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A NECESSARY AND SUFFICIENT CONDITION FOR THE CONVEXITY OF THE ONE–PARAMETER GENERALIZED INVERSE TRIGONOMETRIC SINE FUNCTION ACCORDING TO POWER MEAN

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Abstract. In the article, we present a necessary and sufficient condition such that the one-parameter generalized inverse trigonometric sine function is convex with respect to power mean. As a consequence, we provide the necessary and sufficient condition for the concavity of the one-parameter generalized trigonometric sine function according to power mean.

1. Introduction

Let λ, $T>0$ and p, $q>1$. Then the well-known generalized Dirichlet problem for Laplacian equation on the interval $(0,T)$ or (p,q)-eigenvalue problem with Dirichlet boundary condition and eigenvalue λ along with eigenfunction $u(t)$ is given by

$$\begin{cases} \left(|u'|^{p-2}u'\right)' + \lambda |u|^{q-2}u = 0, & t \in (0,T), \\ u(0) = u(T) = 0. \end{cases}$$

The complete solution for the above problem was given independent by Drábek and Manásevich in [28], and Takeuchi in [59].

Let $T=\pi_{p,q}$. Then the eigenvalue $\lambda = p(p-1)/q$ and the corresponding eigenfunction $u(t) = \sin_{p,q}(t)$, where $\sin_{p,q}$ is the two-parameter generalized trigonometric sine function and

$$\pi_{p,q} = 2\int_0^1 (1-t^q)^{-1/p} dt.$$

In particular, if $p=q=2$, then $\sin_{p,q}$ and $\pi_{p,q}$ reduce to the classical trigonometric sine function $\sin$ and the circumference ratio π, respectively.

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An alternative equivalent definition for $\sin_{p,q}$ can be obtained by the following integral

$$\arcsin_{p,q}(x) = \int_0^x (1-t^q)^{-1/p} dt \quad (p,q>1, x\in[0,1]).$$

We denote its inverse function by $\sin_{p,q}$ defined on $[0,\pi_{p,q}/2]$ due to the function $x\mapsto \arcsin_{p,q}(x)$ is strictly increasing from $[0,1]$ onto $[0,\pi_{p,q}/2]$. By defining $\sin_{p,q}(x) = \sin_{p,q}(\pi_{p,q}-x)$ for $x\in[\pi_{p,q}/2,\pi_{p,q}]$, the function $\sin_{p,q}$ can be extended to $[0,\pi_{p,q}]$. Then one can further extends $\sin_{p,q}$ to $[-\pi_{p,q},\pi_{p,q}]$ as an odd function, and finally to $\mathbb{R}$ by $2\pi_{p,q}$-periodicity.

In [45, 46], Lindqvist and Peetre investigated the one-parameter generalized inverse trigonometric sine function

$$\arcsin_p^*(x) = \arcsin_{\frac{p}{p-1},p}(x) = \int_0^x (1-t^p)^{-\frac{p-1}{p}} dt \quad (p>1, x\in[0,1]) \tag{1.1}$$

and the generalized circumference ratio

$$\pi_p^* = 2\arcsin_p^*(1) = 2\int_0^1 (1-t^p)^{-\frac{p-1}{p}} dt, \tag{1.2}$$

and thereby obtained the one-parameter generalized trigonometric sine function $\sin_p^*$ defined on $[0,\pi_p^*/2]$ as the inverse function of $\arcsin_p^*$. Using the similar extension procedures as $\sin_{p,q}$, the function $\sin_p^*$ can be defined on $\mathbb{R}$.

Recently, the one-parameter generalized trigonometric sine function $\sin_p^*$ has attracted the attention of many researchers [14, 29, 45, 46]. Lindqvist and Peetre [46] proved that the area closed by the p-circle

$$|x|^p + |y|^p = R^p$$

is $\pi_p^* R^2$ and the q-length (l_q metric) of p-circle is $2\pi_p^* R$. In particular, Edmunds, Gurka and Lang [29], and Bakşi, Gurka and Lang [14] gave many important and basis properties for the one-parameter generalized trigonometric sine function $\sin_p^*$. More properties for $\sin_p^*$, $\sin_{p,q}$ and other generalizations for the trigonometric function and their applications, we recommend the literature [18, 19, 21, 22, 28, 36, 42, 43, 44, 47, 57, 59] to the readers.

