left[(v-u)^\mu(\psi(t_2)-u) - (\psi(t_2)-u)^{\mu+1} \right] \psi'(t_2)(g'' \circ \psi)(t_2)dt_2 \right), \end{aligned}$$

and this completes the proof of Lemma 1. \square

REMARK 1. By letting $t_1 = t_2 = t$, then Lemma 1 becomes

$$\begin{aligned} \sigma_{\mu,\psi}(g;u,v) &= \frac{1}{2(\mu+1)(v-u)^\mu} \\ &\quad \times \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \left[(v-u)^{\mu+1} - (v-\psi(t))^{\mu+1} - (\psi(t)-u)^{\mu+1} \right] \psi'(t)(g'' \circ \psi)(t)dt. \end{aligned} \quad (11)$$

COROLLARY 1. *With the similar assumptions of Lemma 1 if $\psi(x) = x$, we have*

$$\begin{aligned} & \frac{g(u) + g(v)}{2} - \frac{\Gamma(\mu + 1)}{2(v - u)^\mu} [I_{u^+}^\mu g(v) + I_{v^-}^\mu g(u)] \\ &= \frac{(v - u)^2}{2(\mu + 1)} \int_0^1 t(1 - t)^\mu \left[g''(tu + (1 - t)v) + g''((1 - t)u + tv) \right] dt, \end{aligned}$$

which is obtained by Dragomir et al. [5].

THEOREM 1. *Let $g : [u, v] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function and $g'' \in L_1[u, v]$ with $0 \leq u < v$. Suppose that $|g''|$ is convex on $[u, v]$, $\psi(x)$ is an increasing and positive monotone function on $(u, v]$ and its derivative $\psi'(x)$ is continuous on (u, v) , then for $\mu \in (0, 1)$ we have*

$$\begin{aligned} |\sigma_{\mu, \psi}(g; u, v)| &\leq \frac{(v - u)^2}{2(\mu + 1)(\mu + 2)} \left(\frac{|g''(u)| + |g''(v)|}{2} \right) \\ &\leq \frac{(v - u)^2}{\mu + 1} \mathcal{B}(2, \mu + 1) \left(\frac{|g''(u)| + |g''(v)|}{2} \right), \end{aligned} \tag{12}$$

where \mathcal{B} denotes the well-known Euler Beta function.

Proof. By using Lemma 1 with $x_1 = \frac{v - \psi(t_1)}{v - u}$ and $x_2 = \frac{\psi(t_2) - u}{v - u}$, we get

$$\begin{aligned} \sigma_{\mu, \psi}(g; u, v) &= \frac{(v - u)^2}{2(\mu + 1)} \left(\int_0^1 (x_1 - x_1^{\mu+1}) g''(x_1 u + (1 - x_1)v) dx_1 \right. \\ &\quad \left. + \int_0^1 (x_2 - x_2^{\mu+1}) g''((1 - x_2)u + x_2 v) dx_2 \right). \end{aligned} \tag{13}$$

Let $t = x_1 = x_2$, then (13) becomes

$$\begin{aligned} & \sigma_{\mu, \psi}(g; u, v) \\ &= \frac{(v - u)^2}{2(\mu + 1)} \left[\int_0^1 (t - t^{\mu+1}) \left(g''(tu + (1 - t)v) + g''((1 - t)u + tv) \right) dt \right]. \end{aligned} \tag{14}$$

Using properties of modulus and the convexity of $|g''|$, we get

$$\begin{aligned} |\sigma_{\mu, \psi}(g; u, v)| &\leq \frac{(v - u)^2}{2(\mu + 1)} \left[\int_0^1 |t - t^{\mu+1}| \left(|g''(tu + (1 - t)v)| + |g''((1 - t)u + tv)| \right) dt \right] \\ &\leq \frac{(v - u)^2}{2(\mu + 1)} \left(|g''(u)| \int_0^1 t(t - t^{\mu+1}) dt + |g''(v)| \int_0^1 (1 - t)(t - t^{\mu+1}) dt \right. \\ &\quad \left. + |g''(v)| \int_0^1 (t - t^{\mu+1}) dt + |g''(u)| \int_0^1 (1 - t)(t - t^{\mu+1}) dt \right) \\ &= \frac{(v - u)^2}{2(\mu + 1)(\mu + 2)} \left(\frac{|g''(u)| + |g''(v)|}{2} \right), \end{aligned}$$

which proves the first inequality of (12).

We know for each $x_1, x_2 \in [0, 1]$ and $\mu \in (0, 1)$ that $|x_1^\mu + x_2^\mu| \leq |x_1 + x_2|^\mu$, so we have

$$\int_0^1 t^2(1-t^\mu) dt + \int_0^1 t(1-t)(1-t^\mu) dt \leq \int_0^1 t(1-t)^\mu dt = \mathcal{B}(2, \mu + 1),$$

which proves the second inequality of (12). Thus the proof of Theorem 1 is completed. \square

COROLLARY 2. *With the similar assumptions of Theorem 1*

1. *if $\psi(x) = x$, we have*

$$\begin{aligned} \left| \frac{g(u) + g(v)}{2} - \frac{\Gamma(\mu + 2)}{2(v-u)^\mu} [I_{u^+}^\mu g(v) + I_{v^-}^\mu g(u)] \right| \\ \leq \frac{(v-u)^2}{2(\mu+1)(\mu+2)} \left(\frac{|g''(u)| + |g''(v)|}{2} \right) \\ \leq \frac{(v-u)^2}{\mu+1} \mathcal{B}(2, \mu+1) \left(\frac{|g''(u)| + |g''(v)|}{2} \right), \end{aligned}$$

this is obtained by Dragomir et al. [5].

