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GENERALIZED HERMITE–HADAMARD TYPE INEQUALITIES FOR DIFFERENTIABLE HARMONICALLY–CONVEX AND HARMONICALLY QUASI–CONVEX FUNCTIONS

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Abstract. Some new Hermite-Hadamard type inequalities for differentiable harmonically-convex and harmonically quasi-convex functions have been discussed, generalizing some existing results in literature. For validity of the results some numerically examples are given.

1. Introduction

In recent years, the Hermite-Hadamard inequality, which is the first fundamental result for convex functions with a natural geometrical interpretation and many applications, has drawn attention much interest in elementary mathematics. The inequality was discovered by C. Hermite and J. Hadamard for convex functions, having considerable attention in Literature is stated as follows [19, p.137]:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}, \tag{1.1}$$

provided that $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex function on an interval I of reals with $a,b \in I$ defined by:

$$f(tx+(1-t)y) \leqslant tf(x)+(1-t)f(y) \tag{1.2}$$

for $x,y \in I$ and $t \in [0,1]$. For concave function, f, the inequalities in (1.1) hold in reverse direction. On the other hand the mathematicians are trying to generalize the definition of convex functions by replacing the weighted arithmetic mean $tx+(1-t)y$ of x,y with the weighted geometric mean and weighted harmonic mean of x,y on the left side of the inequality (1.2) and/or by swapping the weighted arithmetic mean $tf(x)+(1-t)f(y)$ of $f(x)$, $f(y)$ with the weighted geometric mean and weighted harmonic mean $f(x)$, $f(y)$ on the right side of the inequality (1.2), where $x,y \in I$ and $t \in [0,1]$. Over the last two decades these types of swapping of means have led

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to many novel testimonies, stimulating extensions, conspicuous generalizations, innovative Hermite-Hadamard-type inequalities and a lot of applications of the inequalities (1.1) in the literature of mathematical inequalities and in other branches of pure and applied mathematics [3, 4, 5, 8, 12, 13, 16, 17] and the references cited therein. This paper is organized in the following way. After this Introduction, in Section 2 some assumptions and auxiliary results have been discussed, and in Section 3 some new weighted left and right Hermite-Hadamard type integral inequalities have been discussed. The results of Section 3 are believed to supply weighted variant of the findings obtained so far in the field of mathematical inequalities for differentiable harmonically-convex and harmonically quasi-convex functions.

2. Some preliminaries and auxiliary results

DEFINITION 1. [9] A function $f : I \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically-convex function on I if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant tf(y)+(1-t)f(x)$$

holds for all $x,\ y \in I$ and $t \in [0,1]$.

DEFINITION 2. [18] A function $f : I \subseteq (0,\infty) \to \mathbb{R}$ is said to be harmonically qausi-convex function on I if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leqslant \sup\{f(x), f(y)\}$$

holds for all $x,\ y \in I$, $t \in [0,1]$.

For more about harmonically-convex functions and harmonically quasi-convex functions, we refer the readers [1, 2, 6, 7] and the references cited therein.

DEFINITION 3. [10] A function $g : [a,b] \subseteq \mathbb{R}\setminus\{0\} \to \mathbb{R}$ is said to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ if

$$g(x) = g\left(\frac{1}{\frac{1}{a}+\frac{1}{b}-\frac{1}{x}}\right)$$

holds for all $x \in [a,b]$.

