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SHERMAN'S OPERATOR INEQUALITY

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Abstract. In this paper we deal with Sherman's inequality and its complementary inequalities for operator convex functions, whose arguments are the bounded self-adjoint operators from C^*-algebra on a Hilbert space and positive linear mappings between C^*-algebras. We introduce the so called Sherman's operator and study its properties. Applying an extended idea of convexity to operator functions of several variables, we obtain multidimensional Sherman's operator inequality. We define multidimensional Sherman's operator and study its properties. At the end, we observe applications to some operator inequalities related to connections, solidarities, and multidimensional weighted geometric mean.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ be C^*-algebra of all bounded linear operators defined on a complex Hilbert space $\mathscr{H}$ with the identity operator $1_{\mathscr{H}}$. For self-adjoint operators $A,B \in \mathscr{B}(\mathscr{H})$ the order relation $A \leqslant B$ means that $\langle A\xi,\xi\rangle \leqslant \langle B\xi,\xi\rangle$ $(\xi \in \mathscr{H})$. In particular, if $0 \leqslant A$, then A is called positive. If a positive operator A is invertible, then we say that it is strictly positive and write $0 < A$. Let $\mathscr{B}_h(\mathscr{H})$ be the set of all bounded self-adjoint operators on $\mathscr{H}$ and its positive cone $\mathscr{B}^+(\mathscr{H})$ (resp. $\mathscr{B}^{++}(\mathscr{H})$) of all positive operators (resp. all positive invertible operators).

A real valued continuous function f defined on an interval J is said to be operator convex if

$$f(\lambda A + (1-\lambda)B) \leqslant \lambda f(A) + (1-\lambda)f(B)$$

for all $\lambda \in [0,1]$ and every self-adjoint operators $A,B \in \mathscr{B}_h(\mathscr{H})$ whose spectra are contained in J. A function f is called operator concave if $-f$ is operator convex.

A linear map $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ is said to be positive (resp. strictly positive) if $\Phi(A) \geqslant 0$ (resp. $\Phi(A) > 0$) whenever A is positive (resp. strictly positive). It is unital if Φ preserves the identity operator. It can be easily seen that a positive linear map Φ is strictly positive if and only if $\Phi(1_{\mathscr{H}}) > 0$.

It is known that if $\Phi : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$ is a unital positive linear mapping and f is an operator convex function on an interval J, then

$$f(\Phi(X)) \leqslant \Phi(f(X)) \tag{1.1}$$

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for every self-adjoint operator X with the spectrum in J. Inequality (1.1) is known as the Choi-Davis-Jensen inequality, see [9, Theorem 1.20]. Additionally, if J containing 0 and $f(0)=0$, then (1.1) holds for every positive linear mapping $\Phi:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{K})$ with $0<\Phi(1_{\mathscr{H}})\leqslant 1_{\mathscr{K}}$, see [15, Proposition 2.1].

Now let us remind ourselves of Sherman's inequality and some definitions regarding it.

For two vectors $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)\in\mathbb{R}^n$, let $x_{[1]}\geqslant\ldots\geqslant x_{[n]}$ and $y_{[1]}\geqslant\ldots\geqslant y_{[n]}$ be the entries of $\mathbf{x}$ and $\mathbf{y}$, respectively, arranged in decreasing order. We say that $\mathbf{x}$ *majorizes* $\mathbf{y}$, in symbol, $\mathbf{y}\prec\mathbf{x}$ if

$$\sum_{i=1}^{k} y_{[i]}\leqslant\sum_{i=1}^{k} x_{[i]}\ \text{ for } k=1,2,\ldots,n-1 \quad\text{and}\quad \sum_{i=1}^{n} x_i=\sum_{i=1}^{n} y_i.$$

An $n\times l$ real matrix $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nl}(\mathbb{R})$ is said to be *column stochastic* (resp. *row stochastic*) if $s_{ij}\geqslant 0$ for $i=1,2,\ldots,n$, $j=1,2,\ldots,l$ and all column sums (resp. row sums) of $\mathbf{S}$ are equal to 1, i.e., $\sum_{i=1}^{n}s_{ij}=1$ for $j=1,2,\ldots,l$ (resp. $\sum_{j=1}^{l}s_{ij}=1$ for $i=1,2,\ldots,n$).

An $n\times n$ real matrix $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nn}(\mathbb{R})$ is said to be *doubly stochastic* if it is column stochastic and row stochastic.

It is known that for $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$

$$\mathbf{y}\prec\mathbf{x}\quad\text{if and only if}\quad\mathbf{y}=\mathbf{x}\mathbf{S} \tag{1.2}$$

for some doubly stochastic $n\times n$ matrix $\mathbf{S}$, see e.g. [20].

Moreover, the next result plays a very important role in the majorization theory, see [10]. If $f:J\to\mathbb{R}$ is a real convex function defined on an interval $J\subset\mathbb{R}$, then for $\mathbf{x}=(x_1,x_2,\ldots,x_n)$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)\in J^n$

$$\mathbf{y}\prec\mathbf{x}\quad\text{implies}\quad\sum_{i=1}^{n} f(y_i)\leqslant\sum_{i=1}^{n} f(x_i). \tag{1.3}$$

The inequality in (1.3) is known as *the majorization inequality*.

Next, S. Sherman [25] considered the concept of weighted majorization

$$\langle\mathbf{a},\mathbf{y}\rangle\prec\langle\mathbf{b},\mathbf{x}\rangle \tag{1.4}$$

between vectors $\mathbf{x}=(x_1,x_2,\ldots,x_l)\in J^l$, $\mathbf{y}=(y_1,y_2,\ldots,y_n)\in J^n$ with nonnegative weights $\mathbf{a}=(a_1,a_2,\ldots,a_l)\in[0,\infty)^l$, $\mathbf{b}=(b_1,b_2,\ldots,b_n)\in[0,\infty)^n$. The concept of weighted majorization is defined by assuming the existence of a row stochastic matrix $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nl}(\mathbb{R})$ such that

$$\mathbf{y}=\mathbf{x}\mathbf{S}^T\quad\text{and}\quad\mathbf{a}=\mathbf{b}\mathbf{S}, \tag{1.5}$$

i.e.

$$y_i=\sum_{j=1}^{l}x_js_{ij}\quad(i=1,2,\ldots,n)\quad\text{and}\quad a_j=\sum_{i=1}^{n}b_is_{ij}\quad(j=1,2,\ldots,l).$$

Sherman proved that under condition (1.5) the inequality

$$\sum_{i=1}^{n} b_i f(y_i) \leqslant \sum_{j=1}^{l} a_j f(x_j) \tag{1.6}$$

holds for every convex function $f : J \to \mathbb{R}$. If f is a concave function, then the reverse inequality is valid in (1.6).

Sherman's inequality (1.5) contains as special cases classical Jensen's inequality and the majorization inequality, see [14].

Some generalizations of Sherman's inequality were recently obtained (see [1], [2], [3], [11]–[14], [16], [20]–[23]).

In this paper we deal with Sherman's operator inequality and its complementary inequalities with applications. We deal with Sherman's inequality whose real arguments are substituted with the bounded self-adjoint operators acting on a Hilbert space. We give extensions of Sherman's inequality (1.6) to self-adjoint operators and positive linear maps. We also get upper bounds for obtained inequalities, so-called converse inequalities. As easy consequences of obtained results we get Jensen's operator inequality (2.1) and the majorization operator inequalities as well as their conversions. We introduce the so called Sherman's operator and study its properties. Applying an extended idea of convexity to operator functions of several variables, we obtain multidimensional Sherman's operator inequality which reduces to multidimensional Jensen's and the majorization operator inequalities. Moreover, we define multidimensional Sherman's operator and study its properties. In the end, we discuss applications to some operator inequalities related to connections, solidarities and multidimensional weighted geometric mean.

2. Sherman's operator inequality

First of all, we recall a version of Choi-Davis-Jensen inequality (1.1) for multiple operators and mappings, which is known as Jensen's operator inequality.

THEOREM A. *Let* $(X_1, X_2, \dots, X_n)$ *be an* n*-tuple of self-adjoint operators in* $\mathscr{B}_h(\mathscr{H})$ *with spectra in an interval* J, $(\Phi_1, \Phi_2, \dots, \Phi_n)$ *be an* n*-tuple of positive linear mappings* $\Phi_i : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$, $i = 1, 2, \dots, n$, *and* $(a_1, a_2, \dots, a_n) \in [0, \infty)^n$ *be a vector of non-negative numbers. If* $f \in \mathscr{C}(J)$ *is an operator convex function and one of the following conditions is true:*

(A_1) $\Phi = \sum_{i=1}^{n} a_i \Phi_i$ *is a unital mapping,*

(A_2) $\sum_{i=1}^{n} a_i = 1$, *J containing* 0, $f(0) = 0$ *and* $0 < \Phi_i(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$ $(i = 1, 2, \dots, n)$,

(A_3) $\sum_{i=1}^{n} a_i = 1$, *J containing* 0, $f(0) = 0$ *and* $0 < \sum_{i=1}^{n} \Phi_i(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$, *then*

$$f\left(\sum_{i=1}^{n} a_i\Phi_i(X_i)\right) \leqslant \sum_{i=1}^{n} a_i\Phi_i\big(f(X_i)\big). \tag{2.1}$$

If f is operator concave, then the reverse inequality is valid in (2.1).

Theorem A follows from [9], [7] and [15]. We present the proof to interested readers.

Proof. We will prove only the operator convex case.

(A_1) Since $a_i \geqslant 0$, then $\Psi_i = a_i\Phi_i$ is a positive linear mapping and $\sum_{i=1}^{n}\Psi_i(1_{\mathscr{H}}) = 1_{\mathscr{K}}$. So, using the Jensen type operator inequality (see [7, Theorem 8.9]):

$$f\left(\sum_{i=1}^{n} \Psi_i(X_i)\right) \leqslant \sum_{i=1}^{n} \Psi_i\big(f(X_i)\big), \tag{2.2}$$

we obtain (2.1).

(A_2) We obtain (2) again, if we apply (1.1) to a positive linear mapping Φ_i such that $0 < \Phi_i(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$, then we obtain

$$f\left(\sum_{i=1}^{n} a_i\Phi_i(X_i)\right) \leqslant \sum_{i=1}^{n} a_i f\big(\Phi_i(X_i)\big) \leqslant \sum_{i=1}^{n} a_i\Phi_i\big(f(X_i)\big),$$

which gives the desired inequality (2.1).

(A_3) Since $0 < \sum_{i=1}^{n}\Phi_i(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$ implies $0 < \Phi_i(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$, $i = 1,2,\dots,n$, for positive mappings, then applying Theorem A with condition (A_2) gives us (2.1). □

Now, we give a general version of Sherman's operator inequality.