It is well-known that convexity is an indispensable tool in inequality theory [6, 32, 58, 62, 63, 72, 74]. Recently, the generalizations, variants and extensions for the convexity have been the subject of intensive research, for example, the s-convexity [4, 51], m-convexity [70], (s,m)-convexity [2, 31], h-convexity [69], p-convexity [3], ρ-convexity [10], tgs-convexity [30], η-convexity [41], harmonic convexity [1, 9], GG- and GA-convexities [40], prevexity [13, 37] and exponential convexity [48, 50]. In particular, many inequalities can be found in the literature [5, 11, 12, 23, 33, 34, 35, 39, 49, 52, 53, 54, 55, 56, 64, 68, 75] via the convexity theory.

Let $p\in\mathbb{R}$ and $x,y>0$. Then the pth power (Hölder) mean $M_p(x,y)$ of x and y

[24, 27, 66, 73] is defined by

$$M_p(x,y)=\begin{cases}\left(\dfrac{x^p+y^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{xy}, & p=0.\end{cases}$$

Let $I\subseteq(0,\infty)$ be an interval and $f:I\to(0,\infty)$ be a continuous function. Then f is said to be $M_{a,b}$-convex (concave) on I if the inequality

$$f(M_a(x,y))\leqslant(\geqslant)M_b(f(x),f(y)) \tag{1.3}$$

holds for all $x,y\in I$, and f is strictly $M_{a,b}$-convex (concave) if inequality (1.3) is strict except for $x=y$.

In the past few years, numerous authors have studied the $M_{a,b}$-convexity (concavity) properties for the special functions in geometric function theory including generalized trigonometric functions [16, 17, 20, 36, 38], Jacobian sine function [60], Gaussian hypergeometric functions [8, 15, 71], elliptic integrals [25, 26, 61, 65, 67]. For example, in 2012, Bhayo and Vuorinen [20] conjectured that the function $\sin_{p,q}(x)$ is $M_{0,0}$-convex on $(0,1)$ for $p,q\in(1,+\infty)$. Later, Jiang et al. [36] gave a positive answer to this conjecture. In 2013, Baricz, Bhayo and Klén [16] proved that the function $\arcsin_{p,q}(x)$ is $M_{a,b}$-convex on $(0,1)$ if $(a,b)\in\{a,b|a\leqslant 0,b\in\mathbb{R}\}\cup\{a,b|0<a\leqslant b,b\leqslant 1\}$. Recently, Bhayo [19] proved that $\arcsin_{p,p}(x)$ is $M_{a,a}$-convex and $\sin_{p,p}(x)$ is $M_{a,a}$-concave on $(0,1)$ if $p>1$ and $a\geqslant 0$.

The main purpose of the article is to give the maximal regions in the (a,b)-plane such that the function $\arcsin_p^*(x)$ is $M_{a,b}$-convex or $M_{a,b}$-concave on $(0,1)$. As a corollary, a necessary and sufficient condition for the $M_{a,b}$-concavity of the one-parameter generalized trigonometric sine function $\sin_p^*(x)$ is also derived. Our main results are the Theorem 1.1 and Corollary 1.2 as follows.

THEOREM 1.1. *Let $p\in(1,+\infty)$. Then the one-parameter generalized inverse trigonometric sine function $\arcsin_p^*(x)$ is $M_{a,b}$-convex on $(0,1)$ if and only if*

$$(b,a)\in D=\{(b,a)|a\leqslant 1+L(b)\},$$

where

$$L(b)=\inf_{x\in(0,1)}\left[(b-1)\frac{x}{\arcsin_p^*(x)(1-x^p)^{1-1/p}}+(p-1)\frac{x^p}{1-x^p}\right]$$

is a continuous function with $L(b)=b-1$ if $b\geqslant -p$, and $L(b)<b-1$ if $b<-p$. Moreover, there does not exist $(a,b)\in\mathbb{R}^2$ such that $\arcsin_p^*(x)$ is $M_{a,b}$-concave on $(0,1)$.

COROLLARY 1.2. *Let $p\in(1,+\infty)$. Then the one-parameter generalized trigonometric sine function $\sin_p^*(x)$ is $M_{a,b}$-concave on $(0,\pi_p^*/2)$ if and only if*

$$(a,b)\in D^*=\{(a,b)|b\leqslant 1+L(a)\},$$

where the function $L(\cdot)$ is defined as in Theorem 1.1. Moreover, there does not exist $(a,b)\in\mathbb{R}^2$ such that $\sin_p^*(x)$ is $M_{a,b}$-convex on $(0,\pi_p^*/2)$.

2. Lemmas

In order to prove our main results Theorem 1.1 and Corollary 1.2, we need several lemmas which we present in this section.