In what follows we use the followings:

$$G(w,u,v,y;X_y) := \int_a^{X_y(u,v)} \frac{(x-a)w(x)}{ax^3}dx - \int_{X_y(u,v)}^b \frac{(b-x)w(x)}{bx^3}dx. \tag{2.1}$$

Let $w : [a,b] \subseteq (0,\infty) \to [0,\infty)$ be a continuous function such that:

$$\int_a^b \frac{w(x)}{x^2}dx = 1; \quad \frac{1}{a_1} = \int_a^b \frac{w(x)}{x^3}dx. \tag{2.2}$$

$$M_i(w;x,a,a_1,b) := \begin{cases} \frac{w(x)}{x^i}, & x \in [a,a_1], \\ -\frac{w(x)}{x^i}, & x \in (a_1,b]. \end{cases}$$

$$\mathscr{A}(w;a,a_1,b) := \int_a^b \frac{b\ln(\frac{x}{a_1}) - a_1 + x}{b} M_2(w;x,a,a_1,b)dx. \tag{2.3}$$

$$\mathscr{B}(w;a,a_1,b) := \int_a^b \frac{a_1 - x + a\ln\left(\frac{a_1}{x}\right)}{a} M_2(w;x,a,a_1,b)dx. \tag{2.4}$$

$$\mathscr{A}_1(w;u,v) := \int_a^b \frac{x - v\ln x}{v} M_2\left(w;x,u,\frac{2uv}{u+v},v\right)dx. \tag{2.5}$$

$$\mathscr{B}_1(w;u,v) := \int_u^v \frac{u\ln x - x}{v} M_2\left(w;x,u,\frac{2uv}{u+v},v\right)dx. \tag{2.6}$$

$$\mathscr{C}(a,a_1,b) := a_1 \int_a^b M_2(w;x,a,a_1,b)dx - \int_a^b M_1(w;x,a,a_1,b)dx. \tag{2.7}$$

$$\mathscr{D}(u,v) := \int_v^u M_1\left(w;x,u,\frac{2uv}{u+v},v\right)dx. \tag{2.8}$$

$$Z_y(u,v) = \frac{2uv}{(1-y)u + (1+y)v}; \quad X_y(u,v) := \frac{uv}{(1-y)u + yv}. \tag{2.9}$$

DEFINITION 4. A function $\sigma : \mathbb{R} \to [0,1]$ defined by:

$$\sigma(u) := \begin{cases} 0, & u < 0; \\ 1, & u > 0 \end{cases}$$

is called Heavyside function.

LEMMA 1. *Let* $w : [a,b] \subseteq (0,\infty) \to [0,\infty)$ *be harmonically symmetric with respect to* $\frac{2ab}{a+b}$, *then* $a_1 = \frac{2ab}{a+b}$.

Proof. By definition 3 and the relation (2.2), we have

$$\begin{aligned}\int_a^b \frac{w(x)}{x^3}dx &= \int_a^b \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{x}\right)^3 w(x) \frac{dx}{x^2\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{x}\right)^2} \\ &= \int_a^b \left(\frac{1}{a} + \frac{1}{b} - \frac{1}{x}\right) \frac{w(x)dx}{x^2} \\ &= \frac{a+b}{ab} \int_a^b \frac{w(x)}{x^2}dx - \int_a^b \frac{w(x)}{x^3}dx,\end{aligned}$$

which yields the desired result. □

LEMMA 2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I°, interior of I, and $f' \in L([a,b])$, where $[a,b] \subseteq I^\circ$ with $a < b$. Let $w : [a,b] \to [0,\infty)$ be a continuous function, then*

$$\begin{aligned}\varphi(w,f) := {} & \frac{1}{b-a}\left[af(a)\int_a^b \frac{(b-x)w(x)}{x^3}dx \right.\\ & \left. + bf(b)\int_a^b \frac{(x-a)w(x)}{x^3}dx\right] - \int_a^b \frac{f(x)w(x)}{x^2}dx \\ = {} & \frac{b(a_1-a)}{a_1(b-a)}\int_0^1 X_y^2(a,a_1)\,G(w,a,a_1,y;X_y)\,f'(X_y(a,a_1))\,dy \\ & + \frac{a(b-a_1)}{a_1(b-a)}\int_0^1 X_y^2(b,a_1)\,G(w,b,a_1,y;X_y)\,f'(X_y(b,a_1))\,dy, \quad (2.10)\end{aligned}$$

where $G(w,u,v,y;X_y)$ and $X_y(u,v)$ are defined by (2.1) and (2.9) respectively.