THEOREM 2.1. *Let $(X_1, X_2, \dots, X_l)$ be an l-tuple of self-adjoint operators in $\mathscr{B}_h(\mathscr{H})$ with spectra in J, $(\Phi_1, \Phi_2, \dots, \Phi_l)$ be an l-tuple of positive linear mappings $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$, $j = 1,2,\dots,l$, and $(a_1, a_2, \dots, a_l) \in [0,\infty)^l$, $(b_1, b_2 \dots, b_n) \in [0,\infty)^n$ be vectors, $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nl}(\mathbb{R})$ be a matrix with non negative entries such that $\mathbf{a} = \mathbf{b}S$.*

If $f \in \mathscr{C}(J)$ is an operator convex function and one of the following conditions is true:

(i) $\Psi_i = \sum_{j=1}^{l} s_{ij}\Phi_j$ *is a unital mapping,* $i = 1,2,\dots,n$,

(ii) $\mathbf{S}$ *is row stochastic,* $0 \in J$, $f(0) = 0$ *and* $0 < \Phi_j(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$, $j = 1,2,\dots,l$,

(iii) $\mathbf{S}$ *is row stochastic,* $0 \in J$, $f(0) = 0$ *and* $0 < \sum_{j=1}^{l}\Phi_j(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$, *then*

$$\sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij}\Phi_j(X_j)\right) \leqslant \sum_{j=1}^{l} a_j\Phi_j\left(f(X_j)\right). \tag{2.3}$$

If f is operator concave, then the reverse inequality is valid in (2.3)*.*

Proof. We will prove only the operator convex case. Applying Theorem A and since $\mathbf{a}=\mathbf{b}S$ we have

$$\sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij}\Phi_j(X_j)\right) \leqslant \sum_{i=1}^{n} b_i\left(\sum_{j=1}^{l} s_{ij}\Phi_j\left(f(X_j)\right)\right)$$
$$= \sum_{j=1}^{l}\left(\sum_{i=1}^{n} b_i s_{ij}\right)\Phi_j\left(f(X_j)\right) = \sum_{j=1}^{l} a_j\Phi_j\left(f(X_j)\right). \quad \square$$

REMARK 2.2. a) As special cases of (A_1) in Theorem A we obtain that (2.1) is valid if $\sum_{i=1}^{n} a_i = 1$, *Φ_i is unital mapping* $(i=1,2,\ldots,n)$ is true. It follows that (2.3) is valid if $\mathbf{S}$ *is row stochastic, Φ_j is unital* $(j=1,2,\ldots,l)$.

b) Setting $n=1$ and $\mathbf{b}=(1)$ in Theorem 2.1 we get Theorem A, i.e. Sherman's operator inequality (2.3) reduces to Jensen's operator inequality (2.1). Moreover, setting $\mathbf{a}=(1,1,\ldots,1)$ the inequality (2.3) reduces to the form of (2.2). Also, setting $\Phi_j(A)=A$ in (2.3) we get Jensen's operator inequality $f\left(\sum_{i=1}^{n} b_i X_i\right) \leqslant \sum_{ij=1}^{n} b_i f(X_i)$..

Let $\mathbf{X}=(X_1,X_2,\ldots,X_l)$ be an l-tuple and $\mathbf{Y}=(Y_1,Y_2,\ldots,Y_n)$ be an n-tuple of self-adjoint operators in $\mathscr{B}_h(\mathscr{H})$, and $\mathbf{a}=(a_1,a_2,\ldots,a_l)\in[0,\infty)^l$, $\mathbf{b}=(b_1,b_2,\ldots,b_n)$ $\in[0,\infty)^n$ be nonnegative weights. Inspired by (1.2) and (1.4), we introduce the following two symbols:

- We will use a symbol $\mathbf{Y}\prec_\circ\mathbf{X}$ if $n=l$ and $\mathbf{Y}=\mathbf{XS}$, i.e. $Y_i=\sum_{j=1}^{n} s_{ij}X_j$ $(i=1,2,\ldots,n)$ for some doubly stochastic $n\times n$ matrix $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nn}(\mathbb{R})$.
- We will use a symbol $\langle\mathbf{a},\mathbf{Y}\rangle \prec_\circ \langle\mathbf{b},\mathbf{X}\rangle$ if $\mathbf{Y}=\mathbf{XS}^T$ and $\mathbf{a}=\mathbf{bS}$, i.e. $Y_i=\sum_{j=1}^{l} s_{ij}X_j$ $(i=1,2,\ldots,n)$ and $a_j=\sum_{i=1}^{n} b_i s_{ij}$ $(j=1,2,\ldots,l)$ for some row stochastic matrix $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nl}(\mathbb{R})$.

By applying Theorem 2.1 on the identity mappings we get the operator version of Sherman's inequality and the weighted majorization operator inequality.

COROLLARY 2.3. *Let $\mathbf{X}=(X_1,X_2,\ldots,X_l)$ be an l-tuple and $\mathbf{Y}=(Y_1,Y_2,\ldots,Y_n)$ be an n-tuple of self-adjoint operators in $\mathscr{B}_h(\mathscr{H})$ with spectra in J, $\mathbf{a}=(a_1,a_2,\ldots,a_l)$ $\in[0,\infty)^l$, $\mathbf{b}=(b_1,b_2,\ldots,b_n)\in[0,\infty)^n$ be vectors and $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nl}(\mathbb{R})$ be a row* *stochastic matrix such that* $\langle \mathbf{a}, \mathbf{Y} \rangle \prec_\circ \langle \mathbf{b}, \mathbf{X} \rangle$. *If* $f \in \mathscr{C}(J)$ *is an operator convex function, then*

$$\sum_{i=1}^{n} b_i f(Y_i) \leqslant \sum_{j=1}^{l} a_j f(X_j). \tag{2.4}$$

Specially, if $l = n$ *and* $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nn}(\mathbb{R})$ *is a doubly stochastic matrix, then the weighted multidimensional majorization inequality holds, so that*

$$\mathbf{Y} \prec_\circ \mathbf{X} \qquad \text{implies} \qquad \sum_{i=1}^{n} a_i f(Y_i) \leqslant \sum_{i=1}^{n} a_i f(X_i). \tag{2.5}$$

If f *is operator concave, then the reverse inequality is valid in* (2.4) *and* (2.5).

Proof. By applying Theorem 2.1 on the identity mappings, i.e. $\Phi_j(A) = A$, for all $A \in \mathscr{B}(H)$, $j = 1, 2, \ldots, l$, we obtain (2.4). Next, setting $n = l$ in (2.4) and all weights a_i and b_j are equal and nonnegative, the condition $\mathbf{a} = \mathbf{bS}$ assures stochasticity on columns of $\mathbf{S}$, so in that case we deal with a doubly stochastic matrix and (2.5) holds. □

Setting $l = n$, $\mathbf{b} = (1, 1, \ldots, 1)$ and if $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nn}(\mathbb{R})$ is doubly stochastic, we obtain that $\mathbf{a} = (1, 1, \ldots, 1)$. By applying Theorem 2.1 we get the following versions of the majorization operator inequality with mappings.

COROLLARY 2.4. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$, *and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be* n*-tuples of self-adjoint operators in* $\mathscr{B}_h(\mathscr{H})$ *with spectra in* J, $(\Phi_1, \Phi_2, \ldots, \Phi_n)$ *be an* n*-tuple of positive linear mappings* $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$, $j = 1, 2, \ldots, n$, $\mathbf{\Phi}(\mathbf{X}) = (\Phi_1(X_1), \Phi_2(X_2), \ldots, \Phi_n(X_n))$ *and* $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nn}(\mathbb{R})$ *be a doubly stochastic matrix such that* $\mathbf{Y} \prec_\circ \mathbf{\Phi}(\mathbf{X})$.

If $f \in \mathscr{C}(J)$ *is an operator convex function and one of the following conditions is true:*

(a) $\Psi_i = \sum_{j=1}^{n} s_{ij} \Phi_j$ *is a unital mapping,* $i = 1, 2, \ldots, n$,

(b) $0 \in J$, $f(0) = 0$ *and* $0 < \sum_{j=1}^{n} \Phi_j(1_{\mathscr{H}}) \leqslant 1_{\mathscr{K}}$,

then

$$\sum_{i=1}^{n} f(Y_i) \leqslant \sum_{i=1}^{n} \Phi_i(f(X_i)). \tag{2.6}$$

If f *is operator concave, then the reverse inequalities are valid in* (2.6).

3. Inequalities complementary to Sherman's operator inequality

We now give a general version of complementary to Sherman's operator inequality (2.3).

THEOREM 3.1. *Let X_j, Φ_j, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{S}$ be as in Theorem 2.1 and $J=[m,M]$, $m<M$. Let $m_Y=\min_{1\leqslant i\leqslant n}\{m_i\}$, $M_Y=\max_{1\leqslant i\leqslant n}\{M_i\}$, where m_i and M_i, $m_i\leqslant M_i$, are the bounds of the self-adjoint operator $Y_i=\sum_{j=1}^{l}s_{ij}\Phi_j(X_j)$ $(i=1,2,\ldots,n)$. Let $f\in\mathscr{C}([m,M])$, $g\in\mathscr{C}([m_Y,M_Y])$, $F:U\times V\to\mathbb{R}$ be bounded, where $f([m,M])\subseteq U$, $g([m_Y,M_Y])\subseteq V$ and the condition (i) in Theorem 2.1 is true.*

If f is convex and F is operator monotone in the first variable, then

$$\begin{aligned}
&F\left[\sum_{j=1}^{l}a_j\Phi_j\left(f(X_j)\right),\sum_{i=1}^{n}b_i g\left(\sum_{j=1}^{l}s_{ij}\Phi_j(X_j)\right)\right]\\
&\leqslant \max_{m_Y\leqslant t\leqslant M_Y}F\left[\left(\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)\right)\|\mathbf{b}\|_1,g(t)\|\mathbf{b}\|_1\right]1_{\mathscr{K}} \qquad (3.1)\\
&\leqslant \max_{m\leqslant t\leqslant M}F\left[\left(\frac{M-t}{M-m}f(m)+\frac{t-m}{M-m}f(M)\right)\|\mathbf{b}\|_1,g(t)\|\mathbf{b}\|_1\right]1_{\mathscr{K}},
\end{aligned}$$

where $\|\cdot\|_1$ denotes l_1 norm.

If f is concave, then the reverse inequality is valid in (3.1) with min *instead of* max.

Proof. We only prove the convex case. Since $m\Phi_j(1_{\mathscr{H}})\leqslant\Phi_j(X_j)\leqslant M\Phi_j(1_{\mathscr{H}})$ and $\sum_{j=1}^{l}s_{ij}\Phi_j(1_{\mathscr{H}})=1_{\mathscr{K}}$ $(i=1,2,\ldots,n)$, then $m1_{\mathscr{K}}\leqslant\sum_{j=1}^{l}s_{ij}\Phi_j(X_j)\leqslant M1_{\mathscr{K}}$. Furthermore $m_Y1_{\mathscr{K}}\leqslant\sum_{j=1}^{l}s_{ij}\Phi_j(X_j)\leqslant M_Y1_{\mathscr{K}}$, so $[m_Y,M_Y]\subseteq[m,M]$.

