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OPTIMAL BOUNDS FOR THE SÁNDOR MEAN IN TERMS OF THE COMBINATION OF GEOMETRIC AND ARITHMETIC MEANS

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Abstract. In this paper, we prove that $\lambda = 1/2-\sqrt{1-e^{-2/p}}/2$ and $\mu = 1/2-\sqrt{6p}/(6p)$ are the best possible parameters on the interval $(0,1/2)$ such that the double inequalities

$$G^p[\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a]A^{1-p}(a,b) < X(a,b)$$
$$< G^p[\mu a+(1-\mu)b, \mu b+(1-\mu)a]A^{1-p}(a,b)$$

hold for all $p \in [1,\infty)$ and $a,b>0$ with $a \neq b$, where $G(a,b)$ is the geometric mean, $A(a,b)$ is the arithmetic mean, and $X(a,b)$ is the Sándor mean.

1. Introduction

Let $r \in \mathbb{R}$ and $a,b>0$ with $a \neq b$. Then the geometric mean $G(a,b)$, the arithmetic mean $A(a,b)$, first Seiffert mean $P(a,b)$ [4], Sándor mean $X(a,b)$ [6] and rth power mean $M_r(a,b)$ are defined by

$$G(a,b)=\sqrt{ab}, \quad A(a,b)=\frac{a+b}{2}$$

$$P(a,b)=\frac{a-b}{2\arcsin[(a-b)/(a+b)]}, \quad X(a,b)=A(a,b)e^{G(a,b)/P(a,b)-1}, \tag{1.1}$$

and

$$M_r(a,b)=\left(\frac{a^r+b^r}{2}\right)^{1/r} (r\neq 0), \quad M_0(a,b)=a^{1/2}b^{1/2}.$$

It is well known the rth power mean is continuous and strictly increasing with respect to $r \in \mathbb{R}$ for fixed distinct positive real numbers a and b, and the inequalities

$$H(a,b)=M_{-1}(a,b)<G(a,b)=M_0(a,b)<A(a,b)=M_1(a,b) \tag{1.2}$$

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hold for all $a,b>0$ with $a\neq b$, where $H(a,b)=2ab/(a+b)$ is the harmonic mean of a and b. Recently, the Sándor mean has attracted the attention of several researchers. In [7], Sándor established the inequalities

$$X(a,b)<\frac{P^2(a,b)}{A(a,b)},\frac{A(a,b)G(a,b)}{P(a,b)}<X(a,b)<\frac{A(a,b)P(a,b)}{2P(a,b)-G(a,b)},$$

$$\frac{A(a,b)[P(a,b)+G(a,b)]}{3P(a,b)-G(a,b)}<X(a,b)<A(a,b)\left[\frac{1}{e}+\left(1-\frac{1}{e}\right)\frac{G(a,b)}{P(a,b)}\right],$$

$$A(a,b)+G(a,b)-P(a,b)<X(a,b)<A^{-1/3}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^{4/3},$$

$$P^{1/[\log(\pi/2)]}(a,b)A^{1-1/[\log(\pi/2)]}(a,b)<X(a,b)<P^{-1}(a,b)\left[\frac{A(a,b)+G(a,b)}{2}\right]^2$$

hold for all $a,b>0$ with $a\neq b$.

Chu, Yang and Wu [3] proved that the double inequality

$$M_\alpha(a,b)<X(a,b)<M_\beta(a,b) \tag{1.3}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha\leqslant 1/3$ and $\beta\geqslant \log 2/(1+\log 2)=0.4903\cdots$.

In [8], Zhou et al. proved that the double inequality

$$H_\lambda(a,b)<X(a,b)<H_\mu(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\lambda\leqslant 1/2$ and $\mu\geqslant \log 3/(1+\log 2)=0.6488\cdots$, where $H_q(a,b)=\left[\left(a^q+(ab)^{q/2}+b^q\right)/3\right]^{1/q}$ $(q\neq 0)$ and $H_0(a,b)=\sqrt{ab}$ is the qth power-type Heronian mean of a and b.

