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BOUNDING THE SÁNDOR–YANG MEANS FOR THE COMBINATIONS OF CONTRAHARMONIC AND ARITHMETIC MEANS

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Abstract. In the article, we prove that $t_1 = 1/2 + \sqrt{2^{1/(2p)}e^{(\pi-4)/(4p)} - 1}/2$, $t_2 = 1/2 + \sqrt{6p}/(12p)$, $t_3 = 1/2 + \sqrt{(1+\sqrt{2})^{\sqrt{2}/p}/e^{1/p} - 1}/2$ and $t_4 = 1/2 + \sqrt{3p}/(6p)$ are the best possible parameters on the interval $[1/2, 1]$ such that the double inequalities

$$C^p[t_1u + (1-t_1)v, t_1v + (1-t_1)u]A^{1-p}(u,v) < Q(u,v)e^{\frac{A(u,v)}{\mathscr{T}(u,v)}-1}$$
$$< C^p[t_2u + (1-t_2)v, t_2v + (1-t_2)u]A^{1-p}(u,v),$$
$$C^p[t_3u + (1-t_3)v, t_3v + (1-t_3)u]A^{1-p}(u,v) < A(u,v)e^{\frac{Q(u,v)}{\mathscr{NS}(u,v)}-1}$$
$$< C^p[t_4u + (1-t_4)v, t_4v + (1-t_4)u]A^{1-p}(u,v)$$

hold for all $u, v > 0$ with $u \neq v$ and $p \in [1/2, \infty)$, where $A(u,v) = (u+v)/2$, $Q(u,v) = \sqrt{(u^2+v^2)/2}$, $C(u,v) = (u^2+v^2)/(u+v)$, $\mathscr{T}(u,v) = (u-v)/[2\arctan((u-v)/(u+v))]$ and $\mathscr{NS}(u,v) = (u-v)/[2\sinh^{-1}((u-v)/(u+v))]$ are respectively the arithmetic, quadratic, contraharmonic, Seiffert and Neuman-Sándor means of u and v, and $\sinh^{-1}(x) = \log(x + \sqrt{x^2+1})$ is the inverse hyperbolic sine function.

1. Introduction

A real-valued function $M : (0,\infty) \times (0,\infty) \to (0,\infty)$ is said to be a bivariate mean [3] if

$$\min\{x,y\} \leqslant M(x,y) \leqslant \max\{x,y\}$$

for all $x, y \in (0,\infty)$.

It is well-known that the bivariate means have wide applications in mathematics and other natural sciences [1, 4, 6, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 36, 38], they have attracted the attention of many researchers [7, 8, 9, 10, 11, 39, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 52, 53, 54, 55, 56, 58, 59, 60].

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Let $x,y>0$. Then the Schwab-Borchardt mean $SB(x,y)$ [34, 35] of x and y is defined by

$$SB(u,v)=\begin{cases}\frac{\sqrt{y^2-x^2}}{\arccos(x/y)}, & x<y,\\ x, & x=y,\\ \frac{\sqrt{x^2-y^2}}{\cosh^{-1}(x/y)}, & x>y,\end{cases}$$

where $\cosh^{-1}(x)=\log(x+\sqrt{x^2-1})$ is the inverse hyperbolic cosine functions. Carlson [5] proved that

$$SB(x,y)=\frac{2}{\int_0^\infty \frac{dt}{(t+y^2)\sqrt{t+x^2}}}.$$

We clearly see that the Schwab-Borchardt mean $SB(x,y)$ is strictly increasing in both x and y, and nonsymmetric and homogeneous of degree one with respect to its variables x and y. Many symmetric bivariate means can be derived from the Schwab-Borchardt mean, for example,

$$\mathscr{T}(u,v)=\frac{u-v}{2\arctan\left(\frac{u-v}{u+v}\right)}=SB[A(u,v),Q(u,v)], \tag{1.1}$$