By using convexity of f and functional calculus, we obtain

$$\begin{aligned}
&\sum_{j=1}^{l}a_j\Phi_j\left(f(X_j)\right)\\
&\leqslant\sum_{j=1}^{l}a_j\left(\frac{M\Phi_j(1_{\mathscr{H}})-\Phi_j(X_j)}{M-m}f(m)+\frac{\Phi_j(X_j)-m\Phi_j(1_{\mathscr{H}})}{M-m}f(M)\right)\\
&=\sum_{j=1}^{l}\left(\sum_{i=1}^{n}b_is_{ij}\right)\left(\frac{M\Phi_j(1_{\mathscr{H}})-\Phi_j(X_j)}{M-m}f(m)+\frac{\Phi_j(X_j)-m\Phi_j(1_{\mathscr{H}})}{M-m}f(M)\right)\\
&=\sum_{i=1}^{n}b_i\left(\frac{M1_{\mathscr{K}}-\sum_{j=1}^{l}s_{ij}\Phi_j(X_j)}{M-m}f(m)+\frac{\sum_{j=1}^{l}s_{ij}\Phi_j(X_j)-m1_{\mathscr{K}}}{M-m}f(M)\right).
\end{aligned}$$

Using operator monotonicity of $u \mapsto F(u,v)$ and boundedness of F, it follows that

$$\begin{aligned}
&F\left[\sum_{j=1}^{l} a_j\Phi_j(f(X_j)), \sum_{i=1}^{n} b_i g\left(\sum_{j=1}^{l} s_{ij}\Phi_j(X_j)\right)\right] \\
&\leqslant F\left[\sum_{i=1}^{n} b_i\left(\frac{M1_{\mathscr{K}} - \sum_{j=1}^{l} s_{ij}\Phi_j(X_j)}{M-m} f(m) + \frac{\sum_{j=1}^{l} s_{ij}\Phi_j(X_j) - m1_{\mathscr{K}}}{M-m} f(M)\right),\right. \\
&\quad \left.\sum_{i=1}^{n} b_i g\left(\sum_{j=1}^{l} s_{ij}\Phi_j(X_j)\right)\right] \\
&\leqslant \max_{m_Y \leqslant t \leqslant M_Y} F\left[\sum_{i=1}^{n} b_i\left(\frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M)\right), \sum_{i=1}^{n} b_i g(t)\right] 1_{\mathscr{K}} \\
&= \max_{m_Y \leqslant t \leqslant M_Y} F\left[\left(\frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M)\right) \|\mathbf{b}\|_1, g(t)\|\mathbf{b}\|_1\right] 1_{\mathscr{K}} \\
&\leqslant \max_{m \leqslant t \leqslant M} F\left[\left(\frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M)\right) \|\mathbf{b}\|_1, g(t)\|\mathbf{b}\|_1\right] 1_{\mathscr{K}}. \quad \square
\end{aligned}$$

3.1. Difference type complementary inequalities

In this subsection we consider the difference type complementary to the inequality (2.3).

For convenience, we introduce some abbreviations. Let $f : [m,M] \to \mathbb{R}$, $m < M$, be a convex or a concave function. We denote a linear function through $(m, f(m))$ and $(M, f(M))$ by $f^{cho}_{[m,M]}$, i.e.

$$f^{cho}_{[m,M]}(t) = \frac{M-t}{M-m} f(m) + \frac{t-m}{M-m} f(M), \qquad t \in \mathbb{R}$$

and the slope and the intercept by k_f and l_f, respectively, i.e.

$$k_f = \frac{f(M) - f(m)}{M-m} \qquad \text{and} \qquad l_f = \frac{Mf(m) - mf(M)}{M-m}.$$

Putting $F(u,v) = u - \alpha v$, $\alpha \in \mathbb{R}$, in Theorem 3.1 we obtain the following corollary.

COROLLARY 3.2. *Let X_j, Φ_j, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{S}$, m,M, m_Y, M_Y and g be as in Theorem 3.1 and the condition (i) in Theorem 2.1 be true.*

If $f \in \mathscr{C}([m,M])$ is convex, then

$$\sum_{j=1}^{l} a_j\Phi_j(f(X_j)) \leqslant \alpha \sum_{i=1}^{n} b_i g\left(\sum_{j=1}^{l} s_{ij}\Phi_j(X_j)\right) + \max_{m_Y \leqslant t \leqslant M_Y} \{k_f t + l_f - \alpha g(t)\} \|\mathbf{b}\|_1 1_{\mathscr{K}}. \tag{3.2}$$

If f is concave, then the reverse inequality is valid in (3.1) with min *instead of* max.

REMARK 3.3. Now we give a way of determining the bound

$$C_\alpha := \max_{m_Y \leqslant t \leqslant M_Y} \left\{ k_f t + l_f - \alpha g(t) \right\}$$

placed in Corollary 3.2 (see [19, Corollary 3.2]). Let f be convex.

If αg is concave, then

$$C_\alpha = \max_{m_Y \leqslant t \leqslant M_Y} \left\{ f^{cho}_{[m,M]}(m_Y) - \alpha g(m_Y),\, f^{cho}_{[m,M]}(M_Y) - \alpha g(M_Y) \right\}.$$

But, if αg is convex, then

$$C_\alpha = \begin{cases} f^{cho}_{[m,M]}(m_Y) - \alpha g(m_Y) & \text{if } \alpha g'_-(t) \geqslant k_f \text{ for every } t \in (m_Y, M_Y), \\ f^{cho}_{[m,M]}(t_0) - \alpha g(t_0) & \text{if } \alpha g'_-(t_0) \leqslant k_f \leqslant \alpha g'_+(t_0) \text{ for some } t_0 \in (m_Y, M_Y), \\ f^{cho}_{[m,M]}(M_Y) - \alpha g(M_Y) & \text{if } \alpha g'_+(t) \leqslant k_f \text{ for every } t \in (m_Y, M_Y). \end{cases} \tag{3.3}$$

Putting $g \equiv f$ and $\alpha = 1$ in Corollary 3.2 we obtain the following result.

COROLLARY 3.4. *Let X_j, Φ_j, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{S}$, m, M and m_Y, M_Y be as in Theorem 3.1 and let the condition (i) in Theorem 2.1 be true.*

If $f \in \mathscr{C}([m,M])$ is convex, then

$$\sum_{j=1}^{l} a_j \Phi_j(f(X_j)) \leqslant \sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij} \Phi_j(X_j) \right) + \bar{C} \|\mathbf{b}\|_1 1_{\mathscr{K}}, \tag{3.4}$$

where the value of the constant $\bar{C} := \max_{m_Y \leqslant t \leqslant M_Y} \left\{ k_f t + l_f - f(t) \right\}$ determined as in (3.3) with $g = f$, i.e.

$$\bar{C} = \begin{cases} f^{cho}_{[m,M]}(m_Y) - f(m_Y) & \text{if } f'_-(t) \geqslant k_f \text{ for every } t \in (m_Y, M_Y), \\ f^{cho}_{[m,M]}(t_0) - f(t_0) & \text{if } f'_-(t_0) \leqslant k_f \leqslant f'_+(t_0) \text{ for some } t_0 \in (m_Y, M_Y), \\ f^{cho}_{[m,M]}(M_Y) - f(M_Y) & \text{if } f'_+(t) \leqslant k_f \text{ for every } t \in (m_Y, M_Y). \end{cases} \tag{3.5}$$

If f is concave, then the reverse inequality is valid in (3.4) with constant $\bar{c} := \min_{m_Y \leqslant t \leqslant M_Y} \left\{ k_f t + l_f - f(t) \right\}$ can be determined as in the right side in (3.5) with reverse inequality signs.

REMARK 3.5. If f is a strictly convex differentiable function on $[m_Y, M_Y]$, then

$$\sum_{j=1}^{l} a_j \Phi_j(f(X_j)) \leqslant \sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij} \Phi_j(X_j) \right) + \left(k_f t_0 + l_f - f(t_0) \right) \|\mathbf{b}\|_1 1_{\mathscr{K}},$$

where $t_0 = m_Y$ if $f'(m_Y) \geqslant k_f$, $t_0 = M_Y$ if $f'(M_Y) \leqslant k_f$ and $t_0 = f'^{-1}\left(k_f\right)$ if $f'(m_Y) \leqslant k_f \leqslant f'(M_Y)$.

Now, applying Corollary 3.4 on the identity mappings, we obtain complementary to the inequalities (2.4) and (2.5).

COROLLARY 3.6. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_l)$ *be an* l*-tuple and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be an* n*-tuple of self-adjoint operators in* $\mathscr{B}_h(\mathscr{H})$ *with spectra of* X_j *in* $[m, M]$, $m < M$ *and* m_i *and* M_i, $m_i \leqslant M_i$ *are the bounds of* Y_i, $\mathbf{a} = (a_1, a_2, \ldots, a_l) \in [0, \infty)^l$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in [0, \infty)^n$ *be vectors and* $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nl}(\mathbb{R})$ *be a row stochastic matrix such that* $\langle \mathbf{a}, \mathbf{Y} \rangle \prec_\circ \langle \mathbf{b}, \mathbf{X} \rangle$. *If* $f \in \mathscr{C}([m, M])$ *is convex, then*

$$\sum_{j=1}^{l} a_j f(X_j) \leqslant \sum_{i=1}^{n} b_i f(Y_i) + \bar{C} \|\mathbf{b}\|_1 1_{\mathscr{K}}, \tag{3.6}$$

where the constant $\bar{C}$ *is defined by* (3.5) *with* $m_Y = \min_{1 \leqslant i \leqslant n} \{m_i\}$ *and* $M_Y = \max_{1 \leqslant i \leqslant n} \{M_i\}$.

Specially, if $l = n$ *and* $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nn}(\mathbb{R})$ *is a doubly stochastic matrix, then the weighted multidimensional majorization inequality holds, so that*

$$\mathbf{Y} \prec_\circ \mathbf{X} \qquad \text{implies} \qquad \sum_{i=1}^{n} a_i f(X_i) \leqslant \sum_{i=1}^{n} a_i f(Y_i) + \bar{C} \|\mathbf{a}\|_1 1_{\mathscr{K}}. \tag{3.7}$$

If f *is concave, the reverse inequality is valid in* (3.6) *and* (3.7)*, with constant* $\bar{c} := \min_{m_Y \leqslant t \leqslant M_Y} \left\{k_f t + l_f - f(t)\right\}$ *can be determined as in the right side in* (3.5) *with reverse inequality signs.*

Finally, setting $l = n$, $\mathbf{b} = \mathbf{a} = (1, 1, \ldots, 1)$ in Corollary 3.4, we obtain complementary to the inequality (2.6).