Qian, Chu and Zhang [5] proved that the double inequalities

$$\alpha_1A(a,b)+(1-\alpha_1)H(a,b)<X(a,b)<\beta_1A(a,b)+(1-\beta_1)H(a,b),$$
$$\alpha_2A(a,b)+(1-\alpha_2)G(a,b)<X(a,b)<\beta_2A(a,b)+(1-\beta_2)G(a,b)$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leqslant 1/e$, $\beta_1\geqslant 2/3$, $\alpha_2\leqslant 1/3$ and $\beta_2\geqslant 1/e$.

Let $a,b>0$, $p\in[1,\infty)$, $t\in(0,1/2)$ and

$$GA_{t,p}(a,b)=G^p[ta+(1-t)b,\ tb+(1-t)a]A^{1-p}(a,b). \tag{1.4}$$

It is not difficult to verify that (See [2])

$$GA_{t,1}(a,b)=G[ta+(1-t)b,tb+(1-t)a], \tag{1.5}$$
$$GA_{t,2}(a,b)=H[ta+(1-t)b,tb+(1-t)a] \tag{1.6}$$

and $GA_{t,p}(a,b)$ is strictly increasing with respect to $t\in(0,1/2)$ for fixed $a,b>0$ with $a\neq b$.

From (1.2)–(1.6) and monotonicity of the function, we clearly see that

$$\begin{aligned}GA_{1,2}(a,b) &= H(a,b) = M_{-1}(a,b) < GA_{1,1}(a,b) = G(a,b) = M_0(a,b)\\ &< X(a,b) < M_1(a,b) = A(a,b) = GA_{t,0}(a,b) = GA_{1/2,1/2}(a,b)\end{aligned} \tag{1.7}$$

hold for all $a,b>0$ with $a \neq b$.

Motivated by inequality (1.7), it is natural to ask, for fixed $p \in [1,\infty)$, what are the best possible parameters $\lambda,\mu \in (0,1/2)$ such that the double inequality

$$GA_{\lambda,p}(a,b) < X(a,b) < GA_{\mu,p}(a,b)$$

holds and $a,b>0$ with $a \neq b$? The aim of this paper is to answer this question.

2. Lemmas

In order to prove the desired theorems we need following five Lemmas, which we present in this section.

LEMMA 2.1. (See [1, Theorem 1.25]) *For $-\infty < a < b < +\infty$, let $f,g:[a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \textit{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}.$$

If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2.2. *The inequality*

$$\frac{2}{3p}+e^{-2/p}<1 \tag{2.1}$$

hold for all $p \in [1,\infty)$.

Proof. It is easy to check that inequality $t/3+e^{-t}<1$ for $t \in (0,2]$. By substituting $t=2/p$, we obtain Lemma 2.2 immediately. □

LEMMA 2.3. *The function*

$$\phi(x) = \frac{x^3\sqrt{1-x^2}}{\arcsin(x) - x\sqrt{1-x^2}}$$

decreases on the interval $(0,1)$ from $3/2$ to 0.

Proof. Let $\phi_1(x) = x^3\sqrt{1-x^2}$, $\phi_2(x) = \arcsin(x) - x\sqrt{1-x^2}$. Then elaborated computations lead to $\phi_2'(x) = 2x^2/\sqrt{1-x^2} > 0$ for $x \in (0,1)$, and

$$\phi(x) = \frac{\phi_1(x)}{\phi_2(x)} = \frac{\phi_1(x) - \phi_1(0)}{\phi_2(x) - \phi_2(0)}, \tag{2.2}$$

$$\frac{\phi'_1(x)}{\phi'_2(x)} = -2x^2 + \frac{3}{2}. \tag{2.3}$$

It is well known that the function $x \mapsto -2x^2 + 3/2$ is strictly decreasing on $(0,1)$, hence (2.3) leads to the conclusion that the function $\phi_1'(x)/\phi_2'(x)$ is strictly decreasing on $(0,1)$.