LEMMA 2.1. [7, Theorem 1.25, l'Hôptial Monotone Rule] *Let $a,b \in \mathbb{R}$ with $a<b$, $f,g:[a,b]\to\mathbb{R}$ be two continuous functions and differentiable on (a,b) such that $g'(x)\neq 0$ for each $x\in(a,b)$. Then both the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}$$

are (strictly) increasing (decreasing) on (a,b) if $f'(x)/g'(x)$ is (strictly) increasing (decreasing) on (a,b).

LEMMA 2.2. *Let $p\in(1,+\infty)$. Then the function $x\to f(x)=\mathrm{arcsin}_p^*(x)/x$ is strictly increasing from $(0,1)$ onto $(1,\pi_p^*/2)$.*

Proof. Let $f_1(x)=\mathrm{arcsin}_p^*(x)$ and $f_2(x)=x$. Then $f_1(0)=f_2(0)=0$, and $f_1'(x)/f_2'(x)=(1-x^p)^{1/p-1}$ is strictly increasing on $(0,1)$. It follows from Lemma 2.1 that $f(x)$ is also strictly increasing on $(0,1)$. Moreover, by l'Hôptial's Rule, we have $\lim_{x\to 0^+}f(x)=\lim_{x\to 0^+}f_1'(x)/f_2'(x)=1$, $\lim_{x\to 1^-}f(x)=\pi_p^*/2$. □

LEMMA 2.3. *The following statements are true:*

(1) *If $p\in(1,+\infty)$, then the function $F_p(x)=[1+(p-2)x^p]/(1-x^p)^{1/p}$ is strictly increasing from $(0,1)$ onto $(1,+\infty)$;*

(2) *If $p\in(1,2]$, then the function $G_p(x)=[2(p-2)x^{2p}-2(p-2)x^p+(p-1)]/(1-x^p)^{2/p}$ is strictly increasing from $(0,1)$ onto $(p-1,+\infty)$;*

(3) *If $p\in(2,+\infty)$, then the function $H_p(x)=-(p-2)(p-4)x^{2p}+(p^2-5p+8)x^p+(p-2)$ is positive on $(0,1)$.*

Proof. (1) By differentiation, we have

$$F_p'(x)=\frac{(p-1)x^{p-1}[-(p-2)x^p+(p-1)]}{(1-x^p)^{1+1/p}}.$$

Since $x\mapsto -(p-2)x^p+(p-1)$ is strictly monotone on $(0,1)$, $\lim_{x\to 0^+}[-(p-2)x^p+(p-1)]=(p-1)>0$ and $\lim_{x\to 1^-}[-(p-2)x^p+(p-1)]=1>0$, we know that $F_p'(x)>0$ for all $x\in(0,1)$ and $F_p(x)$ is strictly increasing on $(0,1)$. Moreover, $\lim_{x\to 0^+}F_p(x)=1$, $\lim_{x\to 1^-}F_p(x)=+\infty$.

(2) If $p=2$, then $G_p(x)=G_2(x)=1/(1-x^2)$ is strictly increasing from $(0,1)$ onto $(1,+\infty)$. In the case of $p\in(1,2)$, differentiating G_p yields

$$G_p'(x)=2x^{p-1}(1-x^p)^{-2/p-1}[2(1-p)(p-2)x^{2p}+(3p-2)(p-2)x^p+(-p^2+3p-1)].$$

Due to $(1-p)(p-2)>0$ and $(3p-2)/[4(p-1)]>1$ for $p\in(1,2)$, one has

$$\begin{aligned}&\inf_{x\in(0,1)}[2(1-p)(p-2)x^{2p}+(3p-2)(p-2)x^{p}+(-p^2+3p-1)]\\=&2(1-p)(p-2)+(3p-2)(p-2)+(-p^2+3p-1)=p-1>0.\end{aligned}$$

Consequently, $G_p'(x)>0$ for all $x\in(0,1)$, so that $G_p(x)$ is strictly increasing on $(0,1)$. Moreover, $\lim_{x\to 0^+}G_p(x)=p-1$, $\lim_{x\to 1^-}G_p(x)=+\infty$.