$$\mathscr{N}\mathscr{S}(u,v)=\frac{u-v}{2\sinh^{-1}\left(\frac{u-v}{u+v}\right)}=SB[Q(u,v),A(u,v)] \tag{1.2}$$

and

$$\mathscr{L}(u,v)=\frac{u-v}{\log u-\log v}=SB[A(u,v),G(u,v)]$$

are respectively the Seiffert mean, Neuman-Sándor mean and logarithmic mean of two positive numbers u and v, where

$$G(u,v)=\sqrt{uv},\quad A(u,v)=\frac{u+v}{2},\quad Q(u,v)=\sqrt{\frac{u^2+v^2}{2}} \tag{1.3}$$

are respectively the geometric, arithmetic, and quadratic means, and $\sinh^{-1}(x)=\log(x+\sqrt{x^2+1})$ is the inverse hyperbolic sine function.

In 2012, Sándor [40] introduced a new symmetric mean of two positive numbers u and v defined by

$$X(u,v)=A(u,v)e^{G(u,v)/SB[G(u,v),A(u,v)])-1}.$$

In 2017, as an example of a family of two-parameter hyperbolic means, Yang [51] showed that $SY(x,y)=ye^{x/SB(x,y)-1}$ is a nonsymmetric mean of x and y, and introduced two Sándor-type symmetric means as follows:

$$SY_{AQ}(u,v)=SY(A(u,v),Q(u,v))=Q(u,v)e^{\frac{A(u,v)}{\mathscr{T}(u,v)}-1}, \tag{1.4}$$

$$SY_{QA}(u,v)=SY(Q(u,v),A(u,v))=A(u,v)e^{\frac{Q(u,v)}{\mathscr{N}\mathscr{S}(u,v)}-1}. \tag{1.5}$$

In what follows, we call $SY_{AQ}(u,v)$ and $SY_{QA}(u,v)$ Sándor-Yang means.

Let $u,v>0$, $t\in[1/2,1]$, $p\in[1/2,\infty)$ and

$$CA(t,p;u,v)=C^{p}[tu+(1-t)v,tv+(1-t)u]A^{1-p}(u,v), \tag{1.6}$$

where

$$C(u,v)=\frac{u^2+v^2}{u+v} \tag{1.7}$$

is the contraharmonic mean of u and v. Then from (1.6) and (1.7) we clearly see that

$$CA(t,1/2;u,v)=Q[tu+(1-t)v,tv+(1-t)u], \tag{1.8}$$

$$CA(t,1;u,v)=C[tu+(1-t)v,tv+(1-t)u] \tag{1.9}$$

are respectively the one-parameter quadratic and contraharmonic means, and the function $t\to CA(t,p;u,v)$ is strictly increasing on $[1/2,1]$ for fixed $p\in[1/2,\infty)$ and $u,v>0$ with $u\neq v$.

Recently, the Sándor-Yang means have been the subject of intensive research. Zhao, Qian and Song [57] proved that $\alpha=\log 2/[1+\log 2-\sqrt{2}\log(1+\sqrt{2})]=1.5517\cdots$, $\beta=5/3$, $\lambda=4\log 2/(4+2\log 2-\pi)=1.2351\cdots$ and $\mu=4/3$ are the best possible constants such that the double inequalities

$$M_{\alpha}(u,v)<SY_{QA}(u,v)<M_{\beta}(u,v),\quad M_{\lambda}(u,v)<SY_{AQ}(u,v)<M_{\mu}(u,v) \tag{1.10}$$

hold for all $u,v>0$ with $u\neq v$, where

$$M_r(u,v)=\left(\frac{u^r+v^r}{2}\right)^{1/r}\ (u\neq v),\quad M_0(u,v)=\sqrt{uv}=G(u,v)$$

is the rth power mean of u and v.

It is well known that the inequalities

$$\begin{aligned}H(u,v)&=M_{-1}(u,v)<M_0(u,v)=G(u,v)<M_1(u,v)\\&=A(u,v)<M_2(u,v)=Q(u,v)<C(u,v)\end{aligned} \tag{1.11}$$

hold for all $u,v>0$ with $u\neq v$, and the function $r\to M_r(u,v)$ is strictly increasing for fixed $u,v>0$ with $u\neq v$.

