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LYAPUNOV–TYPE INEQUALITIES FOR DIFFERENTIAL EQUATION INVOLVING ONE–DIMENSIONAL MINKOWSKI–CURVATURE OPERATOR

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Abstract. In this paper, some new Lyapunov-type inequalities for one-dimensional Minkowski-curvature equation with anti-periodic and Sturm-Liouville boundary conditions are presented.

1. Introduction

In this paper, we will give some new Lyapunov-type inequalities for the following one-dimensional Minkowski-curvature equation involving anti-periodic and Sturm-Liouville boundary conditions

$$\begin{cases} (\phi(u'(t)))' + r(t)u(t) = 0, \quad a < t < b, \\ u(a) + u(b) = 0, \quad u'(a) + u'(b) = 0, \end{cases} \tag{1.1}$$

$$\begin{cases} (\phi(u'(t)))' + r(t)u(t) = 0, \quad a < t < b, \\ \alpha u(a) - \beta u'(a) = 0, \quad \gamma u(b) + \delta u'(b) = 0, \end{cases} \tag{1.2}$$

where $\phi(y) = \frac{y}{\sqrt{1-|y|^2}}$, $y \in (-1,1)$, weight function $r(t)$ is continuous on $[a,b]$ and $\alpha,\beta,\gamma,\delta \geqslant 0$, $\beta\gamma + \alpha\delta + \alpha\gamma(b-a) > 0$.

The equation in system (1.1) is driven by a strongly nonlinear differential operator of ϕ-laplacian type, precisely

$$u \mapsto -(\phi(u'))', \quad \text{where } \phi(\xi) := \frac{\xi}{\sqrt{1-\xi^2}}.$$

This is the one-dimensional version of the partial differential operator

$$u \mapsto -\text{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right),$$

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which is usually meant as a mean-curvature operator in Lorentz-Minkowski spaces, it also plays a important role in the theory of nonlinear electromagnetism (cf. [9] and the references therein). Recently, there has been a significant interest in the study of the existence and multiplicity issues of the associated boundary value problems (cf. [3–8, 10–12]). For the second order linear ordinary differential equation, Lyapunov [1] found the following interesting results. If $u(t)$ is a nontrivial solution of the differential system

$$\begin{cases} u''(t)+r(t)u(t)=0, & t\in(a,b), \\ u(a)=0=u(b), \end{cases} \tag{1.3}$$

where $r(t)$ is a continuous and nonnegative function defined in $[a,b]$, then

$$\int_a^b r(t)dt > \frac{4}{b-a}, \tag{1.4}$$

and the constant 4 cannot be replaced by a larger number.

After the appearance of Lyapunov inequality, a large number of generalizations have been proved. Concerning previous works on Lyapunov-type inequality, for the integer order differential equations, we refer readers to [13–19], for fractional differential equations, please refer to [20–25].

Recently, Rui Yang et al. [2] first obtained Lyapunov inequalities for the following one-dimensional Minkowski-curvature problems

$$\begin{cases} (\phi(u'(t)))'+r(t)u(t)=0, & a<t<b, \\ u(a)=0=u(b), \end{cases} \tag{1.5}$$

$$\begin{cases} (\phi(u_1'(t)))'+r_1(t)u_2(t)=0, & a<t<b, \\ (\phi(u_2'(t)))'+r_2(t)u_3(t)=0, & a<t<b, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_n'(t)))'+r_n(t)u_1(t)=0, & a<t<b, \\ u_i(a)=u_i(b)=0,\ i=1,2,\cdots,n, \end{cases} \tag{1.6}$$

and

$$\begin{cases} (\phi(u_1'(t)))'+r_1(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ (\phi(u_2'(t)))'+r_2(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_n'(t)))'+r_n(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ u_i(a)=u_i(b)=0,\ i=1,2,\cdots,n, \end{cases} \tag{1.7}$$

Motivated by the paper [2], the aim of the present paper is to get three types of Lyapunov inequalities for one-dimensional Minkowski-curvature problems.