Proof. According to Definition 4, the following identity holds:

$$f(x) - f(a) = \int_a^b \sigma(x-t)\,f'(t)\,dt. \quad (2.11)$$

Equivalently,

$$\begin{aligned}&\int_a^b \frac{(b-x)w(x)f(x)}{bx^3}dx - f(a)\int_a^b \frac{(b-x)w(x)}{bx^3}dx \\ &\qquad = \int_a^b\int_t^b \frac{(b-x)w(x)}{bx^3}f'(t)dxdt. \quad (2.12)\end{aligned}$$

Analogously, from the following identity (2.13), relation (2.14) holds:

$$f(x) - f(a) = \int_a^b \sigma(x-t)\,f'(t)\,dt. \quad (2.13)$$

$$\begin{aligned}&\int_a^b \frac{(x-a)w(x)f(x)}{ax^3}dx - f(b)\int_a^b \frac{(x-a)w(x)}{ax^3}dx \\ &\qquad = -\int_a^b\int_a^t \frac{(x-a)w(x)}{ax^3}f'(t)dxdt. \quad (2.14)\end{aligned}$$

Combining the relations (2.12) and (2.14) yields:

$$\begin{aligned}&\frac{1}{b-a}\left[af(a)\int_a^b \frac{(b-x)w(x)}{x^3}dx + bf(b)\int_a^b \frac{(x-a)w(x)}{x^3}dx\right] - \int_a^b \frac{f(x)w(x)}{x^2}dx \\ &= \frac{ab}{b-a}\int_a^b\left[\int_a^t \frac{(x-a)w(x)}{ax^3}dx - \int_t^b \frac{(b-x)w(x)}{bx^3}dx\right]f'(t)dt \\ &= \frac{ab}{b-a}\int_a^{a_1}\left[\int_a^t \frac{(x-a)w(x)}{ax^3}dx - \int_t^b \frac{(b-x)w(x)}{bx^3}dx\right]f'(t)dt \\ &\quad + \frac{ab}{b-a}\int_{a_1}^b\left[\int_a^t \frac{(x-a)w(x)}{ax^3}dx - \int_t^b \frac{(b-x)w(x)}{bx^3}dx\right]f'(t)dt. \quad (2.15)\end{aligned}$$

Setting $\frac{aa_1}{(1-y)a+ya_1}$ and $\frac{ba_1}{(1-y)b+ya_1}$, respectively, in the first and second integral of the last identity in (2.15), yields the desired identity (2.10). □

It is remarkable to note

- for $w \equiv \frac{ab}{b-a}$ (2.10) reduces to the identity:

$$\varphi\left(\frac{ab}{b-a}, f\right) = \frac{f(a)+f(b)}{2} - \frac{ab}{b-a}\int_a^b \frac{f(x)}{x^2}dx \tag{2.16}$$
$$= \frac{b-a}{4ab}\left[\int_0^1 yZ_y^2(a,b) f'(Z_y(a,b))\,dy - \int_0^1 yZ_y^2(b,a) f'(Z_y(b,a))\,dy\right],$$

  where $Z_y(u,v)$ is defined by (2.9).

- for w to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ (2.10) reduces to the identity:

$$\frac{f(a)+f(b)}{2} - \int_a^b \frac{f(x)w(x)}{x^2}dx$$
$$= \frac{1}{2}\int_0^1 Z_y^2(a,b)\, G(w,a,b,y;Z_y)\, f'(Z_y(a,b))\,dy$$
$$+ \frac{1}{2}\int_0^1 Z_y^2(b,a)\, G(w,b,a,y;Z_y)\, f'(Z_y(b,a))\,dy, \tag{2.17}$$

  where $G(w,u,v,y;Z_y)$ and $Z_y(u,v)$ are defined by (2.1) and (2.9) respectively.

- Lemma 2 is a generalization of [11, Lemma 2.1].

