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ON POST QUANTUM INTEGRAL INEQUALITIES

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Abstract. In the article, we provide some new post quantum refinements of the Hermite-Hadamard like inequalities involving the class of h-preinvex functions by establishing a new auxiliary result involving the post quantum differentiable function. By discussing some special cases, it is shown that our obtained results are the further generalizations of many previous known results.

1. Introduction and preliminaries

In recent decades, quantum calculus [2, 8, 29, 32, 51, 57, 75] has become a bridge between mathematics and physics, it expanded rapidly due to its great many applications in various branches of pure and applied sciences [26, 40, 48, 52, 55]. Quantum calculus also known as q-calculus which can be viewed as calculus without limits [49, 54, 83]. In quantum calculus, we obtain the q-analogues of mathematical objects which can be recaptured to original by taking $q \to 1^-$. Recently Tariboon et al. [61] introduced the concepts of quantum calculus on definite interval $[a,b] \subset \mathbb{R}$.

Let $J = [a,b] \subseteq \mathbb{R}$ be an interval and $0 < q < 1$ be a constant. Then the q-derivative of a function $\mathscr{X} : J \to \mathbb{R}$ at a point $x \in J$ is defined as follows.

DEFINITION 1.1. ([61]) Let $\mathscr{X} : J \to \mathbb{R}$ be a continuous function and $x \in J$. Then the q-derivative of $\mathscr{X}$ at x is defined as

$$_a\mathrm{D}_q\mathscr{X}(x) = \frac{\mathscr{X}(x) - \mathscr{X}(qx + (1-q)a)}{(1-q)(x-a)} \quad (x \neq a).$$

LEMMA 1.2. ([61]) *Let* $\alpha \in \mathbb{R}$. *Then*

$$_a\mathrm{D}_q(x-a)^{\alpha} = \left(\frac{1-q^{\alpha}}{1-q}\right)(x-a)^{\alpha-1}.$$

The q-integral is defined as follows:

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DEFINITION 1.3. ([61]) Let $\mathscr{X} : J \to \mathbb{R}$ be a continuous function. Then the q-integral $\int_a^x \mathscr{X}(t)_a \mathrm{d}_q t$ is defined by

$$\int_a^x \mathscr{X}(t)_a \mathrm{d}_q t = (1-q)(x-a)\sum_{n=0}^{\infty} q^n \mathscr{X}(q^n x + (1-q^n)a) \tag{1.1}$$

for $x \in J$.

LEMMA 1.4. ([61]) *Let* $\alpha \in \mathbb{R} \setminus \{-1\}$. *Then*

$$\int_a^x (t-a)_a^{\alpha} \mathrm{d}_q t = \left(\frac{1-q}{1-q^{\alpha+1}}\right)(x-a)^{\alpha+1}.$$

Utilizing these new concepts, Tariboon et al. [61] obtained the q-analogues of several classical inequalities, such as Hölder inequality, Chauchy-Shewarz inequality, Grüss-Cebysev inequality and so on. They also obtained the quantum analogues of the inequalities involving the functions having certain convexity properties.

In 2016, Tunç and Göv [62] defined the (p,q) post quantum derivative and integral.

DEFINITION 1.5. ([62]) Let $\mathscr{X} : J \to \mathbb{R}$ be a continuous function such that $x \in J$ and $0 < q < p \leqslant 1$. Then the (p,q)-derivative of the function $\mathscr{X}$ at x is defined by

$$\mathrm{D}_{p,q}^{R} \mathscr{X}(x) = \frac{\mathscr{X}(px + (1-p)a)) - \mathscr{X}(qx + (1-q)a)}{(p-q)(x-a)} \quad (x \neq a). \tag{1.2}$$

DEFINITION 1.6. ([62]) Let $\mathscr{X} : J \to \mathbb{R}$ be a continuous function such that $x \in J$. Then the (p,q)-integral $\int_a^x \mathscr{X}(t) \mathrm{d}_{p,q}^{R} t$ is defined by

$$\int_a^x \mathscr{X}(t) \mathrm{d}_{p,q}^{R} t = (p-q)(x-a)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \mathscr{X}\left(\frac{q^n}{p^{n+1}} x + \left(1 - \frac{q^n}{p^{n+1}}\right) a\right).$$

Note that the above definitions reduce to the the concepts for quantum calculus if $p = 1$.

The following Lemmas 1.7–1.11 given in [36] will play important roles in establishing our main results.

LEMMA 1.7. *Let* $\omega \in [0,1]$ *and* $\tau \in [0,\infty)$. *Then*

$$\int_0^{\omega} v^{\tau} \mathrm{d}_{p,q}^{R} v = (p-q)\sum_{n=0}^{\infty} \left(\frac{\omega}{p}\right)^{\tau+1} \left(\frac{q}{p}\right)^{(\tau+1)n} = \frac{\omega^{\tau+1}(p-q)}{p^{\tau+1} - q^{\tau+1}}$$

and

$$\int_0^\omega (1-v)^\tau \mathrm{d}_{p,q}^R v = (p-q)\omega \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \frac{q^n \omega}{p^{n+1}}\right)^\tau .$$

LEMMA 1.8. *Let* $\psi, \omega \in [0,1]$ *and* $\tau \in [0,\infty)$. *Then*

$$\int_0^\omega v^\tau |qv - (\psi - \psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} \frac{\omega^{\tau+1}(p-q)(\psi-\psi\omega)}{p^{\tau+1}-q^{\tau+1}} - \frac{q\omega^{\tau+2}(p-q)}{p^{\tau+2}-q^{\tau+2}}, & (\psi+q)\omega \leqslant \psi, \\ \left[\begin{array}{c} \frac{2(p-q)(\psi-\psi\omega)^{\tau+2}}{q^{\tau+1}} \left(\frac{1}{p^{\tau+1}-q^{\tau+1}} - \frac{1}{p^{\tau+2}-q^{\tau+2}} \right) \\ + \frac{q\omega^{\tau+2}(p-q)}{p^{\tau+2}-q^{\tau+2}} - \frac{\omega^{\tau+1}(p-q)(\psi-\psi\omega)}{p^{\tau+1}-q^{\tau+1}} \end{array} \right], & (\psi+q)\omega > \psi \end{cases}$$

and

$$\int_0^\omega (1-v)^\tau |qv - (\psi - \psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} (p-q)\omega \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(\psi - \psi\omega - \frac{q^{n+1}}{p^{n+1}}\omega\right) \left(1 - \frac{q^n}{p^{n+1}}\omega\right)^\tau, & (\psi+q)\omega \leqslant \psi, \\ \left[\begin{array}{c} 2(p-q)(\psi-\psi\omega)^2 \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}} \left(1 - \frac{q^n}{p^{n+1}}\right) \left(1 - \frac{q^{n-1}}{p^{n+1}}(\psi-\psi\omega)\right)^\tau \\ -(p-q)\omega \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} (\psi - \psi\omega - \frac{q^{n+1}}{p^{n+1}}\omega) \left(1 - \frac{q^n}{p^{n+1}}\omega\right)^\tau \end{array} \right], & (\psi+q)\omega > \psi. \end{cases}$$