2. *if $\psi(x) = x$ and $\mu = 1$, we have*

$$\left| \frac{g(u) + g(v)}{2} - \frac{1}{v-u} \int_u^v g(x) dx \right| \leq \frac{(v-u)^2}{12} \left(\frac{|g''(u)| + |g''(v)|}{2} \right),$$

which is obtained by Ozdemir et al. [25].

REMARK 2. Let the assumptions of Theorem 1 be hold

1. for $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, we have

$$|\sigma_{\mu, \psi}(g; u, v)| \leq \frac{(v-u)^2}{\mu+1} \mathcal{B}^{\frac{1}{p}}(p+1, \mu p+1) \left(\frac{|g''(u)|^q + |g''(v)|^q}{2} \right)^{\frac{1}{q}},$$

such that $|g''|^q$ is convex function on $[u, v]$.

2. for $q > 1$, we have

$$\begin{aligned} |\sigma_{\mu, \psi}(g; u, v)| \leq \frac{\mu(v-u)^2}{4(\mu+1)(\mu+2)} \left[\left(\frac{2\mu+4}{3\mu+9} |g''(u)|^q + \frac{\mu+5}{3\mu+9} |g''(v)|^q \right)^{\frac{1}{q}} \right. \\ \left. + \left(\frac{\mu+5}{3\mu+9} |g''(u)|^q + \frac{2\mu+4}{3\mu+9} |g''(v)|^q \right)^{\frac{1}{q}} \right], \end{aligned}$$

such that $|g''|^q$ is a convex function on $[u, v]$.

3. for $\frac{1}{p} + \frac{1}{q} = 1, p > 1$

$$|\sigma_{\mu,\psi}(g; u, v)| \leq \frac{(v-u)^2}{\mu+1} \mathcal{B}^{\frac{1}{p}}(p+1, \mu p+1) \left| g'' \left(\frac{a+b}{2} \right) \right|,$$

such that $|g''|^q$ is concave function on $[u, v]$.

4. for $\frac{1}{p} + \frac{1}{q} = 1, p \geq 1$

$$\begin{aligned} & |\sigma_{\mu,\psi}(g; u, v)| \\ & \leq \frac{\mu(v-u)^2}{4(\mu+1)(\mu+2)} \left[\left| g'' \left(\frac{2\mu+4}{3\mu+9}a + \frac{\mu+5}{3\mu+9}b \right) \right| + \left| g'' \left(\frac{\mu+5}{3\mu+9}a + \frac{2\mu+4}{3\mu+9}b \right) \right| \right], \end{aligned}$$

such that $|g''|^q$ is concave function on $[u, v]$.

Proof. By making use of the equality (14) and the same method used for Theorem 3–6 of [5], we can easily obtain the items 1–4, respectively. \square

THEOREM 2. *With the assumptions of Theorem 1, we have*

$$\begin{aligned} |\sigma_{\mu,\psi}(g; u, v)| & \leq \frac{(v-u)|g'(v) - g'(u)|}{2(\mu+1)} \\ & + \frac{(v-u)^2}{2(\mu+1)} \left(\frac{\mu+4}{(\mu+2)(\mu+3)} |g''(u)| + \frac{2\mu+5}{(\mu+2)(\mu+3)} |g''(v)| \right). \end{aligned} \tag{15}$$

Proof. Making use of the equality (11), we have

$$\begin{aligned} |\sigma_{\mu,\psi}(g; u, v)| & \leq \frac{1}{2(\mu+1)(v-u)^\mu} \left| \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} (v-u)^{\mu+1} \psi'(t) (g'' \circ \psi)(t) dt \right| \\ & + \frac{1}{2(\mu+1)(v-u)^\mu} \left| \int_{\psi^{-1}(u)}^{\psi^{-1}(v)} \left[(v-\psi(t))^{\mu+1} + (\psi(t)-u)^{\mu+1} \right] \psi'(t) (g'' \circ \psi)(t) dt \right| \\ & := \omega_1 + \omega_2. \end{aligned} \tag{16}$$

By using a simple integrating by parts, we get

$$\omega_1 = \frac{(v-u)|g'(v) - g'(u)|}{2(\mu+1)}. \tag{17}$$

By changing the variable $x = \frac{v-\psi(t)}{v-u}$ and convexity of $|g''|$, we have

$$\begin{aligned} \omega_2 & = \frac{(v-u)^2}{2(\mu+1)} \int_0^1 (x^{\mu+1} + (1-x)^{\mu+1}) |g''(xa + (1-x)b)| dx \\ & = \frac{(v-u)^2}{2(\mu+1)} \left(\frac{\mu+4}{(\mu+2)(\mu+3)} |g''(u)| + \frac{2\mu+5}{(\mu+2)(\mu+3)} |g''(v)| \right). \end{aligned} \tag{18}$$

Hence, the inequalities (16)–(18) rearrange proof of Theorem 2. \square

3. Conclusion

In this paper, we established some new inequalities of trapezoid type for convex functions involving ψ -Riemann–Liouville fractional integrals. Corollary 1–2 and Remark 2 show that our results generalize and extend the obtained results in [5, 25].

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