The rest of this paper is organized as follows. In Section 2, we obtain Lyapunov inequalities for problems (1.1) and (1.2). In Section 3, we give Lyapunov-type inequalities for cycled system. In Section 4, Lyapunov-type inequalities for a strongly coupled system of one-dimensional Minkowski-curvature problem are proved. In Section 5, We give two examples to illustrate the application of the main results.

2. Lyapunov inequalities for problems (1.1) and (1.2)

We say u a solution of problem (1.1) (or (1.2)) if $u \in C^1[a,b], \|u'\|_\infty < 1$, and $\phi(u'(.))$ is absolutely continuous in any compact subinterval of (a,b), and u satisfies the equation and the boundary conditions in problem (1.1) (or (1.2)).

We first give the Lyapunov inequality for problem (1.1).

THEOREM 2.1. *If the problem* (1.1) *has a nontrivial continuous solution, then one has*

$$\int_a^b |r(t)|dt \geqslant \frac{4}{b-a}. \tag{2.1}$$

Proof. Since the nontrivial solution u of problem (1.1) satisfies the anti-periodic boundary conditions, then we have

$$u(t) = \frac{1}{2}\int_a^t u'(s)ds - \frac{1}{2}\int_t^b u'(s)ds = \int_a^b H(t,s)u'(s)ds,$$

$$u'(t) = \frac{1}{2}\int_a^t u''(s)ds - \frac{1}{2}\int_t^b u''(s)ds = \int_a^b H(t,s)u''(s)ds,$$

where

$$H(t,s) = \begin{cases} \frac{1}{2}, & a \leqslant s \leqslant t, \\ -\frac{1}{2}, & t \leqslant s \leqslant b. \end{cases}$$

Thus

$$\begin{aligned} u(t) &= \int_a^b H(t,s)u'(s)ds \\ &= \int_a^b H(t,s)\left(\int_a^b H(s,\tau)u''(\tau)d\tau\right)ds \\ &= \int_a^b H(t,s)u''(\tau)\left(\int_a^b H(s,\tau)ds\right)d\tau, \end{aligned}$$

and

$$|u(t)| \leqslant \frac{b-a}{4}\int_a^b |u''(\tau)|d\tau, \tag{2.2}$$

by the relation

$$(\phi(u'(t)))' = \left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \frac{u''(t)}{\sqrt{[1-(u'(t))^2]^3}},$$

we have

$$\begin{aligned}\max_{a\leqslant t\leqslant b}|u(t)| &\leqslant \frac{b-a}{4}\int_a^b |u''(t)|dt\\ &\leqslant \frac{b-a}{4}\int_a^b \frac{|u''(t)|}{\sqrt{[1-(u'(t))^2]^3}}|dt\\ &= \frac{b-a}{4}\int_a^b |r(t)||u(t)|dt\\ &\leqslant \frac{b-a}{4}\int_a^b |r(t)|dt\cdot \max_{a\leqslant t\leqslant b}|u(t)|,\end{aligned} \tag{2.3}$$

therefore,

$$\int_a^b |r(t)|dt \geqslant \frac{4}{b-a}. \quad \square \tag{2.4}$$

Secondly, we give the Lyapunov-type inequality for problem (1.2). In fact, we have the following result.

THEOREM 2.2. *If the problem* (1.2) *has a nontrivial continuous solution, then one has*

$$\int_a^b [\alpha(t-a)+\beta][\gamma(b-t)+\delta]|r(t)|dt \geqslant \beta\gamma+\alpha\delta+\alpha\gamma(b-a). \tag{2.5}$$

Proof. Define

$$G(t,s)=\begin{cases}\dfrac{[\alpha(s-a)+\beta][\gamma(b-t)+\delta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}, & a\leqslant s\leqslant t\leqslant b\\[2ex] \dfrac{[\alpha(t-a)+\beta][\gamma(b-s)+\delta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}, & a\leqslant t\leqslant s\leqslant b.\end{cases} \tag{2.6}$$