LEMMA 1.9. *Let* $\psi, \omega \in [0,1]$ *and* $\tau \in [0,\infty)$. *Then*

$$\int_0^1 v^\tau |qv - (1 - \psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} \frac{(p-q)(1-\psi\omega)}{p^{\tau+1}-q^{\tau+1}} - \frac{q(p-q)}{p^{\tau+2}-q^{\tau+2}}, & \psi\omega + q \leqslant 1, \\ \left[\begin{array}{c} \frac{2(p-q)(1-\psi\omega)^{\tau+2}}{q^{\tau+1}} \left(\frac{1}{p^{\tau+1}-q^{\tau+1}} - \frac{1}{p^{\tau+2}-q^{\tau+2}} \right) \\ + \frac{q(p-q)}{p^{\tau+2}-q^{\tau+2}} - \frac{(p-q)(1-\psi\omega)}{p^{\tau+1}-q^{\tau+1}} \end{array} \right], & \psi\omega + q > 1 \end{cases}$$

and

$$\int_0^1 (1-v)^\tau |qv - (1 - \psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} (p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \psi\omega - \frac{q^{n+1}}{p^{n+1}}\right) \left(1 - \frac{q^n}{p^{n+1}}\right)^\tau, & \psi\omega + q \leqslant 1, \\ \left[\begin{array}{c} 2(p-q)(1-\psi\omega)^2 \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}} \left(1 - \frac{q^n}{p^{n+1}}\right) \left(1 - \frac{q^{n-1}}{p^{n+1}}(1-\psi\omega)\right)^\tau \\ -(p-q) \sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1 - \psi\omega - \frac{q^{n+1}}{p^{n+1}}\right) \left(1 - \frac{q^n}{p^{n+1}}\right)^\tau \end{array} \right], & \psi\omega + q > 1. \end{cases}$$

LEMMA 1.10. *Let* $\psi,\omega\in[0,1]$ *and* $\tau\in[0,\infty)$. *Then*

$$\int_0^{\omega} v^{\tau}|qv-(1-\psi\omega)|\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}\frac{\omega^{\tau+1}(p-q)(1-\psi\omega)}{p^{\tau+1}-q^{\tau+1}}-\frac{q\omega^{\tau+2}(p-q)}{p^{\tau+2}-q^{\tau+2}}, & (\psi+q)\omega\leqslant 1,\\ \left[\begin{array}{c}\frac{2(p-q)^2(1-\psi\omega)^{\tau+2}}{q^{\tau+1}}\left(\frac{1}{p^{\tau+1}-q^{\tau+1}}-\frac{1}{p^{\tau+2}-q^{\tau+2}}\right)\\ +\frac{q\omega^{\tau+2}(p-q)}{p^{\tau+2}-q^{\tau+2}}-\frac{\omega^{\tau+1}(p-q)(1-\psi\omega)}{p^{\tau+1}-q^{\tau+1}}\end{array}\right], & (\psi+q)\omega>1\end{cases}$$

and

$$\int_0^{\omega}(1-v)^{\tau}|qv-(1-\psi\omega)|\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}(p-q)\omega\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right)^{\tau}, & (\psi+q)\omega\leqslant 1,\\ \left[\begin{array}{c}2(p-q)(1-\psi\omega)^2\sum_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)(1-\frac{q^{n-1}}{p^{n+1}}(1-\psi\omega))^{\tau}\\ -(p-q)\omega\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right)^{\tau}\end{array}\right], & (\psi+q)\omega>1.\end{cases}$$

LEMMA 1.11. *Let* $\psi,\omega\in[0,1]$ *and* $\theta\in[0,\infty)$. *Then*

$$\int_0^{1}|qv-(1-\psi\omega)|^{\theta}\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\right)^{\theta}, & 0\leqslant\psi\omega\leqslant 1-q,\\ \left[\begin{array}{c}(p-q)(1-\psi\omega)^{\theta+1}\sum_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^{\theta}\\ +(p-q)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{p^{n+1}}-1+\psi\omega\right)^{\theta}\\ -(p-q)(1-\psi\omega)^{\theta+1}\sum_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}-1\right)^{\theta}\end{array}\right], & 1-q<\psi\omega\leqslant 1.\end{cases}$$

Let $I\subseteq\mathbb{R}$ be an interval and $f:I\to\mathbb{R}$ be a real-valued function. Then f is said to be convex (concave) if

$$f(\lambda x+(1-\lambda)y)\leqslant(\geqslant)\lambda f(x)+(1-\lambda)f(y)$$

whenever $x,y\in I$ and $\lambda\in[0,1]$. It is well-known that the convexity (concavity) theory has wild applications in the fields of mathematics and engineering technology [9, 10, 12, 17, 18, 19, 23, 24, 25, 30, 44, 45, 46, 53, 64, 65, 66, 67, 68, 70]. Recently, the generalizations and variants for the convexity have attracted the attention of many researchers, for example, the harmonic convexity [1, 15], *GA* and *GG* convexities

[31, 32], s-convexity [4, 5, 50], strong-convexity [11, 59, 77, 79], ρ-convexity [14], Schur convexity [20, 21], η-convexity [33], preinvexity [34], qausi-convexity [35] and exponential convexity [22, 47]. In particular, many inequalities can be found in the literature [13, 37, 39, 41, 42, 43, 58, 60, 69, 71, 72, 73, 76, 78, 80, 81, 82] via the convexity theory.

The classical $\mathscr{H}\mathscr{H}$ (Hermite-Hadamard) inequality [3, 6, 7, 16, 27, 28, 56] is one of the most important inequalities in the geometric function theory which can be stated as follows.

Let $I \in \mathbb{R}$ be an interval and $h : I \to \mathbb{R}$ be a convex function defined on I. Then the double inequality

$$h\left(\frac{\kappa_1+\kappa_2}{2}\right) \leqslant \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} h(x)dx \leqslant \frac{h(\kappa_1)+h(\kappa_2)}{2} \tag{1.3}$$

holds for all $\kappa_1, \kappa_2 \in I$ with $\kappa_1 \neq \kappa_2$. If the function h is concave on I, then both the inequalities in (1.3) hold in the reverse direction

In order to establish our main results of the article, we need to introduce the class of h-preinvex functions.

DEFINITION 1.12. ([38]) Let $h : (0,1) \to \mathbb{R}$ be a real-valued function and J be an invex set with respect to the bivariate function $\eta(\cdot,\cdot)$. Then the function $\mathscr{X} : J \to \mathbb{R}$ is said to be h-preinvex with respect to $\eta(\cdot,\cdot)$ if the inequality

$$\mathscr{X}(a+v\xi(b,a)) \leqslant \mathrm{h}(1-v)\mathscr{X}(a)+\mathrm{h}(v)\mathscr{X}(b)$$

holds for $a,b \in J$ and $v \in (0,1)$.

REMARK 1.13. Note that if $\xi(b,a) = b-a$, then we get the definition of h-convex function introduced and studied by Varosanec [63]. Also after taking suitable choices of the function $\mathrm{h}(\cdot)$, we can get other classes of preinvexity functions, for example, we get the preinvex function defined in [74] if $\mathrm{h}(v) = v$; If $\mathrm{h}(v) = v^s$, then we obtain the class of s-preinvex functions given in [38].

2. Auxiliary results

In this section, we derive a new post quantum integral identity which will be used as an auxiliary result for obtaining our main results of the article.