LEMMA 3. *Let $A : C([a,b]) \to \mathbb{R}$ be a positive linear functional on $C([a,b])$ and e_i be monomials $e_i(x) = x^i$, $x \in [a,b]$, $i \in \mathbb{N}$ let g be harmonically-convex function on $[a,b]$, then*

$$A(g(e_1)) \leqslant \frac{ab}{b-a}\left[A\left(\frac{1}{e_1} - \frac{1}{b}\right) g(a) + A\left(\frac{1}{a} - \frac{1}{e_1}\right) g(b)\right]. \tag{2.18}$$

Proof. By harmonically-convexity of g on $[a,b]$

$$g(e_1) = g\left(\frac{1}{\frac{\frac{1}{e_1}-\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}.\frac{1}{a} + \frac{\frac{1}{a}-\frac{1}{e_1}}{\frac{1}{a}-\frac{1}{b}}.\frac{1}{b}}\right) = g\left(\frac{ab}{\frac{\frac{1}{e_1}-\frac{1}{b}}{\frac{1}{a}-\frac{1}{b}}.b + \frac{\frac{1}{a}-\frac{1}{e_1}}{\frac{1}{a}-\frac{1}{b}}.a}\right) \tag{2.19}$$
$$\leqslant \frac{ab}{b-a}\left[\left(\frac{1}{e_1} - \frac{1}{b}\right) g(a) + \left(\frac{1}{a} - \frac{1}{e_1}\right) g(b)\right]$$

Application and positivity of the linear functional A on (2.18) yields the desired result (2.19). □

3. Main results

THEOREM 1. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a,b])$, where $[a,b] \subseteq I^\circ$ with $a < b$. If $w : [a,b] \to [0,\infty)$ is a continuous mapping and $|f'|$ is harmonically-convex on $[a,b]$, then*

$$\left| \int_a^b \frac{f(x)\,w(x)}{x^2}dx - f(a_1) \right| \leqslant ab \frac{\mathscr{A}(w;a,a_1,b)\,|f'(a)| + \mathscr{B}(w;a,a_1,b)\,|f'(b)|}{b-a}, \quad (3.1)$$

where $\mathscr{A}(w;a,a_1,b)$ and $\mathscr{B}(w;a,a_1,b)$ are given by the relations (2.3) and (2.4).

Proof. By definition 4 and the relation (2.2), following holds:

$$f(x) - f(a_1) = \int_a^b [\sigma(x-t) - \sigma(a_1-t)]\, f'(t)\,dt,$$

that is

$$\int_a^b \frac{f(x)\,w(x)}{x^2}dx - f(a_1) = \int_a^b \left(\int_t^b \frac{w(x)}{x^2}dx - \sigma(a_1-t) \right) f'(t)\,dt \quad (3.2)$$

holds.

Application of Lemma 3 and the properties of modulus yield:

$$\begin{aligned}
&\left| \int_a^b \frac{f(x)\,w(x)}{x^2}dx - f(a_1) \right| \\
&\leqslant \int_a^b \left| \int_t^b \frac{w(x)}{x^2}dx - \sigma(a_1-t) \right| |f'(t)|\,dt \\
&\leqslant \frac{ab}{b-a} \left[|f'(a)| \int_a^b \left| \int_t^b \frac{w(x)}{x^2}dx - \sigma(a_1-t) \right| \left(\frac{1}{t} - \frac{1}{b} \right) dt \right. \\
&\quad \left. + |f'(b)| \int_a^b \left| \int_t^b \frac{w(x)}{x^2}dx - \sigma(a_1-t) \right| \left(\frac{1}{a} - \frac{1}{t} \right) dt \right].
\end{aligned}$$