Note that

$$\phi(0^+) = \lim_{x\to 0^+} \frac{\phi_1'(x)}{\phi_2'(x)} = \frac{3}{2}, \quad \phi(1^-) = 0. \tag{2.4}$$

Therefore, Lemma 2.3 follows from (2.2), (2.4) and Lemma 2.1 together with the monotonicity of $\phi_1'(x)/\phi_2'(x)$. □

LEMMA 2.4. *The function*

$$\varphi(x) = \frac{x^2 \arcsin(x)}{\arcsin(x) - x\sqrt{1-x^2}}$$

decreases on the interval $(0,1)$ *from* $3/2$ *to* 1.

Proof. Let $\varphi_1(x) = x^2\arcsin(x)$, $\varphi_2(x) = \arcsin(x) - x\sqrt{1-x^2}$. Then simple computations lead to $\varphi_2'(x) = 2x^2/\sqrt{1-x^2} > 0$ for $x \in (0,1)$, and

$$\varphi(x) = \frac{\varphi_1(x)}{\varphi_2(x)} = \frac{\varphi_1(x) - \varphi_1(0)}{\varphi_2(x) - \varphi_2(0)}, \tag{2.5}$$

$$\frac{\varphi'_1(x)}{\varphi'_2(x)} = \frac{\sqrt{1-x^2}\arcsin(x)}{x} + \frac{1}{2}. \tag{2.6}$$

Note that $x < \arcsin(x)$ for $x \in (0,1)$, and

$$\frac{d\left[\frac{\sqrt{1-x^2}\arcsin(x)}{x}\right]}{dx} = \frac{x\sqrt{1-x^2} - \arcsin(x)}{x^2\sqrt{1-x^2}} < 0$$

for all $x \in (0,1)$. Thus the function $x \mapsto \sqrt{1-x^2}\arcsin(x)/x$ is strictly decreasing on $(0,1)$ and so is $\varphi_1'(x)/\varphi_2'(x)$ by (2.6). Moreover,

$$\varphi(0^+) = \lim_{x\to 0^+} \frac{\varphi_1'(x)}{\varphi_2'(x)} = \frac{3}{2}, \quad \varphi(1^-) = 1. \tag{2.7}$$

Therefore, Lemma 2.4 follows from (2.5), (2.7) and Lemma 2.1 together with the monotonicity of $\varphi_1'(x)/\varphi_2'(x)$. □

LEMMA 2.5. *Let* $0<u<1$, $p\in[1,\infty)$ *and*

$$g(u,p;x)=\frac{1}{2}p\log\left(1-ux^2\right)-\frac{\sqrt{1-x^2}\arcsin(x)}{x}+1. \tag{2.8}$$

Then the following statements are true:

1. $g(u,p;x)>0$ *for* $x\in(0,1)$ *if and only if* $u\leqslant 2/(3p)$;
2. $g(u,p;x)<0$ *for* $x\in(0,1)$ *if and only if* $u\geqslant 1-e^{-2/p}$.

Proof. From (2.8) and elaborated computations one has

$$\lim_{x\to 0^+} g(u,p;x)=0, \tag{2.9}$$

$$\lim_{x\to 1^-} g(u,p;x)=\frac{1}{2}p\log(1-u)+1, \tag{2.10}$$

$$\begin{aligned}\frac{\partial g(u,p;x)}{\partial x}&=\frac{\arcsin(x)-x\sqrt{1-x^2}}{x^2\sqrt{1-x^2}}-\frac{pux}{1-ux^2}\\&=\frac{(p-1)x\sqrt{1-x^2}+\arcsin(x)}{\sqrt{1-x^2}\left(1-ux^2\right)}\left[g_p(x)-u\right],\end{aligned} \tag{2.11}$$

where

$$g_p(x)=\frac{1}{(p-1)\frac{x^3\sqrt{1-x^2}}{\arcsin(x)-x\sqrt{1-x^2}}+\frac{x^2\arcsin(x)}{\arcsin(x)-x\sqrt{1-x^2}}}. \tag{2.12}$$