LEMMA 2.1. *Let $0 < p,q < 1$ and $\mathscr{X} : J \to \mathbb{R}$ be a continuous and (p,q)-differentiable function on J° (Here and in what follows we denote J° the interior of J). Then the identity*

$$\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)$$
$$\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{d}^R_{p,q}v$$

$$= \xi(b,a)\left[\int_0^{\omega}(qv+\psi\omega-\psi)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right.$$
$$\left.+\int_{\omega}^{1}(qv+\psi\omega-1)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right]$$

holds for all $\psi,\omega\in[0,1]$ *if* $\mathrm{D}_{p,q}^{R}\mathscr{X}$ *is integrable on* J.

Proof. It suffices to show that

$$\xi(b,a)\left[\int_0^{\omega}(qv+\psi\omega-\psi)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right.$$
$$\left.+\int_{\omega}^{1}(qv+\psi\omega-1)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right]$$
$$= \xi(b,a)\left[\int_0^{1}(qv+\psi\omega-1)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right.$$
$$\left.+\int_0^{\omega}(1-\psi)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right]. \tag{2.1}$$

We clearly see that

$$\int_0^1 v\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v$$
$$=\int_0^1 \frac{\mathscr{X}(a+pv\xi(b,a))-\mathscr{X}(a+qv\xi(b,a))}{(p-q)\xi(b,a)}\mathrm{d}_{p,q}^{R}v$$
$$=\frac{1}{\xi(b,a)}\left[\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\mathscr{X}\left(a+\frac{q^n}{p^n}\xi(b,a)\right)-\frac{p}{q}\sum_{n=0}^{\infty}\frac{q^{n+1}}{p^{n+2}}\mathscr{X}\left(a+\frac{q^{n+1}}{p^{n+1}}\xi(b,a)\right)\right]$$
$$=\frac{1}{\xi(b,a)}\left[\frac{1}{p}\mathscr{X}(a+\xi(b,a))+\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}}\mathscr{X}\left(a+\frac{q^n}{p^n}\xi(b,a)\right)\right.$$
$$\left.-\frac{p}{q}\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}}\mathscr{X}\left(a+\frac{q^n}{p^n}\xi(b,a)\right)\right]$$
$$=\frac{1}{\xi(b,a)}\left[\frac{1}{p}\mathscr{X}(a+\xi(b,a))+\left(1-\frac{p}{q}\right)\sum_{n=1}^{\infty}\frac{q^n}{p^{n+1}}\mathscr{X}\left(a+\frac{q^n}{p^n}\xi(b,a)\right)\right]$$
$$=\frac{1}{\xi(b,a)}\left[\frac{1}{q}\mathscr{X}(a+\xi(b,a))-\left(\frac{p-q}{q}\right)\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\mathscr{X}\left(a+\frac{q^n}{p^n}\xi(b,a)\right)\right]$$

$$= \frac{1}{\xi(b,a)}\left[\frac{1}{q}\mathscr{X}(a+\xi(b,a)) - \frac{1}{pq\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{d}_{p,q}^{R}x\right], \tag{2.2}$$

$$\begin{aligned}
&\int_0^1 \mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v \\
&= \int_0^1 \frac{\mathscr{X}(a+pv\xi(b,a)) - \mathscr{X}(a+qv\xi(b,a))}{v(1-q)\xi(b,a)}\mathrm{d}_{p,q}^{R}v \\
&= \frac{1}{\xi(b,a)}\left[\sum_{n=0}^{\infty}\mathscr{X}\left(a+\frac{q^n}{p^n}\xi(b,a)\right) - \sum_{n=0}^{\infty}\mathscr{X}\left(a+\frac{q^{n+1}}{p^{n+1}}\xi(b,a)\right)\right] \\
&= \frac{1}{\xi(b,a)}[\mathscr{X}(a+\xi(b,a)) - \mathscr{X}(a)]
\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}
&\int_0^{\omega} \mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v \\
&= \int_0^{\omega} \frac{\mathscr{X}(a+pv\xi(b,a)) - \mathscr{X}(a+qv\xi(b,a))}{v(1-q)\xi(b,a)}\mathrm{d}_{p,q}^{R}v \\
&= \frac{1}{\xi(b,a)}\left[\sum_{n=0}^{\infty}\mathscr{X}\left(a+\omega\frac{q^n}{p^n}\xi(b,a)\right) - \sum_{n=0}^{\infty}\mathscr{X}\left(a+\omega\frac{q^{n+1}}{p^{n+1}}\xi(b,a)\right)\right] \\
&= \frac{1}{\xi(b,a)}[\mathscr{X}(a+\omega\xi(b,a)) - \mathscr{X}(a)].
\end{aligned} \tag{2.4}$$

Therefore, the desired result follows easily from (2.1)–(2.4). □

REMARK 2.2. Let $q \to 1^-$. Then Lemma 2.1 leads to

$$\begin{aligned}
&\psi[\omega\mathscr{X}(a+\xi(b,a)) + (1-\omega)\mathscr{X}(a)] + (1-\psi) \\
&\times\mathscr{X}(a+\omega\xi(b,a)) - \frac{1}{\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{d}x \\
&= \xi(b,a)\left[\int_0^{\omega}(k+\psi\omega-\psi)\mathscr{X}'(a+v\xi(b,a))\mathrm{d}k\right. \\
&\left.+\int_{\omega}^{1}(k+\psi\omega-1)\mathscr{X}'(a+v\xi(b,a))\mathrm{d}k\right].
\end{aligned}$$

REMARK 2.3. From Lemma 2.1 we get the following three special cases:
(i) It $\omega = 0$, then one has

$$\mathscr{X}(a)+(q-1)\mathscr{X}(a+\xi(b,a))-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\int_{0}^{1}(qv-1)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v.$$

(ii) If $\omega = 1$, then we get

$$\mathscr{X}(a+\xi(b,a))-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\int_{0}^{1}qv\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v.$$

(iii) If $\omega = p/(p+q)$, then we obtain

$$\psi\left[\frac{p\mathscr{X}(a+\xi(b,a))}{p+q}+\frac{q}{p+q}\mathscr{X}(a)\right]+(1-\psi)$$
$$\times\mathscr{X}\left(a+\frac{p}{p+q}\xi(b,a)\right)-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\left[\int_{0}^{\frac{p}{p+q}}\left(qv-\frac{\psi q}{p+q}\right)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v\right.$$
$$\left.+\int_{\frac{p}{p+q}}^{1}\left(qv+\frac{\psi p}{p+q}-1\right)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v\right].$$

REMARK 2.4. Lemma 2.1 also leads to the conclusion that

(i) Let $\psi = 0$. Then one has

$$\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\left[\int_{0}^{\omega}qv\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v+\int_{\omega}^{1}(qv-1)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v\right].$$

(ii) Let $\psi = 0$ and $\omega = p/(p+q)$. Then we get

$$\mathscr{X}\left(a+\frac{p}{p+q}\xi(b,a)\right)-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}_{p,q}^{R}x$$
$$=\xi(b,a)\left[\int_{0}^{\frac{p}{p+q}}qv\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right.$$
$$\left.+\int_{\frac{p}{p+q}}^{1}(qv-1)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right].$$