Then, by the Sturm-Liouville boundary condition in (1.2), we have

$$\begin{aligned}\int_a^b G(t,s)u''(s)ds &= \int_a^t G(t,s)du'(s)+\int_t^b G(t,s)du'(s)\\ &= G(t,t)u'(t)-G(t,a)u'(a)-\frac{\alpha[\gamma(b-t)+\delta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}\int_a^t u'(s)ds\\ &\quad +G(t,b)u'(b)-G(t,t)u'(t)+\frac{\gamma[\alpha(t-a)+\beta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}\int_t^b u'(s)ds\\ &= \frac{\delta[\alpha(t-a)+\beta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}u'(b)-\frac{\beta[\gamma(b-t)+\delta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}u'(a)\\ &\quad -\frac{\alpha[\gamma(b-t)+\delta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}[u(t)-u(a)]\\ &\quad +\frac{\gamma[\alpha(t-a)+\beta]}{\beta\gamma+\alpha\delta+\alpha\gamma(b-a)}[u(b)-u(t)]\\ &= -u(t),\end{aligned}$$

therefore

$$u(t) = -\int_a^b G(t,s)u''(s)ds,$$

by the relation

$$(\phi(u'(t)))' = \left(\frac{u'(t)}{\sqrt{1-(u'(t))^2}}\right)' = \frac{u''(t)}{\sqrt{[1-(u'(t))^2]^3}},$$

we have

$$\begin{aligned}\max_{a\leqslant t\leqslant b}|u(t)| &\leqslant \int_a^b G(t,t)|u''(t)|dt \\ &\leqslant \int_a^b G(t,t)\frac{|u''(t)|}{\sqrt{[1-(u'(t))^2]^3}}|dt \\ &= \int_a^b G(t,t)|r(t)||u(t)|dt \\ &\leqslant \int_a^b G(t,t)|r(t)|dt \cdot \max_{a\leqslant t\leqslant b}|u(t)|, \end{aligned} \tag{2.7}$$

so,

$$\int_a^b G(t,t)|r(t)|dt \geqslant 1.$$

Therefore, (2.5) holds. □

Let $\alpha = 1,\ \beta = 0,\ \gamma = 1,\ \delta = 0$ in Theorem 2.2, we obtain the Theorem 2.1 in [2].

COROLLARY 2.3. *If the problem*

$$\begin{cases} (\phi(u'(t)))' + r(t)u(t) = 0, & a < t < b, \\ u(a) = 0 = u(b), \end{cases} \tag{2.8}$$

has a nontrivial continuous solution, then one has

$$\int_a^b (t-a)(b-t)|r(t)|dt \geqslant b-a. \tag{2.9}$$

Let $\alpha = 1,\ \beta = 0,\ \gamma = 0,\ \delta = 1$ in Theorem 2.2, we obtain the following Lyapunov-type inequality.

COROLLARY 2.4. *If the problem*

$$\begin{cases} (\phi(u'(t)))' + r(t)u(t) = 0, & a < t < b, \\ u(a) = 0 = u'(b), \end{cases} \tag{2.10}$$

has a nontrivial continuous solution, then one has

$$\int_a^b (t-a)|r(t)|dt \geqslant 1. \tag{2.11}$$

Let $\alpha = 0,\ \beta = 1,\ \gamma = 1,\ \delta = 0$ in Theorem 2.2, we obtain the following Lyapunov-type inequality.

COROLLARY 2.5. *If the problem*

$$\begin{cases} (\phi(u'(t)))' + r(t)u(t) = 0, \quad a < t < b, \\ u'(a) = 0 = u(b), \end{cases} \tag{2.12}$$

has a nontrivial continuous solution, then one has

$$\int_a^b (b-t)|r(t)|dt \geqslant 1. \tag{2.13}$$

3. Lyapunov inequalities for cycled system

In this section, we deal with the cycled system

$$\begin{cases} (\phi(u_1'(t)))' + r_1(t)u_2(t) = 0, \quad a < t < b, \\ (\phi(u_2'(t)))' + r_2(t)u_3(t) = 0, \quad a < t < b, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_{n-1}'(t)))' + r_{n-1}(t)u_n(t) = 0, \quad a < t < b, \\ (\phi(u_n'(t)))' + r_n(t)u_1(t) = 0, \quad a < t < b, \end{cases} \tag{3.1}$$