Lemma 2.3 and 2.4 together with (2.12) show that the function $x\mapsto g_p(x)$ is strictly increasing on $(0,1)$, and

$$\lim_{x\to 0^+} g_p(x)=\frac{2}{3p},\quad \lim_{x\to 1^-} g_p(x)=1. \tag{2.13}$$

From Lemma 2.2 we know that the interval on $(0,1)$ can be expressed by

$$(0,1)=\left(0,\frac{2}{3p}\right]\cup\left(\frac{2}{3p},1-e^{-\frac{2}{p}}\right)\cup\left[1-e^{-\frac{2}{p}},1\right).$$

Following we divide the proof into three cases.

Case 1. $0<u\leqslant 2/(3p)$. Then from (2.11) and (2.13) together with the monotonicity of the function $g_p(x)$ lead to the conclusion that the function $x\mapsto g(u,p;x)$ is strictly increasing on $(0,1)$. Therefore, $g(u,p;x)>0$ for all $x\in(0,1)$ follows from (2.9).

Case 2. $1-e^{-2/p}\leqslant u<1$. Then from (2.10), (2.11), (2.13), and Lemma 2.2 together with the monotonicity of the function $x\mapsto g_p(x)$, we clearly see that

$$\lim_{x\to 1^-} g(u,p;x)\leqslant 0, \tag{2.14}$$

and there exists $x_0 \in (0,1)$ such that the function $x \mapsto g(u,p;x)$ is strictly decreasing on $(0,x_0)$ and strictly increasing on $(x_0,1)$. Therefore, $g(u,p;x) < 0$ for all $x \in (0,1)$ follows easily from (2.9) and (2.14).

Case 3. $2/(3p) < u < 1-e^{-2/p}$. Then it follows from (2.10), (2.11), (2.13) together with the monotonicity of the function $x \mapsto g_p(x)$ that

$$\lim_{x\to 1^-} g(u,p;x) > 0, \tag{2.15}$$

and there exists $x_1 \in (0,1)$ such that the function $x \mapsto g(u,p;x)$ is strictly decreasing on $(0,x_1)$ and strictly increasing on $(x_1,1)$. Combining with (2.9), we conclude that there exists $x^* \in (0,1)$ such that $g(u,p;x) < 0$ for $x \in (0,x^*)$ and $g(u,p;x) > 0$ for $x \in (x^*,1)$. □

3. Main results

THEOREM 3.1. *Let $p \in [1,\infty)$ and $\lambda,\mu \in (0,1/2)$, then the double inequality*

$$GA_{\lambda,p}(a,b) < X(a,b) < GA_{\mu,p}(a,b)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\lambda \leqslant 1/2-\sqrt{1-e^{-2/p}}/2$ and $\mu \geqslant 1/2-\sqrt{6p}/(6p)$.

Proof. Without loss of generality, we can assume that $a>b>0$.
Let $x=(a-b)/(a+b)$. Then $x \in (0,1)$, and equations (1.1) and (1.4) yield

$$X(a,b) = A(a,b)e^{\frac{\sqrt{1-x^2}\arcsin(x)}{x}-1}, \tag{3.1}$$

$$GA_{t,p}(a,b) = A(a,b)\left[1-(1-2t)^2x^2\right]^{p/2}. \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\begin{aligned}\log\left[\frac{GA_{t,p}(a,b)}{X(a,b)}\right] &= \log\left[\frac{GA_{t,p}(a,b)}{A(a,b)}\right]-\log\left[\frac{X(a,b)}{A(a,b)}\right]\\ &= \frac{1}{2}p\log\left[1-(1-2t)^2x^2\right]-\frac{\sqrt{1-x^2}\arcsin(x)}{x}+1. \end{aligned}\tag{3.3}$$

Therefore, Theorem 3.1 follows from Lemma 2.5 and (3.3). □

4. Applications

Let $p=1,2$ in Theorem 3.1, then we obtain the following Theorem 4.1 immediately, which provides the sharp one-parameter harmonic and geometric means bounds for the Sándor mean.