(iii) Let $\psi = 1/3$. Then we have

$$\frac{1}{3}\left[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)+2\mathscr{X}(a+\omega\xi(b,a))\right]$$
$$-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}_{p,q}^{R}x$$
$$=\xi(b,a)\left[\int_{0}^{\omega}\left(qv+\frac{\omega}{3}-\frac{1}{3}\right)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right.$$
$$\left.+\int_{\omega}^{1}\left(qv+\frac{\omega}{3}-1)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a)\right)\mathrm{d}_{p,q}^{R}v\right].$$

(iv) Let $\psi = 1/3$ and $\omega = p/(p+q)$. Then we obtain

$$\frac{1}{3}\left[\frac{p}{p+q}\mathscr{X}(a\xi(b,a))+\frac{q}{p+q}\mathscr{X}(a)+2\mathscr{X}\left(a+\frac{p}{p+q}\xi(b,a)\right)\right]$$
$$-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}_{p,q}^{R}x$$
$$=\xi(b,a)\left[\int_{0}^{\frac{p}{p+q}}\left(qv-\frac{q}{3p+3q}\right)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right.$$
$$\left.+\int_{\frac{p}{p+q}}^{1}\left(qv-\frac{3q+2p}{3p+3q}\right)\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))\mathrm{d}_{p,q}^{R}v\right].$$

(v) Let $\psi = 1/2$. Then

$$\frac{1}{2}\left[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)+\mathscr{X}(a+\omega\xi(b,a))\right]$$
$$-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\left[\int_{0}^{\omega}\left(qv+\frac{\omega}{2}-\frac{1}{2}\right)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v\right.$$
$$\left.+\int_{\omega}^{1}\left(qv+\frac{\omega}{2}-1)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a)\right)\mathrm{d}^{R}_{p,q}v\right].$$

(vi) Let $\psi = 1/2$ and $\omega = p/(1+q)$. Then we get

$$\frac{1}{2}\left[\frac{q}{1+q}\mathscr{X}(a)+\mathscr{X}\left(a+\frac{p\xi(b,a)}{p+q}\right)\right]+\frac{p}{p+q}\mathscr{X}(a+\xi(b,a))$$
$$-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\left[\int_{0}^{\frac{1}{1+q}}\left(qv-\frac{q}{2p+2q}\right)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v\right.$$
$$\left.+\int_{\frac{1}{1+q}}^{1}\left(qv-\frac{2q+p}{2p+2q}\right)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v\right].$$

(vii) Let $\psi = 1$. The one has

$$\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\int_{0}^{1}(qv+\omega-1)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v.$$

$(viii)$ Let $\psi = 1$ and $\omega = 1/(q+1)$. Then we obtain

$$\frac{p}{p+q}\mathscr{X}(a+\xi(b,a))+\frac{q}{p+q}\mathscr{X}(a)-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^{R}_{p,q}x$$
$$=\xi(b,a)\int_{0}^{1}\left(qv-\frac{q}{p+q}\right)\mathrm{D}^{R}_{p,q}\mathscr{X}(a+v\xi(b,a))\mathrm{d}^{R}_{p,q}v.$$

3. Results and discussions

In this section, we give our main results of the article.

THEOREM 3.1. *Let* $0 < q < 1$, $\xi(b,a) > 0$, $\mathscr{X} : [a, a+\xi(b,a)] \to \mathbb{R}$ *be a continuous and* (p,q)*-differentiable function on* $(a, a+\xi(b,a))$ *such that* $\mathrm{D}_{p,q}^{R}\mathscr{X}$ *is integrable on* $[a, a+\xi(b,a)]$ *and* $|\mathrm{D}_{p,q}^{R}\mathscr{X}|$ *is* h*-preinvex on* $[e, a+\xi(b,a)]$. *Then the inequality*

$$\begin{aligned}
&\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a)) + (1-\omega)\mathscr{X}(a)] + (1-\psi) \\
&\times \mathscr{X}(a+\omega\xi(b,a)) - \frac{1}{p\xi(b,a)} \int_{a}^{a+p\xi(b,a)} \mathscr{X}(x)\mathrm{D}_{p,q}^{R}x\Bigg| \\
&\leqslant \xi(b,a)\Big\{[\Phi_1(\psi,\omega;p,q) + \Phi_2(\psi,\omega;p,q) - \Phi_3(\psi,\omega;p,q)]|\mathrm{D}_{p,q}^{R}\mathscr{X}(a)| \\
&+ [\Phi_4(\psi,\omega;p,q) + \Phi_5(\psi,\omega;p,q) - \Phi_6(\psi,\omega;p,q)]|\mathrm{D}_{p,q}^{R}\mathscr{X}(b)|\Big\}
\end{aligned}$$

holds for all $\psi, \omega \in [0,1]$, *where*

$$\Phi_1(\psi,\omega;p,q) = \int_0^{\omega} |qv - (\psi - \psi\omega)|\mathrm{h}(1-v)\mathrm{d}_{p,q}^{R}v,$$

$$\Phi_2(\psi,\omega;p,q) = \int_0^{1} |qv - (1 - \psi\omega)|\mathrm{h}(1-v)\mathrm{d}_{p,q}^{R}v,$$

$$\Phi_3(\psi,\omega;p,q) = \int_0^{\omega} |qv - (1 - \psi\omega)|\mathrm{h}(1-v)\mathrm{d}_{p,q}^{R}v,$$

$$\Phi_4(\psi,\omega;p,q) = \int_0^{\omega} |qv - (\psi - \psi\omega)|\mathrm{h}(v)\mathrm{d}_{p,q}^{R}v,$$

$$\Phi_5(\psi,\omega;p,q) = \int_0^{1} |qv - (1 - \psi\omega)|\mathrm{h}(v)\mathrm{d}_{p,q}^{R}v$$

and

$$\Phi_6(\psi,\omega;p,q) = \int_0^{\omega} |qv - (1 - \psi\omega)|\mathrm{h}(v)\mathrm{d}_{p,q}^{R}v.$$

Proof. It follows from Lemma 2.1, the property of modulus and the h-preinvexity of $|\mathscr{X}|$ that

$$\begin{aligned}
&\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)\mathscr{X}(a+\omega\xi(b,a)) \\
&-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg| \\
&\leqslant \xi(b,a)\left[\int_0^{\omega}|qv+\psi\omega-\psi||\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|\mathrm{d}^R_{p,q}v\right. \\
&\left.+\int_{\omega}^{1}|qv+\psi\omega-1||\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|\mathrm{d}^R_{p,q}v\right] \\
&\leqslant \xi(b,a)\left\{\int_0^{\omega}|qv-(\psi-\psi\omega)\left[\mathrm{h}(1-v)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|+\mathrm{h}(v)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|\right]\mathrm{d}^R_{p,q}v\right. \\
&\left.+\int_{\omega}^{1}|qv-(1-\psi\omega)|\left[\mathrm{h}(1-v)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|+\mathrm{h}(v)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|\right]\mathrm{d}^R_{p,q}v\right\} \\
&= \xi(b,a)\left\{\int_0^{\omega}|qv-(\psi-\psi\omega)|\left[\mathrm{h}(1-v)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|+\mathrm{h}(v)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|\right]\mathrm{d}^R_{p,q}v\right. \\
&+\int_0^{1}|qv-(1-\psi\omega)|\left[\mathrm{h}(1-v)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|+\mathrm{h}(v)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|\right]\mathrm{d}^R_{p,q}v \\
&\left.-\int_0^{\omega}|qv-(1-\psi\omega)|\left[\mathrm{h}(1-v)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|+\mathrm{h}(v)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|\right]\mathrm{d}^R_{p,q}v\right\} \\
&= \xi(b,a)\left\{\left[\int_0^{\omega}|qv-(\psi-\psi\omega)\mathrm{h}(1-v)\mathrm{d}^R_{p,q}v+\int_0^{1}|qv-(1-\psi\omega)|\mathrm{h}(1-v)\mathrm{d}^R_{p,q}v\right.\right. \\
&\left.-\int_0^{\omega}|qv-(1-\psi\omega)|\mathrm{h}(1-v)\mathrm{d}^R_{p,q}v\right]|\mathrm{D}^R_{p,q}\mathscr{X}(a)|+\left[\int_0^{\omega}|qv-(\psi-\psi\omega)\mathrm{h}(v)\mathrm{d}^R_{p,q}v\right. \\
&\left.\left.+\int_0^{1}|qv-(1-\psi\omega)|\mathrm{h}(v)\mathrm{d}^R_{p,q}v-\int_0^{\omega}|qv-(1-\psi\omega)|\mathrm{h}(v)\mathrm{d}^R_{p,q}v\right]|\mathrm{D}^R_{p,q}\mathscr{X}(b)|\right\}.
\end{aligned}$$