From (1.3), (1.6) and (1.7) we clearly see that

$$CA(1/2,p;u,v)=A(u,v), \tag{1.12}$$

$$CA(1,p;u,v)=A(u,v)\left[\frac{Q(u,v)}{A(u,v)}\right]^{2p}\geqslant Q(u,v) \tag{1.13}$$

for all $u,v>0$ with $u\neq v$ and $p\in[1/2,\infty)$.

Inequalities (1.10)–(1.13) lead to

$$\begin{aligned}CA(1/2,p;u,v)&=A(u,v)=M_1(u,v)<SY_{AQ}(u,v)\\&<SY_{QA}(u,v)<M_2(u,v)=Q(u,v)\leqslant CA(1,p;u,v).\end{aligned} \tag{1.14}$$

Motivated by inequality (1.14), it is natural to ask what are the best parameters $t_1 = t_1(p)$, $t_2 = t_2(p)$, $t_3 = t_3(p)$ and $t_4 = t_4(p) \in [1/2, 1]$ such that the double inequalities

$$CA(t_1, p; u, v) < SY_{AQ}(u, v) < CA(t_2, p; u, v),$$

$$CA(t_3, p; u, v) < SY_{QA}(u, v) < CA(t_4, p; u, v)$$

hold for all $u, v > 0$ with $u \neq v$ and $p \in [1/2, \infty)$? The aim of this article is to answer this question.

2. Lemmas

In order to prove our main results, we need introduce and establish three lemmas which we present in this section.

LEMMA 2.1. (See [2]) *Let $x_1, x_2 \in \mathbb{R}$ with $x_1 < x_2$, $F, G : [x_1, x_2] \to \mathbb{R}$ be continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2) with $G'(x) \neq 0$ on (x_1, x_2). Then the functions*

$$\frac{F(x) - F(x_1)}{G(x) - G(x_1)}, \quad \frac{F(x) - F(x_2)}{G(x) - G(x_2)}$$

are (strictly) increasing (decreasing) on (x_1, x_2) if $F'(x)/G'(x)$ is (strictly) increasing (decreasing) on (x_1, x_2).

LEMMA 2.2. *Let $p \in [1/2, \infty)$, $u, x \in (0, 1)$ and*

$$F(u, p; x) = p\log(1 + ux^2) - \frac{1}{2}\log(1 + x^2) - \frac{\arctan(x)}{x} + 1. \tag{2.1}$$

Then the following statements are true:

(1) *$F(u, p; x) > 0$ for $x \in (0, 1)$ if and only if $u \geqslant 1/(6p)$;*

(2) *$F(u, p; x) < 0$ for $x \in (0, 1)$ if and only if $u \leqslant e^{(\pi+2\log 2-4)/(4p)} - 1$.*

Proof. It follows from (2.1) that

$$F(u, p; 0^+) = 0, \tag{2.2}$$

$$F(u, p; 1^-) = p\log(1 + u) - \frac{1}{2}\log 2 - \frac{\pi}{4} + 1, \tag{2.3}$$

$$\frac{\partial F(u, p; x)}{\partial x} = \frac{(2p-1)x + \arctan(x)}{1 + ux^2}[u - f(x)], \tag{2.4}$$

where

$$f(x) = \frac{x - \arctan(x)}{(2p-1)x^3 + x^2\arctan(x)}.$$

Let $f_1(x) = x - \arctan(x)$ and $f_2(x) = (2p-1)x^3 + x^2\arctan(x)$. Then we clearly see that

$$f_1(0^+) = f_2(0^+) = 0, \quad f(x) = \frac{f_1(x)}{f_2(x)} \tag{2.5}$$

and elaborated computations lead to

$$\frac{f_1'(x)}{f_2'(x)} = \frac{1}{\frac{2[(1+x^2)\arctan(x)]}{x} + 3(2p-1)x^2 + 2(3p-1)}. \tag{2.6}$$