(3) By differentiation, we have

$$H_p'(x)=-2p(p-2)(p-4)x^{2p-1}+p(p^2-5p+8)x^{p-1}=px^{p-1}h_p(x),$$

where

$$h_p(x)=-2(p-2)(p-4)x^p+p^2-5p+8.$$

Note that $h_p(x)$ is monotone on $(0,1)$ and $h_p(0)=p^2-5p+8>0$ for $p>2$, we get $h_p(x)>0$ for all $x\in(0,1)$, or there exists $x_0\in(0,1)$ such that $h_p(x)>0$ for $x\in(0,x_0)$ and $h_p(x)<0$ for $x\in(x_0,1)$. Hence $H_p'(x)>0$ for all $x\in(0,1)$, or there exists $x_0\in(0,1)$ such that $H_p'(x)>0$ for $x\in(0,x_0)$ and $H_p'(x)<0$ for $x\in(x_0,1)$, so that $H_p(x)$ is strictly increasing on $(0,1)$, or is first increasing then decreasing on $(0,1)$. Since $H_p(0)=p-2>0$ and $H_p(1)=-(p-2)(p-4)+p^2-5p+8+(p-2)=2(p-1)>0$ for $p>2$, hence $H_p(x)$ is positive on $(0,1)$. □

LEMMA 2.4. *Let $p\in(1,+\infty)$ and $\omega(x)$ be defined by*

$$\begin{aligned}\omega(x)=&\left[\frac{\operatorname{arcsin}_p^*(x)}{x}\right]^2(1-x^p)^{-2/p}[2(p-2)x^{2p}-2(p-2)x^p+(p-1)]\\&+2\left[\frac{\operatorname{arcsin}_p^*(x)}{x}\right](1-x^p)^{-1/p}[1+(p-2)x^p].\end{aligned}$$

Then $\omega(x)$ is strictly increasing from $(0,1)$ onto $(p+1,+\infty)$.

Proof. If $p\in(1,2]$, then it follows from Lemmas 2.1-2.3 that $\omega(x)$ is strictly increasing on $(0,1)$. In the remanning case that $p\in(2,+\infty)$, differentiating $\omega(x)$ and multiplying both sides by $x^3(1-x^p)^{1+2/p}/[2\operatorname{arcsin}_p^*(x)^2]$ gives

$$\begin{aligned}&\frac{x^3}{2}(1-x^p)^{1+2/p}\frac{\omega'(x)}{[\operatorname{arcsin}_p^*(x)]^2}\\=&\left[\frac{x}{\operatorname{arcsin}_p^*(x)}\right]^2(1-x^p)^{2/p}[1+(p-2)x^p]\\&+\left[\frac{x}{\operatorname{arcsin}_p^*(x)}\right](1-x^p)^{1/p}\big[-(p-2)(p-4)x^{2p}\\&+(p^2-5p+8)x^p+(p-2)\big]-2(p-2)^2x^{3p}+3(p-2)^2x^{2p}\\&-(p^2-6p+6)x^p-(p-1).\end{aligned}$$

It was proved in [20, Theorem 1.1] that the inequality $\frac{x}{\operatorname{arcsin}_p^*(x)} > (1-x^p)^{(p-1)/[p(1+p)]}$ holds for all $x \in (0,1)$. Hence, by Lemma 2.3(3), it is sufficient to prove that

$$\begin{aligned}\omega_1(x) \equiv & -2(p-2)^2x^{3p}+3(p-2)^2x^{2p}-(p^2-6p+6)x^p-(p-1)\\ &+(1-x^p)^{\frac{2}{1+p}}[-(p-2)(p-4)x^{2p}+(p^2-5p+8)x^p+(p-2)]\\ &+(1-x^p)^{\frac{4}{1+p}}[(p-2)x^p+1] \end{aligned} \tag{2.1}$$

is positive for all $x \in (0,1)$.

It follows from

$$(1-x)^\alpha = \sum_{n=0}^{\infty}\frac{(-\alpha,n)}{n!}x^n > 1-\alpha x-\frac{\alpha}{2}(1-\alpha)x^2-\frac{(1-\alpha)(2-\alpha)}{2}x^3$$

for $\alpha \in (0,1)$ that

$$(1-x^p)^{\frac{2}{1+p}} > 1-\frac{2}{1+p}x^p-\frac{p-1}{(1+p)^2}x^{2p}-\frac{p(p-1)}{(1+p)^2}x^{3p}. \tag{2.2}$$

Using inequality (2.2) in (2.1) we obtain

$$\begin{aligned}(p+1)^4x^{-p}\omega_1(x) > \omega_2(x) \equiv\; & 2p^2(p+1)^3+p(p+1)^2(2p^3-4p^2-3p+1)x^p\\ &-p(p-1)(p+1)(2p^3-5p+1)x^{2p}\\ &-(p-1)(p^5-2p^4+3p^3-7p+1)x^{3p}\\ &+(p-1)(p^5-4p^4+p^3+9p^2-5p+2)x^{4p}\\ &+p(3p-4)(p-1)^2x^{5p}\\ &+p^2(p-2)(p-1)x^{6p}. \end{aligned} \tag{2.3}$$