This completes the proof. □

Now, we discuss some special cases of Theorem 3.1.

I. If we take $\mathrm{h}(v)=v$ in Theorem 3.1, then we have the result for preinvex function.

COROLLARY 3.2. *Under the assumptions of Theorem 3.1, if* $\mathrm{h}(v)=v$*, then the inequality*

$$\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)$$

$$\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|$$

$$\leqslant\xi(b,a)\Big\{[\Phi_1^*(\psi,\omega;p,q)+\Phi_2^*(\psi,\omega;p,q)-\Phi_3^*(\psi,\omega;p,q)]\,|\mathrm{D}^R_{p,q}\mathscr{X}(a)|$$

$$+[\Phi_4^*(\psi,\omega;p,q)+\Phi_5^*(\psi,\omega;p,q)-\Phi_6^*(\psi,\omega;p,q)]\,|\mathrm{D}^R_{p,q}\mathscr{X}(b)|\Big\}$$

holds for all $\psi,\omega\in[0,1]$*, where*

$$\Phi_1^*=\int_0^{\omega}(1-v)|qv-(\psi-\psi\omega)|\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}(p-q)\omega\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\psi-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right), & (\psi+q)\omega\leqslant\psi,\\ \left[\begin{array}{c}2(p-q)(\psi-\psi\omega)^2\sum\limits_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(\psi-\psi\omega)\right)\\ -(p-q)\omega\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\psi-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right)\end{array}\right], & (\psi+q)\omega>\psi,\end{cases}$$

$$\Phi_2^*=\int_0^{1}(1-v)|qv-(1-\psi\omega)|\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}(p-q)\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right), & \psi\omega+q\leqslant 1,\\ \left[\begin{array}{c}2(p-q)(1-\psi\omega)^2\sum\limits_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(1-\psi\omega)\right)\\ -(p-q)\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right)\end{array}\right], & \psi\omega+q>1,\end{cases}$$

$$\Phi_3^*=\int_0^{\omega}(1-v)|qv-(1-\psi\omega)|\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}(p-q)\omega\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right), & (\psi+q)\omega\leqslant 1,\\ \left[\begin{array}{c}2(p-q)(1-\psi\omega)^2\sum\limits_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(1-\psi\omega)\right)\\ -(p-q)\omega\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right)\end{array}\right], & (\psi+q)\omega>1,\end{cases}$$

$$\Phi_4^*=\int_0^{\omega}v|qv-(\psi-\psi\omega)|\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}\frac{\omega^2(\psi-\psi\omega)}{p+q}-\frac{q\omega^3}{p+pq+q^2}, & (\psi+q)\omega\leqslant\psi,\\ \frac{2(\psi-\psi\omega)^3}{q^2}\left(\frac{1}{p+q}-\frac{1}{p+pq+q^2}\right)+\frac{q\omega^3}{p+pq+q^2}-\frac{\omega^2(\psi-\psi\omega)}{p+q}, & (\psi+q)\omega>\psi,\end{cases}$$

$$\Phi_5^*=\int_0^1 v|qv-(1-\psi\omega)|\mathrm{d}_{p,q}^R v$$

$$=\begin{cases}\frac{(1-\psi\omega)}{p+q}-\frac{q}{p+pq+q^2}, & \psi\omega+q\leqslant 1,\\ \frac{2(1-\psi\omega)^3}{q^2}\left(\frac{1}{p+q}-\frac{1}{p+pq+q^2}\right)+\frac{q}{p+pq+q^2}-\frac{(1-\psi\omega)}{p+q}, & \psi\omega+q>1\end{cases}$$

and

$$\Phi_6^*=\int_0^\omega v|qv-(1-\psi\omega)|\mathrm{d}_{p,q}^R v$$

$$=\begin{cases}\frac{\omega^2(1-\psi\omega)}{p+q}-\frac{q\omega^3}{p+pq+q^2}, & (\psi+q)\omega\leqslant 1,\\ \frac{2(1-\psi\omega)^3}{q^2}\left(\frac{1}{p+q}-\frac{1}{p+pq+q^2}\right)+\frac{q\omega^3}{p+pq+q^2}-\frac{\omega^2(1-\psi\omega)}{p+q}, & (\psi+q)\omega>1.\end{cases}$$

II. If we take $\mathrm{h}(v)=v^s$ in Theorem 3.1, then we obtain the result for s-preinvex function of Breckner type.

COROLLARY 3.3. *Under the assumptions of Theorem 3.1, if* $\mathrm{h}(v)=v^s$ *with* $s\in[0,1]$*, then the inequality*

$$\Big|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)$$

$$\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}_{p,q}^R x\Big|$$

$$\leqslant\xi(b,a)\Big\{[\Phi_1^{**}(\psi,\omega;p,q)+\Phi_2^{**}(\psi,\omega;p,q)-\Phi_3^{**}(\psi,\omega;p,q)]\,|\mathrm{D}_{p,q}^R\mathscr{X}(a)|$$

$$+[\Phi_4^{**}(\psi,\omega;p,q)+\Phi_5^{**}(\psi,\omega;p,q)-\Phi_6^{**}(\psi,\omega;p,q)]\,|\mathrm{D}_{p,q}^R\mathscr{X}(b)|\Big\}$$

holds for all $\psi,\omega\in[0,1]$*, where*

$$\Phi_1^{**}=\int_0^\omega(1-v)^s|qv-(\psi-\psi\omega)|\mathrm{d}_{p,q}^R v$$

$$=\begin{cases}(p-q)\omega\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\psi-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right)^s, & (\psi+q)\omega\leqslant\psi,\\ \left[\begin{array}{c}2(p-q)(\psi-\psi\omega)^2\sum_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(\psi-\psi\omega)\right)^s\\ -(p-q)\omega\sum_{n=0}^{\infty}\frac{q^n}{p^{n+1}}(\psi-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega)\left(1-\frac{q^n}{p^{n+1}}\omega\right)^s\end{array}\right], & (\psi+q)\omega>\psi,\end{cases}$$