COROLLARY 3.7. *Let* $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ *and* $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_n)$ *be* n*-tuples of self-adjoint operators in* $\mathscr{B}_h(\mathscr{H})$ *with spectra of* X_j *in* $[m, M]$, $m < M$ *and* m_i *and* M_i, $m_i \leqslant M_i$ *are the bounds of* Y_i. *Let* $(\Phi_1, \Phi_2, \ldots, \Phi_n)$ *be an* n*-tuple of positive linear mappings* $\Phi_j : \mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$, $\mathbf{\Phi}(\mathbf{X}) = (\Phi_1(X_1), \Phi_2(X_2), \ldots, \Phi_n(X_n))$ *and* $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nn}(\mathbb{R})$ *be a doubly stochastic matrix such that* $\mathbf{Y} \prec_\circ \mathbf{\Phi}(\boldsymbol{X})$.

If $f \in \mathscr{C}(J)$ *is convex and one of the conditions (a) or (b) in Corollary* 2.4 *is true, then*

$$\sum_{i=1}^{n} \Phi_i(f(X_i)) \leqslant \sum_{i=1}^{n} f(Y_i) + n \cdot \bar{C}\, 1_{\mathscr{K}} \tag{3.8}$$

where the constant $\bar{C}$ *is defined by* (3.5) *with* $m_Y = \min_{1 \leqslant i \leqslant n} \{m_i\}$ *and* $M_Y = \max_{1 \leqslant i \leqslant n} \{M_i\}$.

If f *is concave, then the reverse inequalities are valid in* (3.8)*, with constant* $\bar{c} := \min_{m_Y \leqslant t \leqslant M_Y} \left\{k_f t + l_f - f(t)\right\}$ *can be determined as in the right side in* (3.5) *with reverse inequality signs.*

3.2. Ratio type complementary inequalities

In this subsection we consider the ratio type complementary to the inequality (2.3).

COROLLARY 3.8. *Let X_j, Φ_j, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{S}$, m,M and m_Y,M_Y be as in Theorem 3.1 and let the condition (i) in Theorem 2.1 be true.*

If $f \in \mathscr{C}([m,M])$ is convex and $g \in \mathscr{C}([m_Y,M_Y])$ is strictly positive, then

$$\sum_{j=1}^{l} a_j \Phi_j(f(X_j)) \leqslant \max_{m_Y \leqslant t \leqslant M_Y} \left\{ \frac{k_f t + l_f}{g(t)} \right\} \sum_{i=1}^{n} b_i g\left(\sum_{j=1}^{l} s_{ij} \Phi_j(X_j) \right). \tag{3.9}$$

If f is concave, then the reverse inequality is valid in (3.9) with min *instead of* max.

Proof. We only prove the cases when f is convex. We choose α such that $\beta = 0$ in Corollary 3.2, where $\beta := \max_{m_Y \leqslant t \leqslant M_Y} \{k_f t + l_f - \alpha g(t)\}$. Then $k_f t + l_f - \alpha g(t) \leqslant \beta = 0$ for all $t \in [m_Y, M_Y]$. Since $g > 0$ it follows $\alpha \geqslant \frac{k_f t + l_f}{g(t)}$ for all $t \in [m_Y, M_Y]$, so $\alpha \geqslant \max_{m_Y \leqslant t \leqslant M_Y} \left\{ \frac{k_f t + l_f}{g(t)} \right\}$. Because a function $t \mapsto k_f t + l_f - \alpha g(t)$ is continuous on $[m_Y, M_Y]$, then there is exactly one point $t_0 \in [m_Y, M_Y]$ which achieves the global maximum: $k_f t_0 + l_f - \alpha g(t_0) = 0$. It follows that $\alpha = \frac{k_f t_0 + l_f}{g(t_0)}$. So $\alpha = \max_{m_Y \leqslant t \leqslant M_Y} \left\{ \frac{k_f t + l_f}{g(t)} \right\}$. □

REMARK 3.9. Now we give a way of determining the bound $K := \max_{m_Y \leqslant t \leqslant M_Y} \left\{ \frac{k_f t + l_f}{g(t)} \right\}$ placed in Corollary 3.8 (see [19, Corollary 4.3]). Let f be convex.

If g is concave, then

$$K = \max \left\{ \frac{f^{cho}_{[m,M]}(m_Y)}{g(m_Y)}, \frac{f^{cho}_{[m,M]}(M_Y)}{g(M_Y)} \right\}.$$

But, if g is convex, then

$$K = \begin{cases} \frac{f^{cho}_{[m,M]}(m_Y)}{g(m_Y)} & \text{if } g'_-(t) \geqslant \frac{k_f g(t)}{k_f t + l_f} \text{ for every } t \in (m_Y, M_Y), \\ \frac{f^{cho}_{[m,M]}(t_0)}{g(t_0)} & \text{if } g'_-(t_0) \leqslant \frac{k_f g(t_0)}{k_f t_0 + l_f} \leqslant g'_+(t_0) \text{ for some } t_0 \in (m_Y, M_Y), \\ \frac{f^{cho}_{[m,M]}(M_Y)}{g(M_Y)} & \text{if } g'_+(t) \leqslant \frac{k_f g(t)}{k_f t + l_f} \text{ for every } t \in (m_Y, M_Y). \end{cases} \tag{3.10}$$

Putting $g \equiv f$ in Corollary 3.8 we obtain the following result.

COROLLARY 3.10. *Let X_j, Φ_j, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{S}$, m,M and m_Y,M_Y be as in Theorem 3.1. Let $f \in \mathscr{C}([m,M])$ be strictly positive on $[m_Y,M_Y]$ and let the condition (i) in Theorem 2.1 be true. If f is convex, then*

$$\sum_{j=1}^{l} a_j \Phi_j(f(X_j)) \leqslant \bar{K} \sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij}\Phi_j(X_j)\right), \tag{3.11}$$

where the value of the constant $\bar{K} := \max\limits_{m_Y \leqslant t \leqslant M_Y} \left\{\frac{k_f t + l_f}{f(t)}\right\}$ determined as in (3.10) with $g = f$, i.e.

$$\bar{K} = \begin{cases} \frac{f^{cho}_{[m,M]}(m_Y)}{f(m_Y)} & \text{if } f'_-(t) \geqslant \frac{k_f f(t)}{k_f t + l_f} \text{ for every } t \in (m_Y, M_Y), \\ \frac{f^{cho}_{[m,M]}(t_0)}{f(t_0)} & \text{if } f'_-(t_0) \leqslant \frac{k_f f(t_0)}{k_f t_0 + l_f} \leqslant f'_+(t_0) \text{ for some } t_0 \in (m_Y, M_Y), \\ \frac{f^{cho}_{[m,M]}(M_Y)}{f(M_Y)} & \text{if } f'_+(t) \leqslant \frac{k_f f(t)}{k_f t + l_f} \text{ for every } t \in (m_Y, M_Y). \end{cases} \tag{3.12}$$

But, if f is concave, then the reverse inequality is valid in (3.11) with constant $\bar{k} := \min\limits_{m_Y \leqslant t \leqslant M_Y} \left\{\frac{k_f t + l_f}{f(t)}\right\}$ can be determined as in the right side in (3.12) with reverse inequality signs.

REMARK 3.11. Let X_j, Φ_j, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{S}$, m,M and m_Y,M_Y be as in Theorem 3.1. Let $f : [m,M] \to \mathbb{R}$ be a continuous function and $f(m), f(M) > 0$. If f is strictly positive and strictly convex twice differentiable on $[m_Y,M_Y]$, then

$$\sum_{j=1}^{l} a_j \Phi_j(f(X_j)) \leqslant \frac{k_f t_0 + l_f}{f(t_0)} \sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij}\Phi_j(X_j)\right),$$

where $t_0 \in (m_Y,M_Y)$ is defined as the unique solution of $k_f f(t) = (k_f t + l_f) f'(t)$ provided $(k_f m_Y + l_f) f'(m_Y)/f(m_Y) \leqslant k_f \leqslant (k_f M_Y + l_f) f'(M_Y)/f(M_Y)$, otherwise t_0 is defined as m_Y or M_Y provided $k_f \leqslant (k_f m_Y + l_f) f'(m_Y)/f(m_Y)$ or $k_f \geqslant (k_f M_Y + l_f) f'(M_Y)/f(M_Y)$, respectively.

Now, applying Corollary 3.10 on the identity mappings, we obtain another complementary to the inequalities (2.4) and (2.5).

COROLLARY 3.12. *Let $\mathbf{X} = (X_1, X_2, \dots, X_l)$ be an l-tuple and $\mathbf{Y} = (Y_1, Y_2, \dots, Y_n)$ be an n-tuple of self-adjoint operators in $\mathscr{B}_h(\mathscr{H})$ with spectra of X_j in $[m,M]$, $m < M$ and m_i and M_i, $m_i \leqslant M_i$ are the bounds of Y_i, and $m_Y = \min\limits_{1 \leqslant i \leqslant n}\{m_i\}$, $M_Y = \max\limits_{1 \leqslant i \leqslant n}\{M_i\}$. Let $\mathbf{a} = (a_1, a_2, \dots, a_l) \in [0,\infty)^l$, $\mathbf{b} = (b_1, b_2, \dots, b_n) \in [0,\infty)^n$ be vectors and $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nl}(\mathbb{R})$ be a row stochastic matrix such that $\langle \mathbf{a}, \mathbf{Y} \rangle \prec_\circ \langle \mathbf{b}, \mathbf{X} \rangle$.*

If $f \in \mathscr{C}([m,M])$ is convex and strictly positive on $[m_Y,M_Y]$, then

$$\sum_{j=1}^{l} a_j f(X_j) \leqslant \bar{K} \sum_{i=1}^{n} b_i f(Y_i), \tag{3.13}$$

where the constant $\bar{K}$ *is defined by* (3.12).

Specially, if $l=n$ *and* $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nn}(\mathbb{R})$ *is a doubly stochastic matrix, then the weighted multidimensional majorization inequality holds, so that*

$$\mathbf{Y}\prec_{\circ}\mathbf{X} \qquad \textit{implies} \qquad \sum_{i=1}^{n} a_i f(X_i)\leqslant \bar{K}\sum_{i=1}^{n} a_i f(Y_i). \tag{3.14}$$

If f *is concave, the reverse inequality is valid in* (3.13) *and* (3.14) *with constant* $\bar{k}:=\min_{m_Y\leqslant t\leqslant M_Y}\left\{\frac{k_f t+l_f}{f(t)}\right\}$ *can be determined as in the right side in* (3.12) *with reverse inequality signs.*

Finally, setting $l=n$, $\mathbf{b}=\mathbf{a}=(1,1,\ldots,1)$ in Corollary 3.4 we obtain complementary to the inequality (2.6).