But,

$$\begin{aligned}
&\int_a^b \left| \int_t^b \frac{w(x)}{x^2}dx - \sigma(a_1-t) \right| \left(\frac{1}{t} - \frac{1}{b} \right) dt \\
&= \int_a^{a_1} \left| \int_t^b \frac{w(x)}{x^2}dx - 1 \right| \left(\frac{1}{t} - \frac{1}{b} \right) dt + \int_{a_1}^b \left(\int_t^b \frac{w(x)}{x^2}dx \right) \left(\frac{1}{t} - \frac{1}{b} \right) dt \\
&= \int_a^{a_1} \left| \int_t^b \frac{w(x)}{x^2}dx - \int_a^b \frac{w(x)}{x^2}dx \right| \left(\frac{1}{t} - \frac{1}{b} \right) dt \\
&\quad + \int_{a_1}^b \left(\int_t^b \frac{w(x)}{x^2}dx \right) \left(\frac{1}{t} - \frac{1}{b} \right) dt \\
&= \int_a^{a_1} \left(\int_a^t \frac{w(x)}{x^2}dx \right) \left(\frac{1}{t} - \frac{1}{b} \right) dt + \int_{a_1}^b \left(\int_t^b \frac{w(x)}{x^2}dx \right) \left(\frac{1}{t} - \frac{1}{b} \right) dt
\end{aligned}$$

$$= \left(\ln a_1 - \frac{a_1}{b}\right)\int_a^{a_1} \frac{w(x)}{x^2}dx - \int_a^{a_1}\left(\ln x - \frac{x}{b}\right)\frac{w(x)}{x^2}dx$$
$$- \left(\ln a_1 - \frac{a_1}{b}\right)\int_{a_1}^{b} \frac{w(x)}{x^2}dx + \int_{a_1}^{b}\left(\ln x - \frac{x}{b}\right)\frac{w(x)}{x^2}dx.$$

Analogously,

$$\int_a^b \left|\int_t^b \frac{w(x)}{x^2}dx - \sigma(a_1 - t)\right|\left(\frac{1}{a} - \frac{1}{t}\right)dt$$
$$= \left(\frac{a_1}{a} - \ln a_1\right)\left[\int_a^{a_1} \frac{w(x)}{x^2}dx - \int_{a_1}^{b} \frac{w(x)}{x^2}dx\right]$$
$$+ \int_{a_1}^{b}\left(\frac{x}{a} - \ln x\right)\frac{w(x)}{x^2}dx - \int_a^{a_1}\left(\frac{x}{a} - \ln x\right)\frac{w(x)}{x^2}dx.$$

This concludes the proof. □

COROLLARY 1. *Let the conditions of Theorem 1 be satisfied for* w *to be harmonically symmetric with respect to* $\frac{2ab}{a+b}$ *on* $[a,b]$, *then*

$$\left|\int_a^b \frac{f(x)w(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\right| \leqslant ab\frac{\mathscr{A}_1(w;a,b)|f'(a)| + \mathscr{B}_1(w;a,b)|f'(b)|}{b-a}, \quad (3.3)$$

where $\mathscr{A}_1(w;a,b)$ *and* $\mathscr{B}_1(w;a,b)$ *are given by the relations* (2.5) *and* (2.6).

Proof. The proof is a direct consequence of Theorem 1 for w to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ on $[a,b]$ so that $a_1 = \frac{2ab}{a+b}$. □

COROLLARY 2. *Let the conditions of Theorem 1 be satisfied for* w *to be a continuous function and* g *harmonically symmetric with respect to* $\frac{2ab}{a+b}$ *on* $[a,b]$, *then*

$$\left|\int_a^b \frac{f(x)g(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx\right|$$
$$\leqslant ab\frac{\mathscr{A}_1(g;a,b)|f'(a)| + \mathscr{B}_1(g;a,b)|f'(b)|}{b-a}, \quad (3.4)$$

where $\mathscr{A}_1(g;a,b)$ *and* $\mathscr{B}_1(g;a,b)$ *are given by the relations* (2.5) *and* (2.6).