Noting that $x > \arctan(x)$ for $x \in (0,1)$, and

$$\frac{d\left[\frac{(1+x^2)\arctan(x)}{x}\right]}{dx} = \frac{x^2\arctan(x) + x - \arctan(x)}{x^2} > 0$$

for all $x \in (0,1)$. Thus the function $x \mapsto [(1+x^2)\arctan(x)]/x$ is strictly increasing and maps $(0,1)$ onto $(1,\pi/2)$. It follows from (2.6) that the function $f_1'(x)/f_2'(x)$ is strictly decreasing on $(0,1)$. Therefore, $f(x)$ is strictly decreasing on $(0,1)$ by Lemma 2.1 and (2.5). Moreover, making use of L'Hôpital's rule we get

$$f(0^+) = \lim_{x\to 0}\frac{f_1'(x)}{f_2'(x)} = \frac{1}{6p}, \tag{2.7}$$

$$f(1^-) = \frac{4-\pi}{4(2p-1)+\pi}. \tag{2.8}$$

We divide the proof into two cases.

Case 1 $u \in [1/(6p),1)$. Then from (2.4) and (2.7) together with the monotonicity of $f(x)$ we clearly see that the function $x \to F(u,p;x)$ is strictly increasing on $(0,1)$. Therefore, $F(u,p;x) > 0$ for all $x \in (0,1)$ follows from (2.2).

Case 2 $u \in (0,1/(6p))$. Then it follows from (2.4), (2.7) and (2.8) together with the monotonicity of $f(x)$ that either the function $x \to F(u,p;x)$ is strictly decreasing on the whole interval $(0,1)$ or there exists $x^* \in (0,1)$ such that $F(u,p;x)$ is strictly decreasing on $(0,x^*)$ and strictly increasing on $(x^*,1)$. Consequently, in both cases, inequality $F(u,p;x) \geqslant 0$ does not hold for all $x \in (0,1)$, and $F(u,p;x) \leqslant 0$ for all $x \in (0,1)$ if and only if $F(u,p;1) \leqslant 0$, namely $u \leqslant e^{(\pi+2\log 2-4)/(4p)} - 1$ by (2.3). □

LEMMA 2.3. *Let* $p \in [1/2,\infty)$, $v,x \in (0,1)$ *and*

$$G(v,p;x) = p\log(1+vx^2) - \frac{\sqrt{1+x^2}\sinh^{-1}(x)}{x} + 1. \tag{2.9}$$

Then the following statements are true:

(1) $G(v,p;x) > 0$ *for* $x \in (0,1)$ *if and only if* $v \geqslant 1/(3p)$*;*

(2) $G(v,p;x) < 0$ *for* $x \in (0,1)$ *if and only if* $v \leqslant ((1+\sqrt{2})^{\sqrt{2}}/e)^{1/p} - 1$.

Proof. It follows from (2.9) that

$$G(v,p;0^+) = 0, \tag{2.10}$$

$$G(v,p;1^-) = p\log(1+v) - \sqrt{2}\log(1+\sqrt{2}) + 1 \tag{2.11}$$

and

$$\frac{\partial G(v,p;x)}{\partial x}=\frac{(2p-1)x\sqrt{1+x^2}+\sinh^{-1}(x)}{(1+vx^2)\sqrt{1+x^2}}[v-g(x)], \tag{2.12}$$

where

$$g(x)=\frac{x\sqrt{1+x^2}-\sinh^{-1}(x)}{(2p-1)x^3\sqrt{1+x^2}+x^2\sinh^{-1}(x)}.$$