THEOREM 4.1. ([8, Theorems 3.3 and 3.4]) *Let* $\lambda_1, \lambda_2, \mu_1, \mu_2 \in (0, 1/2)$. *Then the double inequalities*

$$H[\lambda_1 a + (1-\lambda_1) b, \lambda_1 b + (1-\lambda_1) a] < X(a,b) < H[\mu_1 a + (1-\mu_1) b, \mu_1 b + (1-\mu_1) a],$$
$$G[\lambda_2 a + (1-\lambda_2) b, \lambda_2 b + (1-\lambda_2) a] < X(a,b) < G[\mu_2 a + (1-\mu_2) b, \mu_2 b + (1-\mu_2) a]$$

hold for all $a,b>0$ *with* $a \neq b$ *if and only if* $\lambda_1 \leqslant 1/2 - \sqrt{1-1/e}/2$, $\mu_1 \geqslant 1/2 - \sqrt{3}/6$, $\lambda_2 \leqslant 1/2 - \sqrt{1-1/e^2}/2$ *and* $\mu_2 \geqslant 1/2 - \sqrt{6}/6$.

Theorem 3.1 and (1.1) also lead to Theorem 4.2, which gives the sharp bound for the first Seiffert mean in terms of the combination of geometric and arithmetic means.

THEOREM 4.2. *Let* $p \in [1, \infty)$ *and* $\alpha, \beta \in (0, 1/2)$. *Then the double inequality*

$$\frac{G(a,b)}{\log[GA_{\alpha,p}(a,b)] - \log[A(a,b)] + 1} < P(a,b) < \frac{G(a,b)}{\log\left[GA_{\beta,p}(a,b)\right] - \log[A(a,b)] + 1}$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha \geqslant 1/2 - \sqrt{6p}/(6p)$ *and* $\beta \leqslant 1/2 - \sqrt{1-e^{-2/p}}/2$.

Let $a > b > 0$, $x = (a-b)/(a+b)$, $\alpha = 1/2 - \sqrt{6p}/(6p)$ and $\beta = 1/2 - \sqrt{1-e^{-2/p}}/2$. Then from Theorem 4.2 we obtain the new one-parameter bounds for the inverse sine function as follows: for all $x \in (0,1)$ and $p \in [1, \infty)$,

$$\begin{aligned} \frac{x}{\sqrt{1-x^2}}\left[\frac{1}{2}p\log\left(1-x^2+e^{-2/p}x^2\right)+1\right] &< \arcsin(x) \\ &< \frac{x}{\sqrt{1-x^2}}\left[\frac{1}{2}p\log\left(1-\frac{2}{3p}x^2\right)+1\right]. \end{aligned} \tag{4.1}$$

Making use of Lemma 2.1, it is not difficult to verify that the function $t \to \{\log[1-x^2+(1/e^2)^t x^2]\}/t$ is strictly increasing on $(0,1)$ for fixed $x \in (0,1)$, while $t \to [\log(1-tx^2)]/t$ is decreasing. Changing the variables of the above two functions, we shall find that the function on the left-hand side of (4.1) is strictly decreasing on $p \in [1,\infty)$, while the function on the right-hand side is strictly increasing on $p \in [1, \infty)$. Therefore, the best estimates in (4.1) are arrived at for $p = 1$.

THEOREM 4.3. *The double inequality*

$$\begin{aligned} \frac{x}{\sqrt{1-x^2}}\left[\frac{1}{2}\log\left(1-x^2+(1/e^2)x^2\right)+1\right] &< \arcsin(x) \\ &< \frac{x}{\sqrt{1-x^2}}\left[\frac{1}{2}\log\left(1-\frac{2}{3}x^2\right)+1\right]. \end{aligned}$$

holds for all $x \in (0,1)$.

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