$$\Phi_2^{**} = \int_0^1 (1-v)^s |qv - (1-\psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} (p-q)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1-\psi\omega - \frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right)^s, & \psi\omega + q \leqslant 1, \\ \left[\begin{array}{c} 2(p-q)(1-\psi\omega)^2 \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}} \left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(1-\psi\omega)\right)^s \\ -(p-q)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\right)\left(1-\frac{q^n}{p^{n+1}}\right)^s \end{array}\right], & \psi\omega + q > 1, \end{cases}$$

$$\Phi_3^{**} = \int_0^{\omega} (1-v)^s |qv - (1-\psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} (p-q)\omega\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}} \left(1-\psi\omega - \frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right)^s, & (\psi+q)\omega \leqslant 1, \\ \left[\begin{array}{c} 2(p-q)(1-\psi\omega)^2 \sum_{n=0}^{\infty} \frac{q^{n-1}}{p^{n+1}} \left(1-\frac{q^n}{p^{n+1}}\right)\left(1-\frac{q^{n-1}}{p^{n+1}}(1-\psi\omega)\right)^s \\ -(p-q)\omega\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\omega\right)\left(1-\frac{q^n}{p^{n+1}}\omega\right)^s \end{array}\right], & (\psi+q)\omega > 1, \end{cases}$$

$$\Phi_4^{**} = \int_0^{\omega} v^s |qv - (\psi-\psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} \frac{\omega^{s+1}(p-q)(\psi-\psi\omega)}{p^{s+1}-q^{s+1}} - \frac{q\omega^{s+2}(p-q)}{p^{s+2}-q^{s+2}}, & (\psi+q)\omega \leqslant \psi, \\ \left[\begin{array}{c} \frac{2(p-q)(\psi-\psi\omega)^{s+2}}{q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}} - \frac{1}{p^{s+2}-q^{s+2}}\right) \\ +\frac{q\omega^{s+2}(p-q)}{p^{s+2}-q^{s+2}} - \frac{\omega^{s+1}(p-q)(\psi-\psi\omega)}{p^{s+1}-q^{s+1}} \end{array}\right], & (\psi+q)\omega > \psi, \end{cases}$$

$$\Phi_5^{**} = \int_0^{1} v^s |qv - (1-\psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} \frac{(p-q)(1-\psi\omega)}{p^{s+1}-q^{s+1}} - \frac{q(p-q)}{p^{s+2}-q^{s+2}}, & \psi\omega + q \leqslant 1, \\ \left[\begin{array}{c} \frac{2(p-q)(1-\psi\omega)^{s+2}}{q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}} - \frac{1}{p^{s+2}-q^{s+2}}\right) \\ +\frac{q(p-q)}{p^{s+2}-q^{s+2}} - \frac{(p-q)(1-\psi\omega)}{p^{s+1}-q^{s+1}} \end{array}\right], & \psi\omega + q > 1 \end{cases}$$

and

$$\Phi_6^{**} = \int_0^{\omega} v^s |qv - (1-\psi\omega)| \mathrm{d}_{p,q}^R v$$

$$= \begin{cases} \frac{\omega^{s+1}(p-q)(1-\psi\omega)}{p^{s+1}-q^{s+1}} - \frac{q\omega^{s+2}(p-q)}{p^{s+2}-q^{s+2}}, & (\psi+q)\omega \leqslant 1, \\ \left[\begin{array}{c} \frac{2(p-q)^2(1-\psi\omega)^{s+2}}{q^{s+1}}\left(\frac{1}{p^{s+1}-q^{s+1}} - \frac{1}{p^{s+2}-q^{s+2}}\right) \\ +\frac{q\omega^{s+2}(p-q)}{p^{s+2}-q^{s+2}} - \frac{\omega^{s+1}(p-q)(1-\psi\omega)}{p^{s+1}-q^{s+1}} \end{array}\right], & (\psi+q)\omega > 1. \end{cases}$$

THEOREM 3.4. *Let $0<q<1,\ r>1,\xi(b,a)>0$ and $\mathscr{X}:[a,a+\xi(b,a)]\to\mathbb{R}$ be a continuous and (p,q)-differentiable function on $(a,a+\xi(b,a))$ such that $\mathrm{D}^R_{p,q}\mathscr{X}$ is integrable on $[a,a+\xi(b,a)]$ and $|\mathrm{D}^R_{p,q}\mathscr{X})|^r$ is* h*-preinvex on $[a,a+\xi(b,a)]$. Then the inequality*

$$\begin{aligned}&\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)\\&\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|\\&\leqslant\xi(b,a)\mathscr{K}^{1-\frac{1}{r}}\left[\Phi_2(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r+\Phi_5(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}\\&\quad+(1-\psi)\omega^{1-\frac{1}{r}}\left[\mathscr{L}_1(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r+\mathscr{L}_2(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}\end{aligned}$$

holds for all $\psi,\omega\in[0,1]$, where Φ_2 and Φ_5 are given in Theorem 3.1 and

$$\mathscr{K}=\int_0^1|qv-(1-\psi\omega)|\mathrm{d}^R_{p,q}v=\begin{cases}\frac{p}{p+q}-\psi\omega, & \psi\omega+q\leqslant 1,\\ \left[\begin{array}{c}\frac{2(1-\psi\omega)^2}{q}\left(1-\frac{1}{p+q}\right)\\+\psi\omega-\frac{p}{p+q}\end{array}\right], & \psi\omega+q>1,\end{cases}$$

$$\mathscr{L}_1=\int_0^\omega \mathrm{h}(1-v)\mathrm{d}^R_{p,q}v$$

and

$$\mathscr{L}_2=\int_0^\omega \mathrm{h}(v)\mathrm{d}^R_{p,q}v.$$

Proof. From Lemma 2.1 and power mean integral inequality we clearly see that

$$\begin{aligned}&\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)\\&\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|\\&\leqslant\xi(b,a)\Bigg[\left(\int_0^1|qv-(1-\psi\omega)|\mathrm{d}^R_{p,q}v\right)^{1-\frac{1}{r}}\\&\quad\times\left(\int_0^1|qv-(1-\psi\omega)||\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|^r\mathrm{d}^R_{p,q}v\right)^{\frac{1}{r}}\\&\quad+(1-\psi)\left(\int_0^\omega 1\mathrm{d}^R_{p,q}v\right)^{1-\frac{1}{r}}\left(|\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|^r\mathrm{d}^R_{p,q}v\right)^{\frac{1}{r}}\Bigg].\end{aligned}\tag{3.1}$$