Noting that

$$\begin{aligned} & p^5-4p^4+p^3+9p^2-5p+2\\ =&[4-5(p-2)-(p-2)^2]+[9(p-2)^2+6(p-2)^4+(p-2)^5]\\ =&\frac{89}{32}+\frac{65}{15}\left(p-\frac{5}{2}\right)+\frac{91}{4}\left(p-\frac{5}{2}\right)^2+\frac{47}{2}\left(p-\frac{5}{2}\right)^3+\frac{17}{2}\left(p-\frac{5}{2}\right)^4+\left(p-\frac{5}{2}\right)^5. \end{aligned}$$

The first identity shows that $p^5-4p^4+p^3+9p^2-5p+2>0$ for $p \in (2,5/2)$, while the second one implies that $p^5-4p^4+p^3+9p^2-5p+2>0$ for $p \in [5/2,+\infty)$. Thus $p^5-4p^4+p^3+9p^2-5p+2>0$ for all $p \in (2,+\infty)$. This, together with (2.3), leads to the conclusion that

$$\begin{aligned}\omega_2(x) >&2p^2(p+1)^3+p(p+1)^2(2p^3-4p^2-3p+1)x^p-p(p-1)(p+1)\\&\times(2p^3-5p+1)x^{2p}-(p-1)(p^5-2p^4+3p^3-7p+1)x^{3p}\\>&(p+1)^2[2p^2(p+1)+p(2p^3-4p^2-3p+1)]x^p-\big(p-1\big)\\&\times\big[p(p+1)(2p^3-5p+1)+(p^5-2p^4+3p^3-7p+1)\big]x^{2p}\\=&p(p+1)^2(2p^3-2p^2-p+1)x^p-(p-1)(3p^5-2p^3-4p^2-6p+1)x^{2p}\\>&\Big[p(p+1)^2(2p^3-2p^2-p+1)-(p-1)(3p^5-2p^3-4p^2-6p+1)\Big]x^{2p}\\=&(p-1)(5p^5+4p^4-p^3-6p^2-7p+1)x^{2p}>0\end{aligned}$$

for all $x\in(0,1)$. Consequently, $\omega(x)$ is strictly increasing on $(0,1)$. Moreover,

$$\lim_{x\to 0^+}\omega(x)=p+1,\qquad \lim_{x\to 1^-}\omega(x)=+\infty.$$

This completes the proof. □

LEMMA 2.5. *Let $p\in(1,+\infty)$ and $\varphi(x)$ be defined by*

$$\varphi(x)=\frac{p(p-1)[\arcsin_p^*(x)]^2x^{p-1}(1-x^p)^{-1/p}}{\arcsin_p^*(x)[1+(p-2)x^p]-x(1-x^p)^{1/p}}.$$

Then $\varphi(x)$ is strictly increasing from $(0,1)$ onto $(p+1,+\infty)$.

Proof. Let

$$\varphi_1(x)=[p(p-1)[\arcsin_p^*(x)]^2x^{p-1}(1-x^p)^{-1/p}][1+(p-2)x^p]^{-1},$$

$$\varphi_2(x)=\arcsin_p^*(x)-x(1-x^p)^{1/p}[1+(p-2)x^p]^{-1}.$$

Then $\varphi_1(0)=\varphi_2(0)=0$,

$$\begin{aligned}\varphi_1'(x)=&p(p-1)x^{p-2}(1-x^p)^{1/p-1}[1+(p-2)x^p]^{-2}\big\{2x\arcsin_p^*(x)\\&\times(1-x^p)^{-1/p}[1+(p-2)x^p]+\big[\arcsin_p^*(x)\big]^2(1-x^p)^{-2/p}\\&\times[2(p-2)x^{2p}-2(p-2)x^p+(p-1)]\big\},\end{aligned}$$

$$\varphi_2'(x)=p(p-1)x^p(1-x^p)^{1/p-1}[1+(p-2)x^p]^{-2}$$

and

$$\begin{aligned}\frac{\varphi_1'(x)}{\varphi_2'(x)}=&\left[\frac{\arcsin_p^*(x)}{x}\right]^2(1-x^p)^{-2/p}[2(p-2)x^{2p}-2(p-2)x^p+(p-1)]\\&+2\left[\frac{\arcsin_p^*(x)}{x}\right](1-x^p)^{-1/p}[1+(p-2)x^p].\end{aligned}$$

It follows from Lemma 2.4 that $\varphi_1'(x)/\varphi_2'(x)$ is strictly increasing on $(0,1)$ for $p\in(1,+\infty)$. Applying Lemma 2.1 we conclude that $\varphi(x)$ is also strictly increasing on $(0,1)$. Moreover,

$$\lim_{x\to0^+}\varphi(x)=\lim_{x\to0^+}\frac{\varphi_1'(x)}{\varphi_2'(x)}=p+1,\quad \lim_{x\to1^-}\varphi(x)=+\infty.$$

This completes the proof. □

REMARK 2.6. The proof of Lemma 2.6 implies that $\varphi_2'(x)>0$ and $\varphi_2(0)=0$. Thus $\operatorname{arcsin}_p^*(x)[1+(p-2)x^p]-x(1-x^p)^{1/p}=[1+(p-2)x^p]\varphi_2(x)>0$ for all $x\in(0,1)$ and $p\in(1,+\infty)$.