COROLLARY 3.13. *Let* $\mathbf{X}=(X_1,X_2,\ldots,X_n)$ *and* $\mathbf{Y}=(Y_1,Y_2,\ldots,Y_n)$ *be* n*-tuples of self-adjoint operators in* $\mathscr{B}_h(\mathscr{H})$ *with spectra of* X_j *in* $[m,M]$, $m<M$ *and* m_i *and* M_i, $m_i\leqslant M_i$ *are the bounds of* Y_i, *and* $m_Y=\min_{1\leqslant i\leqslant n}\{m_i\}$, $M_Y=\max_{1\leqslant i\leqslant n}\{M_i\}$. *Let* $(\Phi_1,\Phi_2,\ldots,\Phi_n)$ *be an* n*-tuple of positive linear mappings* $\Phi_j:\mathscr{B}(\mathscr{H})\to\mathscr{B}(\mathscr{K})$, $\mathbf{\Phi(X)}=(\Phi_1(X_1),\Phi_2(X_2),\ldots,\Phi_n(X_n))$ *and* $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nn}(\mathbb{R})$ *be a doubly stochastic matrix such that* $\mathbf{Y}\prec_{\circ}\mathbf{\Phi}(\mathbf{X})$.

If $f\in\mathscr{C}(J)$ *is convex and strictly positive on* $[m_Y,M_Y]$ *and if one of the conditions (a) or (b) in Corollary* 2.4 *is true, then*

$$\sum_{i=1}^{n}\Phi_i(f(X_i))\leqslant \bar{K}\sum_{i=1}^{n} f(Y_i), \tag{3.15}$$

where the constant $\bar{K}$ *is defined by is defined by* (3.12).

If f *is concave, then the reverse inequalities are valid in* (3.15) *with constant* $\bar{k}:=\min_{m_Y\leqslant t\leqslant M_Y}\left\{\frac{k_f t+l_f}{f(t)}\right\}$ *can be determined as in the right side in* (3.12) *with reverse inequality signs.*

4. Sherman's operator and its properties

In this section we define Sherman's operator, deduced from Sherman's operator inequality (2.4).

Let $f\in\mathscr{C}(J)$ be an operator convex function and let $\mathscr{F}_o(J)$ denote the set of all operator convex functions on interval J. Let $\mathscr{B}_h^J(\mathscr{H})$ denote the convex set of bounded self-adjoint operators on the Hilbert space $\mathscr{H}$ with spectra in J. The set of all $n\times l$ row stochastic matrices is denoted by $\mathbb{S}_{nl}(\mathbb{R})$.

We define Sherman's operator $\mathscr{S}:\mathscr{F}_o(J)\times[\mathscr{B}_h^J(\mathscr{H})]^l\times[0,\infty)^n\times\mathbb{S}_{nl}(\mathbb{R})\to\mathscr{B}^+(\mathscr{H})$ as

$$\mathscr{S}(f,\mathbf{X},\mathbf{b},\mathbf{S})=\sum_{j=1}^{l}\sum_{i=1}^{n} b_i s_{ij} f(X_j)-\sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij}X_j\right), \tag{4.1}$$

where $\mathbf{X} = (X_1, X_2, \ldots, X_l)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n)$ and $\mathbf{S} = (s_{ij})$.

Note that operator $\mathscr{S}$ is well-defined. Namely, positivity of operator $\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \mathbf{S})$ follows from Sherman's operator inequality (2.4).

Now, we state and prove our first result that shows the properties of concavity and monotonicity of Sherman's operator.

THEOREM 4.1. *Suppose $\mathscr{S}$ is an operator defined by (4.1). Then it satisfies the following properties:*

(i) $\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \cdot)$ is concave on $\mathbb{S}_{nl}(\mathbb{R})$, that is

$$\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \lambda \mathbf{S} + (1-\lambda)\mathbf{T}) \geqslant \lambda \mathscr{S}(f, \mathbf{X}, \mathbf{b}, \mathbf{S}) + (1-\lambda)\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \mathbf{T}). \tag{4.2}$$

(ii) If $\mathbf{b}, \mathbf{c} \in [0, \infty)^n$ with $\mathbf{b} \geqslant \mathbf{c}$ (i.e. $b_i \geqslant c_i$, $i = 1, 2, \ldots, n$), then

$$\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \mathbf{S}) \geqslant \mathscr{S}(f, \mathbf{X}, \mathbf{c}, \mathbf{S}) \geqslant 0, \tag{4.3}$$

i.e. $\mathscr{S}(f, \mathbf{X}, \cdot, \mathbf{S})$ is increasing on $[0, \infty)^n$.

But, if $f \in \mathscr{C}(J)$ is operator concave, then $-\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \mathbf{S}) \in \mathscr{B}^+(\mathscr{H})$, $\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \cdot)$ is convex on $\mathbb{S}_{nl}(\mathbb{R})$ and $\mathscr{S}(f, \mathbf{X}, \cdot, \mathbf{S})$ is decreasing on $[0, \infty)^n$.

Proof. (i) We start with Sherman's operator (4.1) equipped with a $\lambda \mathbf{S} + (1-\lambda)\mathbf{T}$. Clearly, $\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \lambda \mathbf{S} + (1-\lambda)\mathbf{T})$ can be rewritten in the following form:

$$\begin{aligned}
&\mathscr{S}(f, \mathbf{X}, \mathbf{b}, \lambda \mathbf{S} + (1-\lambda)\mathbf{T}) \\
&\quad = \sum_{j=1}^{l} \sum_{i=1}^{n} b_i (\lambda s_{ij} + (1-\lambda) t_{ij}) f(X_j) - \sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} (\lambda s_{ij} + (1-\lambda) t_{ij}) X_j \right) \\
&\quad = \lambda \sum_{j=1}^{l} \sum_{i=1}^{n} b_i s_{ij} f(X_j) + (1-\lambda) \sum_{j=1}^{l} \sum_{i=1}^{n} b_i t_{ij} f(X_j) \\
&\qquad - \sum_{i=1}^{n} b_i f\left(\lambda \sum_{j=1}^{l} s_{ij} X_j + (1-\lambda) \sum_{j=1}^{l} t_{ij} X_j \right).
\end{aligned} \tag{4.4}$$

On the other hand, operator convexity of the function f and non-negativity of weights provide inequality

$$\begin{aligned}
&\sum_{i=1}^{n} b_i f\left(\lambda \sum_{j=1}^{l} s_{ij} X_j + (1-\lambda) \sum_{j=1}^{l} t_{ij} X_j \right) \\
&\qquad \leqslant \lambda \sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} s_{ij} X_j \right) + (1-\lambda) \sum_{i=1}^{n} b_i f\left(\sum_{j=1}^{l} t_{ij} X_j \right).
\end{aligned} \tag{4.5}$$

Now, considering (4.4) and (4.5), we get operator concave property (4.2), due to definition (4.1).

(ii) First, we note that $\mathscr{S}(f,\mathbf{X},\cdot,\mathbf{S})$ is linear, that is

$$\mathscr{S}(f,\mathbf{X},\alpha\mathbf{b}+\beta\mathbf{c},\mathbf{S}) = \alpha\,\mathscr{S}(f,\mathbf{X},\mathbf{b},\mathbf{S}) + \beta\,\mathscr{S}(f,\mathbf{X},\mathbf{c},\mathbf{S}) \tag{4.6}$$

for every $\alpha,\beta \in \mathbb{R}$.

If $\mathbf{b} = \mathbf{c}$, then relation (4.3) holds trivially. If $\mathbf{b} > \mathbf{c}$, then $\mathbf{b} \in [0,\infty)^n$ can be represented as a sum $\mathbf{b} = (\mathbf{b}-\mathbf{c})+\mathbf{c}$, where $\mathbf{b}-\mathbf{c},\mathbf{c} \in [0,\infty)^n$. So, from linear property (4.6) and positive property $\mathscr{S}(f,\mathbf{X},\cdot,\mathbf{S}) \geqslant 0$ we get

$$\begin{aligned}\mathscr{S}(f,\mathbf{X},\mathbf{b},\mathbf{S}) &= \mathscr{S}(f,\mathbf{X},(\mathbf{b}-\mathbf{c})+\mathbf{c},\mathbf{S}) \\ &= \mathscr{S}(f,\mathbf{X},\mathbf{b}-\mathbf{c},\mathbf{S}) + \mathscr{S}(f,\mathbf{X},\mathbf{c},\mathbf{S}) \\ &\geqslant \mathscr{S}(f,\mathbf{X},\mathbf{c},\mathbf{S}) \geqslant 0,\end{aligned}$$

and consequently (4.3) holds. □

The concavity and monotonicity properties of Sherman's operator are very important properties, considering the numerous applications that will follow from them. First, regarding the monotonicity property (4.3), we give the consequence of Theorem 4.1, which includes the lower and upper bound for Sherman's operator, which are expressed in terms of an associated non-weighted Jensen's operator.

THEOREM 4.2. *Suppose $\mathscr{S}$ is an operator defined by* (4.1)*. Then*

$$\begin{aligned}0 &\leqslant \min_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant l}} \{b_i s_{ij}\}\, \mathscr{S}(f,\mathbf{X},\mathbf{1}_n,(1/l)_{nl}) \leqslant \mathscr{S}(f,\mathbf{X},\mathbf{b},\mathbf{S}) \\ &\leqslant \max_{\substack{1\leqslant i\leqslant n\\1\leqslant j\leqslant l}} \{b_i s_{ij}\}\, \mathscr{S}(f,\mathbf{X},\mathbf{1}_n,(1/l)_{nl}),\end{aligned} \tag{4.7}$$

where $\mathscr{S}(f,\mathbf{X},\mathbf{1}_n,(1/l)_{nl})$ is a non-weighted Sherman's operator

$$\mathscr{S}(f,\mathbf{X},\mathbf{1}_n,(1/l)_{nl}) = \sum_{j=1}^{l}\sum_{i=1}^{n} f(X_j) - l\sum_{i=1}^{n} f\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right) = n\cdot \mathscr{J}_{\mathscr{N}}(f,\mathbf{X}),$$

and $\mathscr{J}_{\mathscr{N}}(f,\mathbf{X})$ is a non-weighted Jensen's operator

$$\mathscr{J}_{\mathscr{N}}(f,\mathbf{X}) = \sum_{j=1}^{l} f(X_j) - l\, f\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right). \tag{4.8}$$

Proof.

- First let us prove that

$$\min_{1\leqslant i\leqslant n}\{b_i\}\, \mathscr{S}(f,\mathbf{X},\mathbf{1}_n,\mathbf{S}) \leqslant \mathscr{S}(f,\mathbf{X},\mathbf{b},\mathbf{S}) \leqslant \max_{1\leqslant i\leqslant n}\{b_i\}\, \mathscr{S}(f,\mathbf{X},\mathbf{1}_n,\mathbf{S}) \tag{4.9}$$

holds, where $\mathbf{1}_n = (1,1,\dots,1)$ is a constant ordered n-tuple, that is $\mathscr{S}(f,\mathbf{X},\mathbf{1}_n,\mathbf{S}) = \sum_{j=1}^{l}\sum_{i=1}^{n} s_{ij} f(X_j) - \sum_{i=1}^{n} f\left(\sum_{j=1}^{l} s_{ij}X_j\right)$.