Proof. The proof is a direct consequence of Theorem 1 for $w(x) = \frac{g(x)}{\int_a^b \frac{g(x)}{x^2}dx}$ and g harmonically symmetric with respect to $\frac{2ab}{a+b}$ on $[a,b]$. □

EXAMPLE 1. Let $f : \left[\frac{1}{2}, \frac{2}{3}\right] \to \left[\frac{3}{2}, 2\right]$ be two functions defined by $f(x) = w(x) = \frac{1}{x}$ so that $|f'(x)| = \frac{1}{x^2}$ is harmonically-convex on $\left[\frac{1}{2}, \frac{2}{3}\right]$. By using the software Mathematica, we observe that

$$LHS \text{ of } (3.3) = 0.893018$$

$$RHS \text{ of } (3.3) = 3.60811$$

and the deviation between these values is 2.967793.

THEOREM 2. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a,b])$, where $[a,b] \subseteq I^\circ$ with $a < b$. If $w : [a,b] \to [0,\infty)$ is a continuous function and $|f'|^q$ is harmonic-convex on $[a,b]$ for $q \geqslant 1$, then*

$$\left| \int_a^b \frac{f(x)w(x)}{x^2} dx - f(a_1) \right| \leqslant \sqrt[q]{ab \frac{\mathscr{A}(a,a_1,b)|f'(a)|^q + \mathscr{B}(a,a_1,b)|f'(b)|^q}{(b-a)[\mathscr{C}(a,a_1,b)]^{1-q}}}, \tag{3.5}$$

where $\mathscr{A}(a,a_1,b)$, $\mathscr{B}(a,a_1,b)$ and $\mathscr{C}(a,a_1,b)$ are given by the relations (2.3), (2.4) and (2.7).

Proof. Applications of Lemma 3 and power mean inequality to the inequality (3.2) yield:

$$\begin{aligned} \left| \int_a^b \frac{f(x)w(x)}{x^2} dx - f(a_1) \right| &\leqslant \int_a^b \left| \int_t^b \frac{w(x)}{x^2} dx - \sigma(a_1 - t) \right| |f'(t)| \, dt \qquad (3.6) \\ &\leqslant \sqrt[q]{\left[\int_a^b \left| \int_t^b \frac{w(x)}{x^2} dx - \sigma(a_1 - t) \right| dt \right]^{q-1}} \\ &\quad \times \sqrt[q]{\int_a^b \left| \int_t^b \frac{w(x)}{x^2} dx - \sigma(a_1 - t) \right| |f'(t)|^q dt} \\ &\leqslant \left[\int_a^{a_1} \int_a^t \frac{w(x)}{x^2} dx dt + \int_{a_1}^b \int_t^b \frac{w(x)}{x^2} dx dt \right] \\ &\quad \times \sqrt[q]{ab \frac{\mathscr{A}(a,a_1,b)|f'(a)|^q + \mathscr{B}(a,a_1,b)|f'(b)|^q}{b-a}}. \end{aligned}$$

This completes the proof. □

COROLLARY 3. *Let the conditions of Theorem 2 be satisfied for $w(x)$ to be harmonically symmetric with respect to $\frac{2ab}{a+b}$ on $[a,b]$, then*

$$\left| \int_a^b \frac{f(x)w(x)}{x^2} dx - f\left(\frac{2ab}{a+b}\right) \right| \leqslant \sqrt[q]{ab \frac{\mathscr{A}_1(a,b)|f'(a)|^q + \mathscr{B}_1(a,b)|f'(b)|^q}{(b-a)[\mathscr{D}(a,b)]^{1-q}}},$$

where $\mathscr{A}_1(a,b)$ and $\mathscr{B}_1(a,b)$ are defined by (2.5) and (2.6).