It follows from the h-preinvexity of the function $|\mathrm{D}_{p,q}^{R}\mathscr{X}|^{r}$ that

$$\begin{aligned}
&\int_{0}^{1} |qv-(1-\psi\omega)||\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))|^{r}\mathrm{d}_{p,q}^{R}v\\
&\leqslant \int_{0}^{1} |qv-(1-\psi\omega)|\left[\mathrm{h}(1-v)|\mathrm{D}_{p,q}^{R}\mathscr{X}(a)|^{r}+\mathrm{h}(v)|\mathrm{D}_{p,q}^{R}\mathscr{X}(b)|^{r}\right]\mathrm{d}_{p,q}^{R}v\\
&=\left(\int_{0}^{1}\mathrm{h}(1-v)|qv-(1-\psi\omega)|\mathrm{d}_{p,q}^{R}v\right)|\mathrm{D}_{p,q}^{R}\mathscr{X}(a)|^{r}\\
&\quad+\left(\int_{0}^{1}\mathrm{h}(v)|qv-(1-\psi\omega)|\mathrm{d}_{p,q}^{R}v\right)|\mathrm{D}_{p,q}^{R}\mathscr{X}(b)|^{r} \qquad (3.2)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{0}^{\omega}|\mathrm{D}_{p,q}^{R}\mathscr{X}(a+v\xi(b,a))|^{r}\mathrm{d}_{p,q}^{R}v\\
&\leqslant \int_{0}^{\omega}\left[\mathrm{h}(1-v)|\mathrm{D}_{p,q}^{R}\mathscr{X}(a)|^{r}+\mathrm{h}(v)|\mathrm{D}_{p,q}^{R}\mathscr{X}(b)|^{r}\right]\mathrm{d}_{p,q}^{R}v\\
&=\left(\int_{0}^{\omega}\mathrm{h}(1-v)\mathrm{d}_{p,q}^{R}v\right)|\mathrm{D}_{p,q}^{R}\mathscr{X}(a)|^{r}+\left(\int_{0}^{\omega}\mathrm{h}(v)\mathrm{d}_{p,q}^{R}v\right)|\mathrm{D}_{p,q}^{R}\mathscr{X}(b)|^{r}. \qquad (3.3)
\end{aligned}$$

Inequalities (3.1)–(3.3) lead to the conclusion that

$$\begin{aligned}
&\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)\\
&\quad\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_{a}^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}_{p,q}^{R}x\Bigg|\\
&\leqslant \xi(b,a)\Bigg\{\left(\int_{0}^{1}|qv-(1-\psi\omega)|\mathrm{d}_{p,q}^{R}v\right)^{1-\frac{1}{r}}\Bigg[\left(\int_{0}^{1}\mathrm{h}(1-v)|qv-(1-\psi\omega)|\mathrm{d}_{p,q}^{R}v\right)\\
&\quad\times|\mathrm{D}_{p,q}^{R}\mathscr{X}(a)|^{r}+\left(\int_{0}^{1}\mathrm{h}(v)|qv-(1-\psi\omega)|\mathrm{d}_{p,q}^{R}v\right)|\mathrm{D}_{p,q}^{R}\mathscr{X}(b)|^{r}\Bigg]+(1-\psi)\omega^{1-\frac{1}{r}}\\
&\quad\times\left[\left(\int_{0}^{\omega}\mathrm{h}(1-v)\mathrm{d}_{p,q}^{R}v\right)|\mathrm{D}_{p,q}^{R}\mathscr{X}(a)|^{r}+\left(\int_{0}^{\omega}\mathrm{h}(v)\mathrm{d}_{p,q}^{R}v\right)|\mathrm{D}_{p,q}^{R}\mathscr{X}(b)|^{r}\right]^{\frac{1}{r}}\Bigg\}.
\end{aligned}$$

This completes the proof. □

We now discuss some special cases of Theorem 3.4.

I. If we take $\mathrm{h}(v) = v$ in Theorem 3.4, then we obtain the result for h-preinvex function.

COROLLARY 3.5. *Under the assumptions of Theorem 3.4, if $\mathrm{h}(v) = v$, then the inequality*

$$\Bigg| \psi[\omega\mathscr{X}(a+\xi(b,a)) + (1-\omega)\mathscr{X}(a)] + (1-\psi) \times \mathscr{X}(a+\omega\xi(b,a)) - \frac{1}{p\xi(b,a)} \int_a^{a+p\xi(b,a)} \mathscr{X}(x)\mathrm{D}^R_{p,q}x \Bigg|$$

$$\leqslant \xi(b,a)\Bigg\{ \mathscr{K}^{1-\frac{1}{r}} \left[\Phi_2^*(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \Phi_5^*(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}} + (1-\psi)\omega^{1-\frac{1}{r}} \left[\mathscr{L}_1^*(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \mathscr{L}_2^*(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^b\right]^{\frac{1}{r}} \Bigg\}$$

holds for all $\psi,\omega \in [0,1]$, where Φ_2^* and Φ_5^* are given in Corollary 3.2, and

$$\mathscr{L}_1^* = \int_0^\omega (1-v)\mathrm{d}^R_{p,q}v = \omega - \frac{\omega^2}{p+q}$$

and

$$\mathscr{L}_2^* = \int_0^\omega v\mathrm{d}^R_{p,q}v = \frac{\omega^2}{p+q}.$$

II. If we take $\mathrm{h}(v) = v^s$ in Theorem 3.4, then we get the result for s-preinvex function of Breckner type.

COROLLARY 3.6. *Under the assumptions of Theorem 3.4, if $\mathrm{h}(v) = v^s$ with $s \in [0,1]$, then the inequality*

$$\Bigg| \psi[\omega\mathscr{X}(a+\xi(b,a)) + (1-\omega)\mathscr{X}(a)] + (1-\psi) \times \mathscr{X}(a+\omega\xi(b,a)) - \frac{1}{p\xi(b,a)} \int_a^{a+p\xi(b,a)} \mathscr{X}(x)\mathrm{D}^R_{p,q}x \Bigg|$$

$$\leqslant \xi(b,a)\Bigg\{ \mathscr{K}^{1-\frac{1}{r}} \left[\Phi_2^{**}(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \Phi_5^{**}(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}} + (1-\psi)\omega^{1-\frac{1}{r}} \left[\mathscr{L}_1^{**}(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \mathscr{L}_2^{**}(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}} \Bigg\}$$

holds for all $\psi,\omega \in [0,1]$, where Φ_2^{**} and Φ_5^{**} are given in Corollary 3.3, and

$$\mathscr{L}_1^{**}(\omega,p,q) = \int_0^\omega (1-v)^s\mathrm{d}^R_{p,q}v = (p-q)\omega\sum_{n=0}^\infty \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\omega\right)^s$$

and

$$\mathscr{L}_2^{**}(\omega,p,q)=\int_0^{\omega} v^s \mathrm{d}^R_{p,q}v=\frac{\omega^{s+1}(p-q)}{p^{s+1}-q^{s+1}}.$$