Let $g_1(x)=x\sqrt{1+x^2}-\sinh^{-1}(x)$ and $g_2(x)=(2p-1)x^3\sqrt{1+x^2}+x^2\sinh^{-1}(x)$. Then we clearly see that

$$g_1(0^+)=g_2(0^+)=0, \quad g(x)=\frac{g_1(x)}{g_2(x)} \tag{2.13}$$

and

$$\frac{g_1'(x)}{g_2'(x)}=\frac{1}{\frac{\sqrt{1+x^2}\sinh^{-1}(x)}{x}+2(2p-1)x^2+3p-1}. \tag{2.14}$$

Noting that $x>\sinh^{-1}(x)$ for $x\in(0,1)$, and

$$\frac{d\left[\frac{\sqrt{1+x^2}\sinh^{-1}(x)}{x}\right]}{dx}=\frac{x\sqrt{1+x^2}-\sinh^{-1}(x)}{x^2\sqrt{1+x^2}}>0$$

for all $x\in(0,1)$. Thus the function $x\mapsto[\sqrt{1+x^2}\sinh^{-1}(x)]/x$ is strictly increasing and maps $(0,1)$ onto $(1,\sqrt{2}\log(1+\sqrt{2}))$. It follows from (2.14) that $g_1'(x)/g_2'(x)$ is strictly decreasing on $(0,1)$. Therefore, $g(x)$ is strictly decreasing on $(0,1)$ by Lemma 2.1 and (2.13). Moreover, making use of L'Hôpital's rule we get

$$g(0^+)=\lim_{x\to 0}\frac{g_1'(x)}{g_2'(x)}=\frac{1}{3p}, \tag{2.15}$$

$$g(1^-)=\frac{\sqrt{2}-\log(1+\sqrt{2})}{(2p-1)\sqrt{2}+\log(1+\sqrt{2})}:=\lambda. \tag{2.16}$$

We divide the proof into two cases.

Case 1 $v\in[1/(3p),1)$. Then from (2.12) and (2.15) together with the monotonicity of $g(x)$ we know that the function $x\to G(v,p;x)$ is strictly increasing on $(0,1)$. Therefore, $G(v,p;x)>0$ for all $x\in(0,1)$ follows from (2.10).

Case 2 $v\in(0,1/(3p))$. Then it follows from (2.12), (2.15) and (2.16) together with the monotonicity of $g(x)$ that either the function $x\to G(v,p;x)$ is strictly decreasing on the whole interval $(0,1)$ or there exists $x_0^*\in(0,1)$ such that $G(v,p;x)$ is strictly decreasing on $(0,x_0^*)$ and strictly increasing on $(x_0^*,1)$. Consequently, in both cases, inequality $G(v,p;x)\geqslant 0$ does not hold for all $x\in(0,1)$, and $G(v,p;x)\leqslant 0$ for all $x\in(0,1)$ if and only if $G(v,p;1)\leqslant 0$, namely $v\leqslant[(1+\sqrt{2})^{\sqrt{2}}/e]^{1/p}-1$ by (2.11). $\square$

3. Main results

THEOREM 3.1. *Let* $t_1, t_2 \in [1/2, 1]$ *and* $p \in [1/2, \infty)$. *Then the double inequality*

$$CA(t_1, p; u, v) < SY_{AQ}(u, v) < CA(t_2, p; u, v)$$

holds for all $u, v > 0$ *with* $u \neq v$ *if and only if* $t_1 \leqslant 1/2 + \sqrt{2^{1/(2p)}e^{(\pi-4)/(4p)} - 1}/2$ *and* $t_2 \geqslant 1/2 + \sqrt{6p}/(12p)$.