LEMMA 2.7. *Let $x\in(0,1)$, $b\in\mathbb{R}$, $p\in(1,+\infty)$, $\phi_b(x)$ be defined by*

$$\phi_b(x)=(b-1)\frac{x}{\operatorname{arcsin}_p^*(x)(1-x^p)^{1-1/p}}+(p-1)\frac{x^p}{1-x^p}$$

and

$$L(b)=\inf_{x\in(0,1)}\phi_b(x).$$

Then $\phi_b(x)$ is strictly increasing from $(0,1)$ onto $(b-1,+\infty)$ if and only if $b\geqslant -p$, and there exists $\lambda\in(0,1)$ such that $\phi_b(x)$ is strictly decreasing on $(0,\lambda)$ and strictly increasing on $(\lambda,1)$ with the range $(L(b),+\infty)$ if $b<-p$. Moreover, $L(b)=b-1$ for $b\geqslant -p$, and $L(b)<b-1$ for $b<-p$.

Proof. Let

$$\phi_{b1}(x)=(b-1)x(1-x^p)^{1/p}+(p-1)\operatorname{arcsin}_p^*(x)x^p$$

and

$$\phi_{b2}(x)=\operatorname{arcsin}_p^*(x)(1-x^p).$$

Then $\phi_{b1}(0)=\phi_{b2}(0)=0$,

$$\phi_{b1}'(x)=(b-1)(1-x^p)^{1/p}+(p-b)x^p(1-x^p)^{1/p-1}+p(p-1)\operatorname{arcsin}_p^*(x)x^{p-1}$$

and

$$\phi_{b2}'(x)=(1-x^p)^{1/p}-p\operatorname{arcsin}_p^*(x)x^{p-1}.$$

Making use of the l'Hôptial Rule, we get

$$\lim_{x\to0^+}\phi_b(x)=\lim_{x\to0^+}\frac{\phi_{b1}'(x)}{\phi_{b2}'(x)}=b-1,\quad \lim_{x\to1^-}\phi_b(x)=+\infty.$$

Differentiating $\phi_b(x)$ leads to

$$\phi_b'(x)=\frac{\operatorname{arcsin}_p^*(x)[1+(p-2)x^p]-x(1-x^p)^{1/p}}{[\operatorname{arcsin}_p^*(x)]^2(1-x^p)^{2-1/p}}[(b-1)+\varphi(x)],$$

where $\varphi(x)$ is defined as in Lemma 2.5. We divide the proof into two cases.

Case A. $b \geqslant -p$. Then it follows from Lemma 2.5 and Remark 2.6 that $\phi_b'(x) > 0$ for all $x \in (0,1)$, so that $\phi_b(x)$ is strictly increasing from $(0,1)$ onto $(b-1,+\infty)$.

Case B. $b < -p$. Then Lemma 2.5 and Remark 2.6 lead to the conclusion that there exists $\lambda \in (0,1)$ such that $\phi_b(x)$ is strictly decreasing on $(0,\lambda)$ and strictly increasing on $(\lambda,1)$. Thus, $\phi_b(x) \in (L(b),+\infty)$. $\square$

LEMMA 2.8. *Let $x \in (0,1)$, $a,b \in \mathbb{R}$, $p \in (1,+\infty)$, $\eta_{a,b}(x)$ be defined by*

$$\eta_{a,b}(x) = \frac{[\arcsin_p^*(x)]^{b-1}x^{1-a}}{(1-x^p)^{1-1/p}}$$

and $L(b)$ be defined as in Lemma 2.7. Then $\eta_{a,b}(x)$ is strictly increasing on $(0,1)$ if and only if $a \leqslant 1+L(b)$ and $\eta_{a,b}(x)$ is piecewise monotone on $(0,1)$ if $a > 1+L(b)$.