COROLLARY 4. *Let the conditions of Theorem 2 be satisfied for w to be a continuous function and g harmonically symmetric with respect to $\frac{2ab}{a+b}$ on $[a,b]$, then*

$$\left|\int_a^b \frac{f(x)g(x)}{x^2}dx - f\left(\frac{2ab}{a+b}\right)\int_a^b \frac{g(x)}{x^2}dx\right| \leqslant \sqrt[q]{ab\frac{\mathscr{A}_1(a,b)|f'(a)|^q + \mathscr{B}_1(a,b)|f'(b)|^q}{(b-a)[\mathscr{D}(a,b)]^{1-q}}}, \tag{3.7}$$

where $\mathscr{A}_1(a,b)$ and $\mathscr{B}_1(a,b)$ are defined by (2.5) and (2.6).

EXAMPLE 2. Suppose $f(x) = \frac{q}{(q-2)x^{\frac{2}{q}-1}}$, $w(x) = \frac{1}{x}$, $x \in \left[\frac{1}{2}, \frac{2}{3}\right]$. Then $w : \left[\frac{1}{2}, \frac{2}{3}\right] \to [0,\infty)$ is continuous function and $|f'(x)|^q = \frac{1}{x^2}$ is harmonically-convex on $\left[\frac{1}{2}, \frac{2}{3}\right]$ for $q = 3$. By using the software Mathematica, we observe that

$$LHS \text{ of } (3.5) = 1.52799$$
$$RHS \text{ of } (3.5) = 1.7642$$

and the error between the actual and the upper bound is 0.23621.

THEOREM 3. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable function on I° and $f' \in L([a,b])$, where $[a,b] \subseteq I^\circ$ with $a < b$; let $w : [a,b] \to [0,\infty)$ be a continuous function and $|f'|$ is harmonically quasi-convex on $[a,b]$, then*

$$|\varphi(w,f)| \leqslant \frac{b(a_1-a)}{a_1(b-a)}\left(\sup\left\{|f'(a)|, |f'(a_1)|\right\}\right)\int_0^1 X_y^2(a,a_1)\left|G(w,a,a_1,y;X_y)\right|dy$$
$$+\frac{a(b-a_1)}{a_1(b-a)}\left(\sup\left\{|f'(a_1)|, |f'(b)|\right\}\right)\int_0^1 X_y^2(b,a_1)\left|G(w,a_1,b,y;X_y)\right|dy, \tag{3.8}$$

where $G(w,u,v,y;X_y)$ and $X_y(u,v)$ are defined by (2.1) and (2.9) respectively.

Proof. The proof is followed by Lemma 2 and the harmonically-quasi convexity of $|f'|$ on $[a,b]$. □

EXAMPLE 3. Let $f : \left[\frac{1}{2}, \frac{2}{3}\right] \to \left[\frac{3}{2}, 2\right]$ be two functions defined by $f(x) = w(x) = \frac{1}{x}$ so that $|f'(x)| = \frac{1}{x^2}$ is harmonically quasi-convex on $\left[\frac{1}{2}, \frac{2}{3}\right]$. By using the software Mathematica, we observe that

$$LHS \text{ of } (3.8) = 0.0202165$$
$$RHS \text{ of } (3.8) = 2.89896$$

and the deviation between these values is 2.8787435.

THEOREM 4. *Let $f : I \subseteq (0,\infty) \to \mathbb{R}$ be a differentiable mapping on I° and $f' \in L([a,b])$, where $[a,b] \subseteq I^\circ$ with $a < b$. If $w : [a,b] \to [0,\infty)$ is a continuous mapping harmonically symmetric with respect to $\frac{2ab}{a+b}$ and $|f'|$ is harmonically quasi-convex on $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \int_a^b \frac{f(x)w(x)}{x^2}dx\right| \leqslant \sup\left\{|f'(a)|, \left|f'\left(\frac{2ab}{a+b}\right)\right|\right\}\left(\frac{a}{2} - \int_a^{\frac{2ab}{a+b}} \frac{w(x)}{x}dx\right) + \sup\left\{\left|f'\left(\frac{2ab}{a+b}\right)\right|, |f'(b)|\right\}\left(\frac{b}{2} - \int_{\frac{2ab}{a+b}}^b \frac{w(x)}{x}dx\right). \tag{3.9}$$