We compare an ordered n-tuple $\mathbf{b} = (b_1, b_2, \dots, b_n) \in [0,\infty)^n$ with constant ordered n-tuples $\mathbf{b}_{\min} = \big(\min_{1\leqslant i\leqslant n}\{b_i\}, \dots, \min_{1\leqslant i\leqslant n}\{b_i\}\big)$ and $\mathbf{b}_{\max} = \big(\max_{1\leqslant i\leqslant n}\{b_i\}, \dots, \max_{1\leqslant i\leqslant n}\{b_i\}\big)$. Clearly, $\mathbf{b}_{\max} \geqslant \mathbf{b} \geqslant \mathbf{b}_{\min}$. Using the monotonicity property (4.3) of $\mathscr{S}(f,\mathbf{X},\cdot,\mathbf{S})$ yields a series of inequalities:

$$\mathscr{S}(f,\mathbf{X},\mathbf{b}_{\min},\mathbf{S}) \leqslant \mathscr{S}(f,\mathbf{X},\mathbf{b},\mathbf{S}) \leqslant \mathscr{S}(f,\mathbf{X},\mathbf{b}_{\max},\mathbf{S}).$$

Finally, considering $\mathscr{S}(f,\mathbf{X},\mathbf{b}_{\min},\mathbf{S}) = \min_{1\leqslant i\leqslant n}\{b_i\}\,\mathscr{S}(f,\mathbf{X},\mathbf{1}_n,\mathbf{S})$ and $\mathscr{S}(f,\mathbf{X},\mathbf{b}_{\max},\mathbf{S}) = \max_{1\leqslant i\leqslant n}\{b_i\}\,\mathscr{S}(f,\mathbf{X},\mathbf{1}_n,\mathbf{S})$ we get the series of inequalities in (4.9).

- Next, let us prove that

$$\begin{aligned} m_i\left(\sum_{j=1}^{l} f(X_j) - l f\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right)\right) &\leqslant \sum_{j=1}^{l} s_{ij} f(X_j) - f\left(\sum_{j=1}^{l} s_{ij}X_j\right) \\ &\leqslant M_i\left(\sum_{j=1}^{l} f(X_j) - l f\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right)\right) \end{aligned} \tag{4.10}$$

holds for every $i = 1, 2, \dots, n$, where $m_i = \min_{1\leqslant j\leqslant l}\{s_{ij}\}$ and $M_i = \max_{1\leqslant j\leqslant l}\{s_{ij}\}$.

Since $l\cdot m_i + \sum_{j=1}^{l}(s_{ij} - m_i) = 1$, we have

$$\begin{aligned} &f\left(\sum_{j=1}^{l} s_{ij}X_j\right) = f\left(l\cdot m_i\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right) + \sum_{j=1}^{l}(s_{ij}-m_i)X_j\right) \\ &\leqslant l\cdot m_i f\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right) + \sum_{j=1}^{l}(s_{ij}-m_i)f(X_j) \qquad \text{(by Jensen's operator inequality)} \\ &= \sum_{j=1}^{l} s_{ij}f(X_j) - m_i\left(\sum_{j=1}^{l} f(X_j) - l f\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right)\right), \end{aligned}$$

which gives the LHS inequality in (4.10).

Also, since $M_i > 0$ $(i = 1, 2, \dots, n)$ and $\frac{1}{M_i}\sum_{j=1}^{l}\left(\frac{M_i}{l} - \frac{s_{ij}}{l}\right) + \frac{1}{lM_i} = 1$, we have

$$\begin{aligned} &f\left(\frac{1}{l}\sum_{j=1}^{l} X_j\right) = f\left(\frac{1}{M_i}\sum_{j=1}^{l}\left(\frac{M_i}{l} - \frac{s_{ij}}{l}\right)X_j + \frac{1}{lM_i}\left(\sum_{j=1}^{l} s_{ij}X_j\right)\right) \\ &\leqslant \frac{1}{M_i}\sum_{j=1}^{l}\left(\frac{M_i}{l} - \frac{s_{ij}}{l}\right)f(X_j) + \frac{1}{lM_i} f\left(\sum_{j=1}^{l} s_{ij}X_j\right) \qquad \text{(by Jensen's operator inequality)} \\ &= \frac{1}{l}\sum_{j=1}^{l} f(X_j) - \frac{1}{lM_i}\left(\sum_{j=1}^{l} s_{ij}f(X_j) - f\left(\sum_{j=1}^{l} s_{ij}X_j\right)\right), \end{aligned}$$

which gives the RHS inequality in (4.10).

• Finally we prove (4.7). Multiplying the series of inequalities in (4.10) with $b_i \geqslant 0$ and summing over $i=1,\ldots,n$, we get

$$\sum_{i=1}^{n} b_i m_i \left(\sum_{j=1}^{l} f(X_j) - l f\left(\frac{1}{l} \sum_{j=1}^{l} X_j \right) \right) \leqslant \sum_{i=1}^{n} b_i \left(\sum_{j=1}^{l} s_{ij} f(X_j) - f\left(\sum_{j=1}^{l} s_{ij} X_j \right) \right)$$
$$\leqslant \sum_{i=1}^{n} b_i M_i \left(\sum_{j=1}^{l} f(X_j) - l f\left(\frac{1}{l} \sum_{j=1}^{l} X_j \right) \right)$$

i.e.

$$\mathscr{S}(f,\mathbf{X},\underline{\mathbf{b}},(1/l)_{nl}) \leqslant \mathscr{S}(f,\mathbf{X},\mathbf{b},\mathbf{S}) \leqslant \mathscr{S}(f,\mathbf{X},\overline{\mathbf{b}},(1/l)_{nl}), \tag{4.11}$$

where $\underline{\mathbf{b}} = (b_1m_1, b_2m_2, \ldots, b_nm_n)$, $\overline{\mathbf{b}} = (b_1M_1, b_2M_2, \ldots, b_nM_n)$ and $(1/l)_{nl} \in \mathbb{S}_{nl}(\mathbb{R})$ is a row stochastic matrix with constant entries. By using the LHS and the RHS inequality in (4.9), we obtain

$$\min_{1\leqslant i\leqslant n} \{b_i m_i\}\, \mathscr{S}(f,\mathbf{X},\mathbf{1}_n,(1/l)_{nl}) \leqslant \mathscr{S}(f,\mathbf{X},\underline{\mathbf{b}},(1/l)_{nl}), \tag{4.12}$$

$$\mathscr{S}(f,\mathbf{X},\overline{\mathbf{b}},(1/l)_{nl}) \leqslant \max_{1\leqslant i\leqslant n} \{b_i m_i\}\, \mathscr{S}(f,\mathbf{X},\mathbf{1}_n,(1/l)_{nl}). \tag{4.13}$$

Finally, we get (4.7) by combining (4.11), (4.12) and (4.13) and using

$$\min_{1\leqslant i\leqslant n} \{b_i m_i\} = \min_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant l}} \{b_i s_{ij}\}, \qquad \max_{1\leqslant i\leqslant n} \{b_i m_i\} = \max_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant l}} \{b_i s_{ij}\},$$
$$\mathscr{S}(f,\mathbf{X},\mathbf{1}_n,(1/l)_{nl}) = n \cdot \mathscr{J}_{\mathscr{N}}(f,\mathbf{X}). \qquad \square$$

REMARK 4.3. 1) Dragomir et al. in [4] investigated the properties of discrete Jensen's functional

$$J_l(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{l} p_i f(x_i) - P f\left(\frac{\sum_{i=1}^{l} p_i f(x_i)}{P} \right), \tag{4.14}$$

where $f : J \subset \mathbb{R} \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1, x_2, \ldots, x_l) \in J^l$ and $\mathbf{p} = (p_1, p_2, \ldots, p_l)$ is a positive l-tuple of real numbers with $P = \sum_{i=1}^{l} p_i$.

By setting $n = 1$, $b_1 = P$, $s_{1j} = \frac{p_j}{P}$ $(j = 1, 2, \ldots, l)$ in Sherman's operator (4.1), we obtain Jensen's operator that corresponds to (4.14):

$$\mathscr{J}_l(f,\mathbf{X},\mathbf{p}) = \sum_{i=1}^{l} p_i f(X_i) - P f\left(\sum_{i=1}^{l} \frac{p_i}{P} f(X_i) \right), \tag{4.15}$$

where $f \in \mathscr{F}_o(J)$ and $\mathbf{X} = (X_1, X_2, \ldots, X_l) \in [\mathscr{B}_h^J(\mathscr{H})]^l$.

Applying Theorem 4.1, we obtain that $\mathscr{J}_l$ has the properties of superadditivity and monotonicity, since by putting $\lambda = \frac{1}{2}$ in (4.2) we obtain

$$\mathscr{J}_l(f,\mathbf{X},\mathbf{p}+\mathbf{q}) \geqslant \mathscr{J}_l(f,\mathbf{X},\mathbf{p}) + \mathscr{J}_l(f,\mathbf{X},\mathbf{q}).$$

Also, applying Theorem 4.2 we obtain the bounds of $\mathscr{J}_l$, which are expressed in terms of non-weighted Jensen's operator (4.8):

$$0 \leqslant \min_{1\leqslant j\leqslant l} \{p_j\}\, \mathscr{J}_{\mathscr{N}}(f,\mathbf{X}) \leqslant \mathscr{J}_l(f,\mathbf{X},\mathbf{p}) \leqslant \max_{1\leqslant j\leqslant l} \{p_j\}\, \mathscr{J}_{\mathscr{N}}(f,\mathbf{X}). \tag{4.16}$$

2) Setting $l=2$ and $X_1 = D$, $X_2 = \delta 1_{\mathscr{H}}$ in Jensen's operator $\mathscr{J}_l$ defined by (4.15), we obtain Jensen's operator considered in [17, (14)]:

$$\mathscr{J}_2(f,D,\delta,\mathbf{p}) = p_1 f(D) + p_2(\delta)1_{\mathscr{H}} - (p_1+p_2) f\left(\frac{p_1 D + p_2 \delta 1_{\mathscr{H}}}{p_1+p_2}\right),$$

where $\mathbf{p} = (p_1, p_2)$, $a1_{\mathscr{H}} \leqslant D \leqslant b1_{\mathscr{H}}$, $a \leqslant \delta \leqslant b$ and $f \in \mathscr{F}([a,b])$. We remark that f is not operator convex in this case, because $\mathscr{J}_2(f,D,\delta,\mathbf{p})$ we can obtain directly from Jensen's functional $J_2(f,\mathbf{x},\mathbf{p})$ defined by (4.14). We obtain that $\mathscr{J}_2$ has the properties of superadditivity and monotonicity as immediately proven in [17, Theorem 1]. Also, we obtained that

$$0 \leqslant \min\{p_1,p_2\}\ \mathscr{J}_{\mathscr{N}}(f,D,\delta) \leqslant \mathscr{J}_2(f,D,\delta,\mathbf{p}) \leqslant \max\{p_1,p_2\}\ \mathscr{J}_{\mathscr{N}}(f,D,\delta)$$

holds where $\mathscr{J}_{\mathscr{N}}(f,D,\delta) = f(D) + f(\delta)1_{\mathscr{H}} - 2\cdot f\left(\frac{D+\delta 1_{\mathscr{H}}}{2}\right)$, see [17, Corollary 1].