Proof. Without loss of generality, we assume that $u > v > 0$. Let $t \in [1/2, 1]$ and $x = (u-v)/(u+v) \in (0, 1)$. Then from (1.1), (1.3), (1.4), (1.6) and (1.7) we get

$$\frac{CA(t, p; u, v)}{A(u, v)} = \left[1 + (2t-1)^2 x^2\right]^p, \tag{3.1}$$

$$\frac{SY_{AQ}(u, v)}{A(u, v)} = \sqrt{1 + x^2} e^{\arctan(x)/x - 1}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\log\left[\frac{CA(t, p; u, v)}{SY_{AQ}(u, v)}\right] = \log\left[\frac{CA(t, p; u, v)}{A(u, v)}\right] - \log\left[\frac{SY_{AQ}(u, v)}{A(u, v)}\right] \tag{3.3}$$
$$= p\log[1 + (2t-1)^2 x^2] - \frac{1}{2}\log(1 + x^2) - \frac{\arctan(x)}{x} + 1.$$

Therefore, Theorem 3.1 follows easily from Lemma 2.2 and (3.3). ☐

THEOREM 3.2. *Let* $t_3, t_4 \in [1/2, 1]$ *and* $p \in [1/2, \infty)$. *Then the double inequality*

$$CA(t_3, p; u, v) < SY_{QA}(u, v) < CA(t_4, p; u, v)$$

holds for all $u, v > 0$ *with* $u \neq v$ *if and only if* $t_3 \leqslant 1/2 + \sqrt{(1+\sqrt{2})^{\sqrt{2}/p}/e^{1/p} - 1}/2$ *and* $t_4 \geqslant 1/2 + \sqrt{3p}/(6p)$.

Proof. Without loss of generality, we assume that $u > v > 0$. Let $t \in [1/2, 1]$ and $x = (u-v)/(u+v) \in (0, 1)$. Then from (1.2), (1.3) and (1.5) we get

$$\frac{SY_{QA}(u, v)}{A(u, v)} = e^{[\sqrt{1+x^2}\sinh^{-1}(x)]/x - 1}. \tag{3.4}$$

It follows from (3.1) and (3.4) that

$$\log\left[\frac{CA(t, p; u, v)}{SY_{QA}(u, v)}\right] = \log\left[\frac{CA(t, p; u, v)}{A(u, v)}\right] - \log\left[\frac{SY_{QA}(u, v)}{A(u, v)}\right]$$
$$= p\log[1 + (2t-1)^2 x^2] - \frac{\sqrt{1+x^2}\sinh^{-1}(x)}{x} + 1. \tag{3.5}$$

Therefore, Theorem 3.2 follows easily from Lemma 2.3 and (3.5). ☐

From (1.8), (1.9) and Theorems 3.1 and 3.2 we get Corollary 3.3 immediately.

COROLLARY 3.3. *Let* $t_5,t_6,t_7,t_8,t_9,t_{10},t_{11},t_{12}\in[1/2,1]$. *Then the double inequalities*

$$Q[t_5u+(1-t_5)v,t_5v+(1-t_5)u]<SY_{AQ}(u,v)<Q[t_6u+(1-t_6)v,t_6v+(1-t_6)u],$$

$$Q[t_7u+(1-t_7)v,t_7v+(1-t_7)u]<SY_{QA}(u,v)<Q[t_8u+(1-t_8)v,t_8v+(1-t_8)u],$$

$$C[t_9u+(1-t_9)v,t_9v+(1-t_9)u]<SY_{AQ}(u,v)<C[t_{10}u+(1-t_{10})v,t_{10}v+(1-t_{10})u],$$

$$C[t_{11}u+(1-t_{11})v,t_{11}v+(1-t_{11})u]<SY_{QA}(u,v)<C[t_{12}u+(1-t_{12})v,t_{12}v+(1-t_{12})u]$$

hold for all $u,v>0$ *with* $u\neq v$ *if and only if*

$$t_5\leqslant 1/2+\sqrt{2e^{(\pi-4)/2}-1}/2\approx 0.7747,\quad t_6\geqslant 1/2+\sqrt{3}/6\approx 0.7886,$$

$$t_7\leqslant 1/2+\sqrt{(1+\sqrt{2})^{2\sqrt{2}}/e^2-1}/2\approx 0.8990,\quad t_8\geqslant 1/2+\sqrt{6}/6=0.9082,$$

$$t_9\leqslant 1/2+\sqrt{\sqrt{2}e^{(\pi-4)/4}-1}/2\approx 0.6878,\quad t_{10}\geqslant 1/2+\sqrt{6}/12\approx 0.7041,$$

$$t_{11}\leqslant 1/2+\sqrt{(1+\sqrt{2})^{\sqrt{2}}/e-1}/2\approx 0.7643,\quad t_{12}\geqslant 1/2+\sqrt{3}/6\approx 0.7886.$$