Proof. It may be observed that:

$$\frac{b(a_1-a)}{a_1(b-a)}\int_0^1 X_y^2(a,a_1)\left|G(w,a,a_1,y;X_y)\right|dy = \frac{ab}{b-a}\int_a^{a_1}\left|\int_a^t \frac{(x-a)w(x)}{ax^3}dx - \int_t^b \frac{(b-x)w(x)}{bx^3}dx\right|dt. \tag{3.10}$$

$$\frac{a(b-a_1)}{a_1(b-a)}\int_0^1 X_y^2(b,a_1)\left|G(w,a_1,b,y;X_y)\right|dy = \frac{ab}{b-a}\int_{a_1}^{b}\left|\int_a^t \frac{(x-a)w(x)}{ax^3}dx - \int_t^b \frac{(b-x)w(x)}{bx^3}dx\right|dt. \tag{3.11}$$

Consider the function $p : [a,b] \to \mathbb{R}$ defined by:

$$p(t) = \int_a^t \frac{(x-a)w(x)}{ax^3}dx - \int_t^b \frac{(b-x)w(x)}{bx^3}dx,$$

so that $p'(t) = \frac{b-a}{ab}\frac{w(t)}{t^2} > 0$, that is, $p(t)$ is an increasing function on $[a,b]$ with $p(a_1) = 0$; hence in this case (3.10) and (3.11) reduce to

$$\frac{b(a_1-a)}{a_1(b-a)}\int_0^1 X_y^2(a,a_1)\left|G(w,a,a_1,y;X_y)\right|dy = \frac{a^2}{b-a}\int_a^b \frac{(b-x)w(x)}{x^3}dx - \int_a^{a_1} \frac{w(x)}{x}dx = \frac{a}{2} - \int_a^{\frac{2ab}{a+b}} \frac{w(x)}{x}dx. \tag{3.12}$$

$$\begin{aligned}
&\frac{a(b-a_1)}{a_1(b-a)}\int_0^1 X_y^2(b,a_1)\left|G(w,a_1,b,y;X_y)\right|dy \\
&= \frac{b^2}{b-a}\int_a^b \frac{(x-a)w(x)}{x^3}dx - \int_{a_1}^b \frac{w(x)}{x}dx \\
&= \frac{b}{2} - \int_{\frac{2ab}{a+b}}^b \frac{w(x)}{x}dx. \qquad (3.13)
\end{aligned}$$

A combination of (3.12), (3.13) and (3.8) yields the desired inequality (3.9). □

EXAMPLE 4. Let $f,w:\left[\frac{1}{2},\frac{2}{3}\right]\to[0,\infty)$ be two functions defined by $f(x)=\frac{2}{3}x^{\frac{3}{2}}$ and $w(x)=\left(\frac{7}{4}-\frac{1}{x}\right)^2$ so that $|f'(x)|=\sqrt{x}$. Then, obviously w is continuous and harmonically symmetric with respect to $\frac{4}{7}$ and $|f'(x)|=\sqrt{x}$ is harmonically quasi-convex on $\left[\frac{1}{2},\frac{2}{3}\right]$. By using the software Mathematica, we observe that

$$LHS \text{ of } (3.9) = 0.159613$$
$$RHS \text{ of } (3.9) = 0.46639$$

and the deviation between these values is 0.296777.

NOTE 1. It may be noted that

- Theorem 1 is a generalization of [10, Corollary 2.1].
- Corollary 2 is a generalization of [10, Corollary 2.1]
- Theorem 2 is a generalization of [10, Theorem 2.1].
- Theorems 3 and 4 are some generalizations in [14].
- For $w(x)\to\frac{g(x)}{\int_a^b\frac{g(x)}{x^2}dx}$ and g harmonically symmetric with respect to $\frac{2ab}{a+b}$ on $[a,b]$, then Corollary 3 reduces to [10, Theorem 2.1].

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