under anti-periodic boundary conditions

$$u_i(a) + u_i(b) = 0 = u_i'(a) + u_i'(b),\ i = 1,2,\cdots,n, \tag{3.2}$$

and Sturm-Liouville boundary conditions

$$\alpha_i u_i(a) - \beta_i u_i'(a) = 0 = \gamma_i u_i(b) + \delta_i u_i'(b),\ i = 1,2,\cdots,n, \tag{3.3}$$

where weight function $r_i(t)$ is continuous on $[a,b]$, $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geqslant 0,\ \beta_i\gamma_i + \alpha_i\delta_i + \alpha_i\gamma_i(b-a) > 0, i = 1,2,\cdots,n$.

We say $(u_1,u_2,\cdots,u_n)$ a solution of problem (3.1)–(3.2) (or (3.1)–(3.3)) if $u_i \in C^1[a,b], \|u_i'\|_\infty < 1$, and $\phi(u_i'(t))$ is absolutely continuous in any compact subinterval of (a,b), and u_i satisfies the equation and the boundary conditions in problem (3.1)–(3.2) (or (3.1)–(3.3)).

We first deal with a cycled system under anti-periodic boundary conditions.

THEOREM 3.1. *If the problem* (3.1)–(3.2) *has a nontrivial continuous solution, then one has*

$$\prod_{i=1}^{n} \int_a^b |r_i(t)|dt \geqslant \left(\frac{4}{b-a}\right)^n. \tag{3.4}$$

Proof. Denote $M_i = \max_{a\leqslant t\leqslant b}|u_i(t)|$, as in (2.3), repeating this procedure to each equation in (3.1), for $i = 2,3,\cdots,n$, we obtain

$$M_1 \leqslant M_2 \cdot \frac{b-a}{4}\int_a^b |r_1(t)|dt$$
$$M_2 \leqslant M_3 \cdot \frac{b-a}{4}\int_a^b |r_2(t)|dt$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_{n-1} \leqslant M_n \cdot \frac{b-a}{4}\int_a^b |r_{n-1}(t)|dt$$
$$M_n \leqslant M_1 \cdot \frac{b-a}{4}\int_a^b |r_n(t)|dt$$

Multiplying all inequalities, we arrive at (3.4). □

We now deal with a cycled system involving Sturm-Liouville boundary conditions.

THEOREM 3.2. *If the problem* (3.1)–(3.3) *has a nontrivial continuous solution, then one has*

$$\prod_{i=1}^{n}\int_a^b [\alpha_i(t-a)+\beta_i][\gamma_i(b-t)+\delta_i]|r_i(t)|dt \geqslant \prod_{i=1}^{n}[\beta_i\gamma_i+\alpha_i\delta_i+\alpha_i\gamma_i(b-a)]. \quad (3.5)$$

Proof. Define

$$G_i(t,s) = \begin{cases} \dfrac{[\alpha_i(s-a)+\beta_i][\gamma_i(b-t)+\delta_i]}{\beta_i\gamma_i+\alpha_i\delta_i+\alpha_i\gamma_i(b-a)}, & a\leqslant s\leqslant t\leqslant b \\ \dfrac{[\alpha_i(t-a)+\beta_i][\gamma_i(b-s)+\delta_i]}{\beta_i\gamma_i+\alpha_i\delta_i+\alpha_i\gamma_i(b-a)}, & a\leqslant t\leqslant s\leqslant b. \end{cases}$$

and denote $M_i = \max_{a\leqslant t\leqslant b}|u_i(t)|$, as in (2.7), repeating this procedure to each equation in (3.1), for $i = 2,3,\cdots,n$, we obtain

$$M_1 \leqslant M_2 \cdot \int_a^b G_1(t,t)|r_1(t)|dt$$
$$M_2 \leqslant M_3 \cdot \int_a^b G_2(t,t)|r_2(t)|dt$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_{n-1} \leqslant M_n \cdot \int_a^b G_{n-1}(t,t)|r_{n-1}(t)|dt$$
$$M_n \leqslant M_1 \cdot \int_a^b G_n(t,t)|r_n(t)|dt$$

Multiplying all inequalities, we arrive at (3.5). □

Let $\alpha_i = 1$, $\beta_i = 0$, $\gamma_i = 1$, $\delta_i = 0$ in Theorem 3.2, we obtain the Theorem 4.1 in [2].