The obtained results can be applied to operator means. E.g. we can get refinements and conversions of numerous mean inequalities for Hilbert space operators, see [17, §4].

5. Multidimensional Sherman's operator and applications

The main objective of this section is to obtain a unified treatment for the examples of joint concave mappings. With that intention we are going to extend the concept of Sherman's operator to several variables.

5.1. Multidimensional Sherman's operator inequality

Let $k \in \mathbb{N}$ and $\mathscr{H}_j$ be the Hilbert space $(j = 1,2,\ldots,k)$. Suppose $A_j \in \mathscr{B}_h(\mathscr{H}_j)$ $(j=1,2,\ldots,k)$ and let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ denote a k-tuple of self-adjoint operators. Let $\mathscr{D} \subseteq \prod_{j=1}^k \mathscr{B}_h(\mathscr{H}_j)$ be a convex set. Further, let F be a mapping that every $\mathbf{A} \in \mathscr{D}$ assigns the self-adjoint operator on a Hilbert space $\mathscr{H}'$, that is, we suppose the function $F : \mathscr{D} \to \mathscr{B}_h(\mathscr{H}')$ is well-defined.

With the above assumptions, we can establish the definition of operator convexity in multiple variables. A function $F : \mathscr{D} \to \mathscr{B}_h(\mathscr{H}')$ is said to be operator convex in k variables, if the operator inequality

$$F(\lambda\mathbf{A} + (1-\lambda)\mathbf{B}) \leqslant \lambda F(\mathbf{A}) + (1-\lambda)F(\mathbf{B}) \tag{5.1}$$

holds for all $\mathbf{A},\mathbf{B} \in \mathscr{D}$ and $\lambda \in [0,1]$, with respect to operator order in $\mathscr{H}'$. A function F is called operator concave in k variables, if the reverse inequality holds in (5.1) for all $\mathbf{A},\mathbf{B} \in \mathscr{D}$ and $\lambda \in [0,1]$.

Similarly as in the classical Jensen's inequality, inequality (5.1) can be easily extended to multidimensional Jensen's operator inequality (see [18, Proposition 1]) as follows: If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is an n-tuple of nonnegative real numbers, $\sum_{i=1}^n a_i = 1$, and $\mathbf{X}_i \in \mathscr{D}$ $(i = 1,2,\ldots,n)$ then

$$F\left(\sum_{i=1}^n a_i \mathbf{X}_i\right) \leqslant \sum_{i=1}^n a_i F(\mathbf{X}_i) \tag{5.2}$$

holds for any operator convex function in k variables, $F:\mathscr{D}\to\mathscr{B}_h(\mathscr{H}')$. If F is an operator concave function in k variables, then the reverse inequality is valid in (5.2).

Now we extend Sherman's operator inequality (2.3) to multidimensional operator inequality.

In the following, $\mathscr{D}$ denotes a convex set $\mathscr{D}\subseteq\prod_{j=1}^{k}\mathscr{B}_h(\mathscr{H}_j)$.

PROPOSITION 5.1. *Let* $(\mathbf{X}_1,\mathbf{X}_2,\dots,\mathbf{X}_l)$ *be an* l*-tuple and* $(\mathbf{Y}_1,\mathbf{Y}_2,\dots,\mathbf{Y}_n)$ *be an* n*-tuple of operators* $\mathbf{X}_j=(X_{j1},X_{j2},\dots,X_{jk})\in\mathscr{D}$ $(j=1,2,\dots l)$, $\mathbf{Y}_i=(Y_{i1},Y_{i2},\dots,Y_{ik})\in\mathscr{D}$ $(i=1,2,\dots n)$, *let* $\mathbf{a}=(a_1,a_2,\dots,a_l)\in[0,\infty)^l$, $\mathbf{b}=(b_1,b_2,\dots,b_n)\in[0,\infty)^n$ *be vectors and* $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nl}(\mathbb{R})$ *be a row stochastic matrix such that*

$$\mathbf{Y}_i=\sum_{j=1}^{l}s_{ij}\mathbf{X}_j\ \ (i=1,2,\dots,n)\quad\textit{and}\quad a_j=\sum_{i=1}^{n}b_is_{ij}\ \ (j=1,2,\dots,l).$$

If $F:\mathscr{D}\to\mathscr{B}_h(\mathscr{H}')$ *is an operator convex function in* k *variables, then*

$$\sum_{i=1}^{n}b_iF(\mathbf{Y}_i)\leqslant\sum_{j=1}^{l}a_jF(\mathbf{X}_j).\tag{5.3}$$

Especially, if $l=n$ *and* $\mathbf{S}=(s_{ij})\in\mathbb{M}_{nn}(\mathbb{R})$ *is a doubly stochastic matrix, then the weighted multidimensional majorization inequality holds, so that*

$$\mathbf{Y}_i=\sum_{j=1}^{n}s_{ij}\mathbf{X}_j\ \ (i=1,2,\dots,n)\qquad\textit{implies}\qquad\sum_{i=1}^{n}a_iF(\mathbf{Y}_i)\leqslant\sum_{i=1}^{n}a_iF(\mathbf{X}_\mathbf{i}).\tag{5.4}$$

If F *is operator concave in* k *variables, then the reverse inequality is valid in* (5.3) *and* (5.4).

Proof. Since F is an operator convex function in k variables and $\mathbf{S}$ is row stochastic, then

$$\sum_{i=1}^{n}b_iF(\mathbf{Y}_i)=\sum_{i=1}^{n}b_iF\left(\sum_{j=1}^{l}s_{ij}\mathbf{X}_j\right)\leqslant\sum_{i=1}^{n}b_i\left(\sum_{j=1}^{l}s_{ij}F(\mathbf{X}_j)\right)$$
$$=\sum_{j=1}^{l}\left(\sum_{i=1}^{n}b_is_{ij}\right)F(\mathbf{X}_j)=\sum_{j=1}^{l}a_jF(\mathbf{X}_j),$$

which gives (5.3).

Setting $n=l$ in (5.3) and all weights a_i and b_j are equal and nonnegative, the condition $\mathbf{a}=\mathbf{bS}$ assures stochasticity on columns of $\mathbf{S}$, so in that case we deal with doubly stochastic matrix. Then (5.4) holds. □

REMARK 5.2. a) Setting $n=1$ and $\mathbf{b}=(1)$ the multidimensional Sherman operator inequality (5.3) reduces to the multidimensional Jensen operator inequality (5.1).

b) Setting $\mathbf{a} = (1,1,\ldots,1)$ in the inequality (5.4), we get the multidimensional majorization inequality in the form:

$$\mathbf{Y}_i = \sum_{j=1}^{n} s_{ij}\mathbf{X}_j \ \ (i = 1,2,\ldots,n) \qquad \text{implies} \qquad \sum_{i=1}^{n} F(\mathbf{Y}_i) \leqslant \sum_{i=1}^{n} F(\mathbf{X_i}),$$

where $\mathbf{S} = (s_{ij}) \in \mathbb{M}_{nn}(\mathbb{R})$ is a doubly stochastic matrix.

If F is operator concave in k variables, then the reverse inequality is valid in previous inequalities.

5.2. Multidimensional Sherman's operator

Now, we define multidimensional Sherman's operator, deduced from multidimensional Sherman's operator inequality.

Let $F : \mathscr{D} \to \mathscr{B}_h(\mathscr{H}')$ is an operator convex function in k variables and let $\mathscr{F}_k(\mathscr{D})$ denote the set of all operator convex functions in k variables on $\mathscr{D}$.

Considering relation (5.3), we define multidimensional Sherman's operator $\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{S},\mathbf{b}) : \mathscr{F}_k(\mathscr{D}) \times \mathscr{D} \times [0,\infty)^n \times \mathbb{S}_{nl}(\mathbb{R}) \to \mathscr{B}_h(\mathscr{H}')$ as

$$\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\mathbf{S}) = \sum_{j=1}^{l}\sum_{i=1}^{n} b_i s_{ij} F(\mathbf{X}_j) - \sum_{i=1}^{n} b_i F\left(\sum_{j=1}^{l} s_{ij}\mathbf{X}_j\right), \tag{5.5}$$

where $\mathbf{X} = (\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_l)$ is l-tuple of operators $\mathbf{X}_j = (X_{j1},X_{j2},\ldots,X_{jk}) \in \mathscr{D}$ $(j = 1,2,\ldots,l)$, $\mathbf{b} = (b_1,b_2,\ldots,b_n)$ and $\mathbf{S} = (s_{ij})$.

Note that operator $\mathscr{S}_{\mathscr{M}}$ is well-defined and positive. Positivity of operator $\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\mathbf{S})$ follows from multidimensional Sherman's operator inequality (5.3). Now, we prove the properties of concavity and monotonicity of this operator.

THEOREM 5.3. *Suppose that $\mathscr{S}_{\mathscr{M}}$ is an operator defined by (5.5). Then it satisfies the following properties:*

(i) $\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\cdot)$ *is concave on* $\mathbb{S}_{nl}(\mathbb{R})$*, that is*

$$\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\lambda\mathbf{S}+(1-\lambda)\mathbf{T}) \geqslant \lambda\,\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\mathbf{S}) + (1-\lambda)\,\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\mathbf{T}). \tag{5.6}$$

(ii) If $\mathbf{b},\mathbf{c} \in [0,\infty)^n$ *with* $\mathbf{b} \geqslant \mathbf{c}$ *(i.e.* $b_i \geqslant c_i$, $i = 1,2,\ldots,n$*), then*

$$\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\mathbf{S}) \geqslant \mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{c},\mathbf{S}) \geqslant 0, \tag{5.7}$$

i.e. $\mathscr{S}_{\mathscr{M}}(f,\mathbf{X},\cdot,\mathbf{S})$ *is increasing on* $[0,\infty)^n$.

But, if F is operator concave in k variables, then $-\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\mathbf{S}) \in \mathscr{B}^+(\mathscr{H}')$, $\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\mathbf{b},\cdot)$ *is convex on* $\mathbb{S}_{nl}(\mathbb{R})$ *and* $\mathscr{S}_{\mathscr{M}}(F,\mathbf{X},\cdot,\mathbf{S})$ *is decreasing on* $[0,\infty)^n$.