Proof. Lemma 2.8 follows from Lemma 2.7 and the logarithmic differentiation of $\eta_{a,b}(x)$

$$\begin{aligned}\frac{\eta_{a,b}'(x)}{\eta_{a,b}(x)} &= \frac{1}{x}\left[\frac{(b-1)x}{\arcsin_p^*(x)(1-x^p)^{1-1/p}} + (1-a) + \frac{(p-1)x^p}{1-x^p}\right]\\ &= \frac{1}{x}[\phi_b(x)+1-a],\end{aligned}$$

where $\phi_b(x)$ is defined as in Lemma 2.7. $\square$

3. Proofs of Theorem 1.1 and Corollary 1.2

Proof of Theorem 1.1. We divide the proof of Theorem 1.1 into two cases.

Case 1 $b \neq 0$. Without loss of generality, we assume that $0 < x \leqslant y < 1$. Let

$$J(x,y) = [\arcsin_p^*(M_a(x,y))]^b - \frac{[\arcsin_p^*(x)]^b + [\arcsin_p^*(y)]^b}{2} \tag{3.1}$$

and $t = M_a(x,y)$. Then $\partial t/\partial x = (x/t)^{a-1}/2$. If $x < y$, then $t > x$. Differentiating $J(x,y)$ with respect to x gives

$$\begin{aligned}\frac{\partial J(x,y)}{\partial x} =& \frac{b[\arcsin_p^*(t)]^{b-1}(1-t^p)^{1/p-1}(\frac{x}{t})^{a-1}}{2} - \frac{1}{2}b[\arcsin_p^*(x)]^{b-1}(1-x^p)^{1/p-1}\\ =& \frac{b}{2}x^{a-1}[\eta_{a,b}(t)-\eta_{a,b}(x)],\end{aligned} \tag{3.2}$$

where $\eta_{a,b}(x)$ is defined as in Lemma 2.8. Next, we divide the proof into two subcases.

Subcase 1.1 $a \leqslant 1+L(b)$. Then it follows from (3.2) and Lemma 2.8 that $\partial J(x,y)/\partial x > 0$ if $b > 0$, and $\partial J(x,y)/\partial x < 0$ if $b < 0$. Hence $J(x,y) < J(y,y) = 0$ if $b > 0$, and $J(x,y) > J(y,y) = 0$ if $b < 0$. Thus from (3.1) we have

$$\arcsin_p^*(M_a(x,y)) \leqslant M_b(\arcsin_p^*(x), \arcsin_p^*(y))$$

for $a \leqslant 1+L(b)$ and $b \neq 0$, with equality if and only if $x=y$. Therefore, $\operatorname{arcsin}_p^*(x)$ is strictly $M_{a,b}$-convex for $(b,a) \in \{(b,a) | a \leqslant 1+L(b), b \neq 0\}$.

Subcase 1.2 $a > 1+L(b)$. Then making use of (3.1), (3.2) and Lemma 2.8 together with the similar argument as in Subcase 1.1 we know that $\operatorname{arcsin}_p^*(x)$ is neither $M_{a,b}$-concave nor $M_{a,b}$-convex on $(0,1)$.

Case 2 $b=0$. Then we assume that $0 < x \leqslant y < 1$. Let

$$I(x,y) = \frac{[\operatorname{arcsin}_p^*(M_a(x,y))]^2}{\operatorname{arcsin}_p^*(x)\operatorname{arcsin}_p^*(y)} \tag{3.3}$$

and $t = M_a(x,y)$. Then $\partial t/\partial x = (x/t)^{a-1}/2$. Logarithmic differentiating $I(x,y)$ with respect to x gives

$$\begin{aligned}\frac{\partial I(x,y)}{\partial x}\frac{1}{I(x,y)} =& x^{a-1}\left[\frac{t^{1-a}}{\operatorname{arcsin}_p^*(t)(1-t^p)^{1-1/p}} - \frac{x^{1-a}}{\operatorname{arcsin}_p^*(x)(1-x^p)^{1-1/p}}\right] \\ =& x^{a-1}[\eta_{a,b}(t) - \eta_{a,b}(x)],\end{aligned} \tag{3.4}$$

where $\eta_{a,b}(x)$ is defined as in Lemma 2.8. Noting that $b = 0 > -p$, $L(b) = -1$. We divide the proof into two subcases.

Subcase 2.1 $a \leqslant 1+L(b) = 0$. It follows from (3.4) and Lemma 2.8 that $\partial I(x,y)/\partial x > 0$ and $I(x,y) \leqslant I(y,y) = 1$. Therefore,

$$\operatorname{arcsin}_p^*(M_a(x,y)) \leqslant M_b(\operatorname{arcsin}_p^*(x), \operatorname{arcsin}_p^*(y))$$

follows from (3.3) with equality if and only if $x=y$, and $\operatorname{arcsin}_p^*(x)$ is strictly $M_{a,b}$-convex for $(b,a) \in \{(b,a) | a \leqslant 0,\ b=0\}$.