COROLLARY 3.3. *If the problem*

$$\begin{cases} (\phi(u_1'(t)))' + r_1(t)u_2(t) = 0, \quad a < t < b, \\ (\phi(u_2'(t)))' + r_2(t)u_3(t) = 0, \quad a < t < b, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_{n-1}'(t)))' + r_{n-1}(t)u_n(t) = 0, \quad a < t < b, \\ (\phi(u_n'(t)))' + r_n(t)u_1(t) = 0, \quad a < t < b, \\ u_i(a) = 0 = u_i(b), \ i = 1, 2, \cdots, n. \end{cases} \tag{3.6}$$

has a nontrivial continuous solution, then one has

$$\prod_{i=1}^{n} \int_a^b (t-a)(b-t)|r_i(t)|dt \geqslant (b-a)^n. \tag{3.7}$$

Let $\alpha_i = 1, \ \beta_i = 0, \ \gamma_i = 0, \ \delta_i = 1$ in Theorem 3.2, we obtain the following Lyapunov-type inequality.

COROLLARY 3.4. *If the problem*

$$\begin{cases} (\phi(u_1'(t)))' + r_1(t)u_2(t) = 0, \quad a < t < b, \\ (\phi(u_2'(t)))' + r_2(t)u_3(t) = 0, \quad a < t < b, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_{n-1}'(t)))' + r_{n-1}(t)u_n(t) = 0, \quad a < t < b, \\ (\phi(u_n'(t)))' + r_n(t)u_1(t) = 0, \quad a < t < b, \\ u_i(a) = 0 = u_i'(b), \ i = 1, 2, \cdots, n. \end{cases} \tag{3.8}$$

has a nontrivial continuous solution, then one has

$$\prod_{i=1}^{n} \int_a^b (t-a)|r_i(t)|dt \geqslant 1. \tag{3.9}$$

Let $\alpha_i = 0, \ \beta_i = 1, \ \gamma_i = 1, \ \delta_i = 0$ in Theorem 3.2, we obtain the following Lyapunov-type inequality.

COROLLARY 3.5. *If the problem*

$$\begin{cases} (\phi(u_1'(t)))' + r_1(t)u_2(t) = 0, \quad a < t < b, \\ (\phi(u_2'(t)))' + r_2(t)u_3(t) = 0, \quad a < t < b, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_{n-1}'(t)))' + r_{n-1}(t)u_n(t) = 0, \quad a < t < b, \\ (\phi(u_n'(t)))' + r_n(t)u_1(t) = 0, \quad a < t < b, \\ u_i'(a) = 0 = u_i(b), \ i = 1, 2, \cdots, n. \end{cases} \tag{3.10}$$

has a nontrivial continuous solution, then one has

$$\prod_{i=1}^{n} \int_a^b (b-t)|r_i(t)|dt \geqslant 1. \tag{3.11}$$

4. Lyapunov inequalities for strongly coupled system

In this section, we deal with the strongly coupled system

$$\begin{cases} (\phi(u_1'(t)))' + r_1(t)(u_1(t)+u_2(t)+\cdots+u_n(t)) = 0, & a<t<b, \\ (\phi(u_2'(t)))' + r_2(t)(u_1(t)+u_2(t)+\cdots+u_n(t)) = 0, & a<t<b, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ (\phi(u_n'(t)))' + r_n(t)(u_1(t)+u_2(t)+\cdots+u_n(t)) = 0, & a<t<b, \end{cases} \tag{4.1}$$

under anti-periodic boundary conditions

$$u_i(a)+u_i(b) = 0 = u_i'(a)+u_i'(b),\ i=1,2,\cdots,n, \tag{4.2}$$

and Sturm-Liouville boundary conditions

$$\alpha_i u_i(a) - \beta_i u_i'(a) = 0 = \gamma_i u_i(b) + \delta_i u_i'(b),\ i=1,2,\cdots,n, \tag{4.3}$$

where weight function $r_i(t)$ is continuous on $[a,b]$, $\alpha_i,\ \beta_i,\ \gamma_i,\ \delta_i \geqslant 0,\ \beta_i\gamma_i + \alpha_i\delta_i + \alpha_i\gamma_i(b-a) > 0, i=1,2,\cdots,n$.