Proof. We use the same technique as in the proof of Theorem 4.1. We omit the details. □

The concavity and monotonicity properties of multidimensional Sherman's operator are very significant properties, considering the numerous applications that will follow from them. As a first application of Theorem 5.3 we establish the lower and upper bounds for multidimensional operator (5.5), which are expressed in terms of associated non-weighted Jensen's operator.

COROLLARY 5.4. *Suppose that* $\mathscr{S}_{\mathscr{M}}$ *is an operator defined by* (5.5). *Then*

$$0 \leqslant n \cdot \min_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l}} \{b_i s_{ij}\} \, \mathscr{J}_{\mathscr{N}}(F, \mathbf{X}) \leqslant \mathscr{S}_{\mathscr{M}}(F, \mathbf{X}, \mathbf{b}, \mathbf{S}) \leqslant n \cdot \max_{\substack{1 \leqslant i \leqslant n \\ 1 \leqslant j \leqslant l}} \{b_i s_{ij}\} \, \mathscr{J}_{\mathscr{N}}(F, \mathbf{X}), \tag{5.8}$$

where $\mathscr{J}_{\mathscr{N}}$ *is a non-weighted Jensen's operator defined by* (4.8)*, i.e.*

$$\mathscr{J}_{\mathscr{N}}(F, \mathbf{X}) = \sum_{j=1}^{l} F(\mathbf{X}_j) - lF\left(\sum_{j=1}^{l} \frac{1}{l} \mathbf{X}_j\right).$$

Proof. We use the same technique as in the proof of Theorem 4.2. We omit the details. □

REMARK 5.5. 1) Setting $n = 1$, $b_1 = P$, $s_{1j} = \frac{p_j}{P}$ $(j = 1, 2, \ldots, l)$ in (5.5), we obtain Jensen's operator considered in [18, (6)]:

$$\mathscr{J}_{\mathscr{M}}(F, \mathbf{X}, \mathbf{p}) = \sum_{j=1}^{l} p_j F(\mathbf{X}_j) - PF\left(\frac{1}{P} \sum_{j=1}^{l} p_j \mathbf{X}_j\right), \tag{5.9}$$

where $\mathbf{p} = (p_1, p_2, \ldots, p_l)$. By applying Theorem 5.3 and Corollary 5.4, we obtain properties and bounds of operator $\mathscr{J}_{\mathscr{M}}$ as in [18, Theorem 1, Corolary 1]. We omit the details.

2) Let $\mathscr{D} \subseteq \mathscr{B}_h(\mathscr{H}_1) \times \mathscr{B}_h(\mathscr{H}_2)$, $F : \mathscr{D} \to \mathscr{B}_h(\mathscr{H}')$ be an operator function in two variables. Putting $\mathbf{X}_j = (A_j, B_j)$ $(j = 1, 2, \ldots, l)$ in Sherman's operator (5.5) we obtain

$$\mathscr{S}_{\mathscr{M}}(F, \mathbf{A}, \mathbf{B}, \mathbf{b}, \mathbf{S}) = \sum_{j=1}^{l} \sum_{i=1}^{n} b_i s_{ij} F(A_j, B_j) - \sum_{i=1}^{n} b_i F\left(\sum_{j=1}^{l} s_{ij} A_j, \sum_{j=1}^{l} s_{ij} B_j\right), \tag{5.10}$$

where $\mathbf{A} = (A_1, A_2, \ldots, A_l)$, $\mathbf{B} = (B_1, B_2, \ldots, B_l)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in [0, \infty)^n$ and $\mathbf{S} = (s_{ij}) \in \mathbb{S}_{nl}(\mathbb{R})$. Also, Jensen's operator $\mathscr{J}_{\mathscr{M}}$ (5.9) reduces to (see [18, (12)]):

$$\mathscr{J}_{\mathscr{M}_2}(F, \mathbf{A}, \mathbf{B}, \mathbf{p}) = \sum_{j=1}^{l} p_j F(A_j, B_j) - P \cdot F\left(\frac{1}{P} \sum_{j=1}^{l} p_j A_j, \frac{1}{P} \sum_{j=1}^{l} p_j B_j\right).$$

a) Replacing the function F with connection $\sigma : \mathscr{B}^+(\mathscr{H}) \times \mathscr{B}^+(\mathscr{H}) \to \mathscr{B}^+(\mathscr{H})$ in the operator (5.10) and using that every connection is positive homogeneous, i.e. $\alpha(A\sigma B) = (\alpha A)\sigma(\alpha B)$ for all $\alpha > 0$ and $A, B \in \mathscr{B}^+(\mathscr{H})$ (see [18]), then we obtain the operator

$$\mathscr{S}_\sigma(\mathbf{A}, \mathbf{B}, \mathbf{a}) = \sum_{j=1}^{l} a_j(A_j \sigma B_j) - \left(\sum_{j=1}^{l} a_j A_j\right) \sigma \left(\sum_{j=1}^{l} a_j B_j\right), \tag{5.11}$$

since $\mathbf{a} = \mathbf{bS}$. So, applying Theorem 5.3 and Corollary 5.4, we obtain results as in [18, §4.1]. Examples of solidarity are the weighted arithmetic mean $A\nabla_s B$, the weighted harmonic mean $A!_s B$ and the weighted geometric mean $A\sharp_s B$ (see [9]). In these cases the connection operator (5.11) reduces to appropriate operators, see [18, Remark 3].

b) Replacing the function F with the solidarity $s : \mathscr{B}^{++}(\mathscr{H}) \times \mathscr{B}^{++}(\mathscr{H}) \to \mathscr{B}^{++}(\mathscr{H})$ in the operator (5.10) and using that every solidarity is also positive homogeneous, i.e. $\alpha(AsB) = (\alpha A)s(\alpha B)$ for all $\alpha > 0$ and $A, B \in \mathscr{B}^{++}(\mathscr{H})$ (see [6]), then we obtain the operator

$$\mathscr{S}_s(\mathbf{A}, \mathbf{B}, \mathbf{a}) = \sum_{j=1}^{l} a_j(A_j s B_j) - \left(\sum_{j=1}^{l} a_j A_j\right) s \left(\sum_{j=1}^{l} a_j B_j\right). \tag{5.12}$$

Applying Theorem 5.3 and Corollary 5.4, we obtain results as in [18, §4.2]. Examples of solidarity are the well-known relative operator entropy $S(A|B)$ and the Tsallis relative operator entropy $T_\lambda(A|B)$, see [8]. In these cases the solidarity operator (5.12) reduces to appropriate operators, see [18, Remark 4].

5.3. Multidimensional weighted geometric mean

Fujii et.al. in [5] established the weighted geometric mean $G[n,t]$, $0 \leqslant t \leqslant 1$, for an n-tuple of positive invertible operators $A_1, A_2, \ldots, A_n$. Let $G[2,t](A_1, A_2) = A_1 \sharp_t A_2$. For $n \geqslant 3$, $G[n,t]$ is defined inductively as follows: Define $A_i^{(1)} = A_i$ $(i = 1, 2, \cdots, n)$ and

$$A_i^{(r)} = G[n-1,t]\big(A_i^{(r-1)}, \ldots, A_{i-1}^{(r-1)}, A_{i+1}^{(r-1)}, \ldots, A_n^{(r-1)}\big),$$

inductively for r. Then, there exist the limit $\lim_{r\to\infty} A_i^{(r)}$ in the Thompson metric and it does not depend on i (see [5]). Thus, the above mentioned weighted geometric mean is defined as $G[n,t](A_1, A_2, \ldots, A_n) = \lim_{r\to\infty} A_i^{(r)}$. Such defined geometric mean is also jointly concave, i.e.

$$G[n,t]\left(\sum_{i=1}^{n} \lambda_i \mathbf{A}_i\right) \geqslant \sum_{i=1}^{n} \lambda_i G[n,t](\mathbf{A}_i), \tag{5.13}$$

where $\lambda_i \geqslant 0$, $\sum_{i=1}^{n} \lambda_i = 1$, and every $\mathrm{A}_i = (A_1, A_2, \ldots, A_n)$ $(i = 1, 2, \ldots, n)$ is an ordered n-tuple of positive invertible operators on a Hilbert space.

Now, we define a multidimensional geometric operator of Sherman type, deduced from multidimensional Sherman's operator (5.5)

$$\mathscr{S}_{G[n,t]}(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n,\mathbf{b},\mathbf{S}) = \sum_{j=1}^{n}\sum_{i=1}^{n} b_i s_{ij} G[n,t](\mathbf{X}_j) - \sum_{i=1}^{n} b_i G[n,t]\left(\sum_{j=1}^{n} s_{ij}\mathbf{X}_j\right), \tag{5.14}$$

where $G[n,t]$ is a multidimensional weighted geometric mean, $(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n)$ is an n-tuple of operators $\mathbf{X}_j = (X_{j1},X_{j2},\ldots,X_{jn}) \in \mathscr{D} \subseteq [\mathscr{B}^{++}(\mathscr{H})]^n$ $(j=1,2,\ldots,n)$, $\mathbf{b} = (b_1,b_2,\ldots,b_n) \in [0,\infty)^n$ and $\mathbf{S} = (s_{ij}) \in \mathbb{S}_{nn}(\mathbb{R})$.

The operator (5.14) is a generalisation of the multidimensional geometric operator of Jensen type considered in [18, §4.3]:

$$\mathscr{J}_{G[n,t]}(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n,\mathbf{p}) := \sum_{j=1}^{n} p_j G[n,t](\mathbf{X}_j) - G[n,t]\left(\sum_{j=1}^{n} p_j\mathbf{X}_j\right).$$

Since the weighted geometric mean is jointly concave, then similarly as in Corollary 5.4, we obtain that the operator (5.14) can mutually be bounded by the simpler operator of the same type. Such estimates can be regarded as both a refinement and converse of inequality (5.13). This result is also a generalisation of the result given in [18, Corollary 4].

COROLLARY 5.6. *Suppose that $\mathscr{S}_{G[n,t]}$ is a geometric operator of Sherman type defined by (5.14). Then*

$$\begin{aligned} & n\cdot \min_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \{b_i s_{ij}\}\, \mathscr{S}^{\mathscr{N}}_{G[n,t]}(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n) \\ & \leqslant \mathscr{S}_{G[n,t]}(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n,\mathbf{b},\mathbf{S}) \leqslant n\cdot \max_{\substack{1\leqslant i\leqslant n\\ 1\leqslant j\leqslant n}} \{b_i s_{ij}\}\, \mathscr{S}^{\mathscr{N}}_{G[n,t]}(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n), \end{aligned}$$

where $\mathscr{S}^{\mathscr{N}}$ is non-weighted geometric operator:

$$\mathscr{S}^{\mathscr{N}}_{G[n,t]}(\mathbf{X}_1,\mathbf{X}_2,\ldots,\mathbf{X}_n) = \sum_{j=1}^{n} G[n,t](\mathbf{X}_j) - G[n,t]\left(\sum_{j=1}^{n} \mathbf{X}_j\right).$$

Proof. Follows from Corollary 5.4. □

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