Subcase 2.2 $a > 0$. Then making use of (3.3), (3.4) and Lemma 2.8 together with the similar argument as in Subcase 2.1 we know that $\operatorname{arcsin}_p^*(x)$ is neither $M_{a,b}$-concave nor $M_{a,b}$-convex on $(0,1)$. □

Proof of Corollary 1.2. If $\operatorname{arcsin}_p^*(x)$ is $M_{a,b}$-convex on $(0,1)$, then the inequality

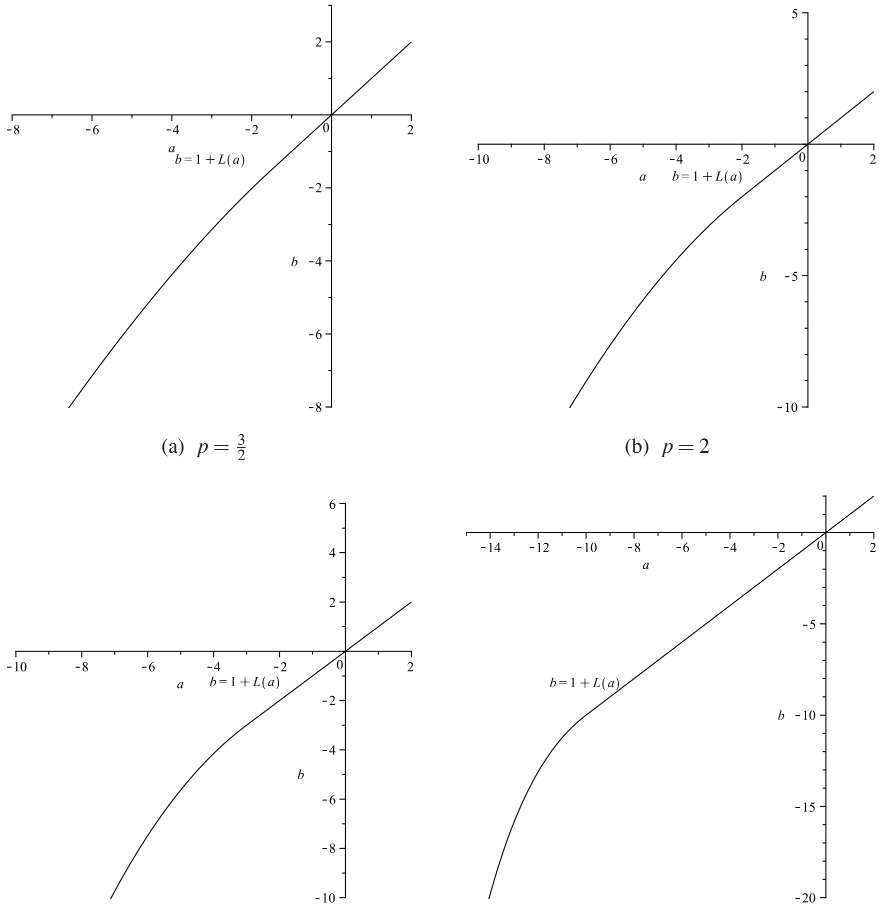
$$\operatorname{arcsin}_p^*(M_a(x,y)) \leqslant M_b(\operatorname{arcsin}_p^*(x), \operatorname{arcsin}_p^*(y))$$

holds for any $x, y \in (0,1)$. Let $x = \sin_p^*(u)$, $y = \sin_p^*(v)$ for $u, v \in (0, \pi_p^*/2)$. Then

$$M_a(\sin_p^*(u), \sin_p^*(v)) \leqslant \sin_p^*(M_b(u,v)),$$

so that $\sin_p^*(x)$ is $M_{b,a}$-concave on $(0, \pi_p^*/2)$. Therefore, by Theorem 1.1, Corollary 1.2 holds true. □

REMARK 3.1. According to Lemma 2.7, we draw some boundary curves of the regions D^* in Corollary 1.2 (see Figure 1) for $p = 3/2, 2, 3, 10$, where b is the best possible value with respect to fixed $a \in \mathbb{R}$ such that $\sin_p^*$ is $M_{a,b}$-concave on $(0,1)$.



(a) $p = \frac{3}{2}$ (b) $p = 2$

(c) $p = 3$ (d) $p = 10$

Figure 1: *The boundary curves* $b = 1 + L(a)$ *with different parameters* $p = 3/2, 2, 3, 10$

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