We say $(u_1,u_2,\cdots,u_n)$ a solution of problem (4.1)–(4.2) (or (4.1)–(4.3)) if $u_i \in C^1[a,b], \|u_i'\|_\infty < 1$, and $\phi(u_i'(t))$ is absolutely continuous in any compact subinterval of (a,b), and u_i satisfies the equation and the boundary conditions in problem (4.1)–(4.2) (or (4.1)–(4.3)).

We first deal with the following strongly coupled system under anti-periodic conditions.

THEOREM 4.1. *If the problem* (4.1)–(4.2) *has a nontrivial continuous solution, then one has*

$$\sum_{i=1}^{n} \int_a^b |r_i(t)|dt \geqslant \frac{4}{b-a}. \tag{4.4}$$

Proof. Denote $M_i = \max_{a\leqslant t\leqslant b} |u_i(t)|$, as in (2.3), repeating this procedure to each equation in (4.1), for $i=1,2,3,\cdots,n$, we obtain

$$\begin{aligned} M_1 &\leqslant (M_1+M_2+\cdots+M_n)\cdot\frac{b-a}{4}\int_a^b |r_1(t)|dt \\ M_2 &\leqslant (M_1+M_2+\cdots+M_n)\cdot\frac{b-a}{4}\int_a^b |r_2(t)|dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ M_{n-1} &\leqslant (M_1+M_2+\cdots+M_n)\cdot\frac{b-a}{4}\int_a^b |r_{n-1}(t)|dt \\ M_n &\leqslant (M_1+M_2+\cdots+M_n)\cdot\frac{b-a}{4}\int_a^b |r_n(t)|dt \end{aligned}$$

Adding all inequalities, we obtain (4.4). □

Next, we study the strongly coupled system under Sturm-Liouville boundary conditions

THEOREM 4.2. *If the problem* (4.1)–(4.3) *has a nontrivial continuous solution, then one has*

$$\sum_{i=1}^{n}\int_a^b \frac{[\alpha_i(t-a)+\beta_i][\gamma_i(b-t)+\delta_i]}{\beta_i\gamma_i+\alpha_i\delta_i+\alpha_i\gamma_i(b-a)}|r_i(t)|dt \geqslant 1. \tag{4.5}$$

Proof. Define $G_i(t,s)$ and M_i as in Theorem 3.2, as in (2.7), repeating this procedure to each equation in (4.3), for $i=2,3,\cdots,n$, we obtain

$$\begin{aligned} M_1 &\leqslant (M_1+M_2+\cdots+M_n)\cdot\int_a^b G_1(t,t)|r_1(t)|dt \\ M_2 &\leqslant (M_1+M_2+\cdots+M_n)\cdot\int_a^b G_2(t,t)|r_2(t)|dt \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ M_{n-1} &\leqslant (M_1+M_2+\cdots+M_n)\cdot\int_a^b G_{n-1}(t,t)|r_{n-1}(t)|dt \\ M_n &\leqslant (M_1+M_2+\cdots+M_n)\cdot\int_a^b G_n(t,t)|r_n(t)|dt \end{aligned}$$

Adding all inequalities, we arrive at (4.5). □

Let $\alpha_i=1,\ \beta_i=0,\ \gamma_i=1,\ \delta_i=0$ in Theorem 4.2, we obtain the Theorem 4.2 in [2].