THEOREM 3.7. *Let* $0<q<1$, $q,r>1$ *with* $1/r+1/l=1$, $\xi(b,a)>0$, *and* $\mathscr{X}:[a,a+\xi(b,a)]\to\mathbb{R}$ *be a continuous and* (p,q)*-differentiable function on* $(a,a+\xi(b,a))$ *such that* $\mathrm{D}^R_{p,q}\mathscr{X}$ *is integrable on* $[a,a+\xi(b,a)]$ *and* $|\mathrm{D}^R_{p,q}\mathscr{X}|^r$ *is* h*-preinvex on* $[a,a+\xi(b,a)]$. *Then the inequality*

$$\begin{aligned}&\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)\\&\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|\\&\leqslant\xi(b,a)\Big\{\Omega_1^{\frac{1}{l}}\left[\Omega_2(q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r+\Omega_3(q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}\\&+(1-\psi)\omega^{\frac{1}{l}}\left[\mathscr{L}_1(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r+\mathscr{L}_2(\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}\Big\}\end{aligned}$$

holds for all $\psi,\omega\in[0,1]$, *where* $\mathscr{L}_1$ *and* $\mathscr{L}_2$ *are given in Theorem* 3.4, *and*

$$\Omega_1=\int_0^1|qv-(1-\psi\omega)|^l\mathrm{d}^R_{p,q}v$$

$$=\begin{cases}(p-q)\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(1-\psi\omega-\frac{q^{n+1}}{p^{n+1}}\right)^l, & 0\leqslant\psi\omega\leqslant 1-q,\\ \left[\begin{array}{c}(p-q)(1-\psi\omega)^{l+1}\sum\limits_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^l\\ +(p-q)\sum\limits_{n=0}^{\infty}\frac{q^n}{p^{n+1}}\left(\frac{q^{n+1}}{p^{n+1}}-1+\psi\omega\right)^l\\ -(p-q)(1-\psi\omega)^{l+1}\sum\limits_{n=0}^{\infty}\frac{q^{n-1}}{p^{n+1}}\left(\frac{q^n}{p^{n+1}}-1\right)^l\end{array}\right], & 1-q<\psi\omega\leqslant 1,\end{cases}$$

$$\Omega_2(q)=\int_0^1\mathrm{h}(1-v)\mathrm{d}^R_{p,q}v\quad\textit{and}\quad\Omega_3(q)=\int_0^1\mathrm{h}(v)\mathrm{d}^R_{p,q}v.$$

Proof. It follows from Lemma 2.1 and Hölder inequality that

$$\begin{aligned}&\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)\\&\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|\end{aligned}$$

$$\leqslant \xi(b,a)\Bigg[\left(\int_0^1 |qv-(1-\psi\omega)|^l \mathrm{d}^R_{p,q}v\right)^{\frac{1}{l}}\left(\int_0^1 |\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|^r \mathrm{d}^R_{p,q}v\right)^{\frac{1}{r}}$$

$$+(1-\psi)\left(\int_0^\omega 1\mathrm{d}^R_{p,q}v\right)^{\frac{1}{l}}\left(|\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|^r \mathrm{d}^R_{p,q}v\right)^{\frac{1}{r}}\Bigg]. \tag{3.4}$$

Making use of the h-preinvexity of the function $|\mathrm{D}^R_{p,q}\mathscr{X}|^r$, we have

$$\int_0^1 |\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|^r \mathrm{d}^R_{p,q}v$$

$$\leqslant \int_0^1 \left[\mathrm{h}(1-v)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \mathrm{h}(v)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]\mathrm{d}^R_{p,q}v$$

$$=\left(\int_0^1 \mathrm{h}(1-v)\mathrm{d}^R_{p,q}v\right)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \left(\int_0^1 \mathrm{h}(v)\mathrm{d}^R_{p,q}v\right)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r \tag{3.5}$$

and

$$\int_0^\omega |\mathrm{D}^R_{p,q}\mathscr{X}(a+v\xi(b,a))|^r \mathrm{d}^R_{p,q}v$$

$$\leqslant \int_0^\omega \left[\mathrm{h}(1-v)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \mathrm{h}(v)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]\mathrm{d}^R_{p,q}v$$

$$=\left(\int_0^\omega \mathrm{h}(1-v)\mathrm{d}^R_{p,q}v\right)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \left(\int_0^\omega \mathrm{h}(v)\mathrm{d}^R_{p,q}v\right)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r. \tag{3.6}$$

Inequalities (3.4)–(3.6) lead to the conclusion that

$$\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a))+(1-\omega)\mathscr{X}(a)]+(1-\psi)$$

$$\times\mathscr{X}(a+\omega\xi(b,a))-\frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)}\mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|$$

$$\leqslant \xi(b,a)\Bigg\{\left(\int_0^1 |qv-(1-\psi\omega)|^l \mathrm{d}^R_{p,q}v\right)^{\frac{1}{l}}\Bigg[\left(\int_0^1 \mathrm{h}(1-v)\mathrm{d}^R_{p,q}v\right)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r$$

$$+\left(\int_0^1 \mathrm{h}(v)\mathrm{d}^R_{p,q}v\right)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\Bigg] + (1-\psi)\omega^{1-\frac{1}{r}}\Bigg[\left(\int_0^{\omega} \mathrm{h}(1-v)\mathrm{d}^R_{p,q}v\right)$$
$$\times|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \left(\int_0^{\omega} \mathrm{h}(v)\mathrm{d}^R_{p,q}v\right)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\Bigg]^{\frac{1}{r}}\Bigg\}.$$

This completes the proof. □

Now we discuss some special cases of Theorem 3.7.

I. If we take $\mathrm{h}(v) = v$ in Theorem 3.7, then we have the result for preinvex function.

COROLLARY 3.8. *Under the assumptions of Theorem 3.7, if* $\mathrm{h}(v) = v$*, then the inequality*

$$\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a)) + (1-\omega)\mathscr{X}(a)] + (1-\psi)$$
$$\times\mathscr{X}(a+\omega\xi(b,a)) - \frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} \mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|$$
$$\leqslant \xi(b,a)\Bigg\{\Omega_1^{\frac{1}{l}}\left[\left(1-\frac{1}{p+q}\right)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \frac{1}{p+q}|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}$$
$$+(1-\psi)\omega^{\frac{1}{p}}\left[\left(\omega-\frac{\omega^2}{p+q}\right)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \frac{\omega^2}{p+q}|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}\Bigg\}$$

holds for all $\psi,\omega \in [0,1]$.

II. If we take $\mathrm{h}(v) = v^s$ in Theorem 3.7, then we get the result for s-preinvex function of Breckner type.

COROLLARY 3.9. *Under the assumptions of Theorem 3.7, if* $\mathrm{h}(v) = v^s$ *with* $s \in [0,1]$*, then the inequality*

$$\Bigg|\psi[\omega\mathscr{X}(a+\xi(b,a)) + (1-\omega)\mathscr{X}(a)] + (1-\psi)$$
$$\times\mathscr{X}(a+\omega\xi(b,a)) - \frac{1}{p\xi(b,a)}\int_a^{a+p\xi(b,a)} \mathscr{X}(x)\mathrm{D}^R_{p,q}x\Bigg|$$
$$\leqslant \xi(b,a)\Bigg\{\Omega_1^{\frac{1}{l}}\left[\Omega_2^*(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \Omega_3^*(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}$$
$$+(1-\psi)\omega^{\frac{1}{p}}\left[\mathscr{L}_1^{**}(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(a)|^r + \mathscr{L}_2^{**}(\psi,\omega;p,q)|\mathrm{D}^R_{p,q}\mathscr{X}(b)|^r\right]^{\frac{1}{r}}\Bigg\}$$

holds for all $\psi,\omega \in [0,1]$, *where* $\mathscr{L}_1^{**}, \mathscr{L}_2^{**}$ *are given in Corollary* 3.6, *and*

$$\Omega_2^* = \int_0^1 (1-v)^s \mathrm{d}_{p,q}^R v = (p-q)\sum_{n=0}^{\infty} \frac{q^n}{p^{n+1}}\left(1-\frac{q^n}{p^{n+1}}\right)^s$$

and

$$\Omega_1^* = \int_0^1 v^s \mathrm{d}_{p,q}^R v = \frac{p-q}{p^{s+1}-q^{s+1}}.$$

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