Let $x\in(0,1)$, $p\in[1/2,\infty)$, $u=1+x$, $v=1-x$, $t_1=1/2+\sqrt{2^{1/(2p)}e^{(\pi-4)/(4p)}-1}/2$, $t_2=1/2+\sqrt{6p}/(12p)$, $t_3=1/2+\sqrt{(1+\sqrt{2})^{\sqrt{2}/p}/e^{1/p}-1}/2$ and $t_4=1/2+\sqrt{3p}/(6p)$. Then equations (1.1)–(1.7) and Theorems 3.1 and 3.2 lead to the conclusion that the double inequalities

$$1+p\log\left[1+\left(\left(\frac{2}{e^{(4-\pi)/2}}\right)^{\frac{1}{2p}}-1\right)x^2\right]-\frac{1}{2}\log\left(1+x^2\right)$$

$$<\frac{\arctan(x)}{x}<1+p\log\left(1+\frac{x^2}{6p}\right)-\frac{1}{2}\log\left(1+x^2\right), \tag{3.6}$$

$$1+p\log\left[1+\left(\left(\frac{(1+\sqrt{2})^{\sqrt{2}}}{e}\right)^{\frac{1}{p}}-1\right)x^2\right]-\frac{1}{2}\log\left(1+x^2\right)$$

$$<\frac{\sinh^{-1}(x)}{x}<1+p\log\left(1+\frac{x^2}{3p}\right)-\frac{1}{2}\log\left(1+x^2\right) \tag{3.7}$$

hold for all $x\in(0,1)$ and $p\in[1/2,\infty)$.

Let $p=1/2$ in (3.6) and (3.7), then we obtain the following Corollary 3.4.

COROLLARY 3.4. *The double inequalities*

$$1+\frac{1}{2}\log\left[1+\left(\frac{2}{e^{(4-\pi)/2}}-1\right)x^2\right]-\frac{1}{2}\log\left(1+x^2\right)$$
$$<\frac{\arctan(x)}{x}<1+\frac{1}{2}\log\left(1+\frac{x^2}{3}\right)-\frac{1}{2}\log\left(1+x^2\right),$$
$$1+\frac{1}{2}\log\left[1+\left(\frac{(1+\sqrt{2})^{2\sqrt{2}}}{e^2}-1\right)x^2\right]-\frac{1}{2}\log\left(1+x^2\right)$$
$$<\frac{\sinh^{-1}(x)}{x}<1+\frac{1}{2}\log\left(1+\frac{2x^2}{3}\right)-\frac{1}{2}\log\left(1+x^2\right)$$

hold for all $x\in(0,1)$.

REMARK 3.5. One of the referees pointed out the functions on the right-hand sides of (3.6) and (3.7) are strictly increasing on $p\in[1/2,\infty)$, and while the functions on the left-hand sides are strictly decreasing on $p\in[1/2,\infty)$. Actually, for fixed $a>0$ with $a\neq 1$ and $x\in(0,1)$, set $f_1(t)=\log[1+(a^t-1)x^2]/t$ and $f_2(t)=\log[1+tx^2]/t$, then by Lemma 2.1 we easily obtain that $f_1(t)$ is strictly increasing on $(0,\infty)$, and $f_2(t)$ are strictly decreasing on $(0,\infty)$. Letting $t=1/p$ in $f_1(t)$, and $t=1/(3p)$, $t=1/(6p)$ in $f_2(t)$, then the assertions about the monotonicity properties of the functions on two sides of (3.6) and (3.7) with respect to p follow. In conclusion, the upper and lower bounds in Corollary 3.4 are optimal.

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