COROLLARY 4.3. *If the problem*

$$\begin{cases} (\phi(u_1'(t)))'+r_1(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ (\phi(u_2'(t)))'+r_2(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_n'(t)))'+r_n(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ u_i(a)=0=u_i(b),\ i=1,2,\cdots,n. \end{cases} \tag{4.6}$$

has a nontrivial continuous solution, then one has

$$\sum_{i=1}^{n}\int_a^b (t-a)(b-t)|r_i(t)|dt \geqslant b-a. \tag{4.7}$$

Let $\alpha_i=1,\ \beta_i=0,\ \gamma_i=0,\ \delta_i=1$ in Theorem 4.2, we obtain the following Lyapunov-type inequality.

COROLLARY 4.4. *If the problem*

$$\begin{cases} (\phi(u_1'(t)))'+r_1(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ (\phi(u_2'(t)))'+r_2(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ (\phi(u_n'(t)))'+r_n(t)(u_1(t)+u_2(t)+\cdots+u_n(t))=0, & a<t<b, \\ u_i(a)=0=u_i'(b),\ i=1,2,\cdots,n. \end{cases} \tag{4.8}$$

has a nontrivial continuous solution, then one has

$$\sum_{i=1}^{n}\int_a^b (t-a)|r_i(t)|dt \geqslant 1. \tag{4.9}$$

Let $\alpha_i = 0,\ \beta_i = 1,\ \gamma_i = 1,\ \delta_i = 0$ in Theorem 4.2, we obtain the following Lyapunov-type inequality.

COROLLARY 4.5. *If the problem*

$$\begin{cases} (\phi(u_1'(t)))' + r_1(t)(u_1(t)+u_2(t)+\cdots+u_n(t)) = 0, & a<t<b,\\ (\phi(u_2'(t)))' + r_2(t)(u_1(t)+u_2(t)+\cdots+u_n(t)) = 0, & a<t<b,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ (\phi(u_n'(t)))' + r_n(t)(u_1(t)+u_2(t)+\cdots+u_n(t)) = 0, & a<t<b,\\ u_i'(a) = 0 = u_i(b),\ i=1,2,\cdots,n. & \end{cases} \tag{4.10}$$

has a nontrivial continuous solution, then one has

$$\sum_{i=1}^{n}\int_a^b (b-t)|r_i(t)|dt \geqslant 1. \tag{4.11}$$

5. Applications

Let $a=0$ and $b=1$ in (1.1), we now discuss the following problem.

EXAMPLE 5.1. Consider the problem

$$\begin{cases} (\phi(u'(t)))' + r(t)u(t) = 0, \quad 0<t<1,\\ u(0)+u(1) = 0, \quad u'(0)+u'(1) = 0, \end{cases} \tag{5.1}$$

where $r(t) = \frac{2}{t(1-t)\left(\sqrt{(\frac{3}{2}-t)(\frac{1}{2}+t)}\right)^3}$. Taking $u(t) = \frac{1}{2}t(1-t)$, we see that u is a solution of problem (5.1) satisfying $u \in C^1[0,1], \|u'\|_\infty < 1$. Applying Theorem 2.1, we have

$$\int_0^1 \frac{1}{t(1-t)\left(\sqrt{(\frac{3}{2}-t)(\frac{1}{2}+t)}\right)^3}dt \geqslant 2. \tag{5.2}$$

Let $a=0,\ b=1$ and $\alpha=\beta=\gamma=\delta=1$ in (1.2), we now discuss the following problem.

EXAMPLE 5.2. Consider the problem

$$\begin{cases} (\phi(u'(t)))' + r(t)u(t) = 0, \quad 0<t<1,\\ u(0)-u'(0) = 0, \quad u(1)+u'(1) = 0, \end{cases} \tag{5.3}$$

where $r(t) = \frac{2}{(1+t-t^2)\left(\sqrt{(\frac{3}{2}-t)(\frac{1}{2}+t)}\right)^3}$. Taking $u(t) = \frac{1}{2}(1+t-t^2)$, we see that u is a solution of problem (5.3) satisfying $u \in C^1[0,1]$, $\|u'\|_\infty < 1$. Applying Theorem 2.2, we have

$$\int_0^1 \frac{2+t-t^2}{(1+t-t^2)\left(\sqrt{(\frac{3}{2}-t)(\frac{1}{2}+t)}\right)^3} dt \geqslant \frac{3}{2}. \tag{5.4}$$

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