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ON THE FURTHER REFINEMENT OF SOME OPERATOR INEQUALITIES FOR POSITIVE LINEAR MAP

CHANGSEN YANG AND YU LI*

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Abstract. In this paper, some further improvements of operator inequalities are given at the base of Yang et al.'s recent work [Filomat 32:12 (2018), 4333–4340] and [Math. Slovaca 69 (2019), 919–930] for positive linear map. Besides, the corresponding multiple-term refinements for scalars and operators are shown as well.

1. Introduction

Throughout the paper, $\mathscr{B}(\mathscr{H})$ denotes the C^**-algebra* of all bounded linear operators on a complex Hilbert space $\mathscr{H}$. A operator $A \in \mathscr{B}(\mathscr{H})$ is called positive, if

$$\langle Ax, x\rangle \geqslant 0$$

for all $x \in \mathscr{H}$, and we write $A \geqslant 0$. The set of all positive operators on a complex Hilbert space $\mathscr{H}$ is denoted by $\mathscr{B}^+(\mathscr{H})$. Also, the set of all positive invertible operators on a complex Hilbert space $\mathscr{H}$ is denoted with $\mathscr{B}^{++}(\mathscr{H})$. If $A \in \mathscr{B}^{++}(\mathscr{H})$, in symbols $A > 0$. For the notations adopted in this paper, we defined $v-$weighted arithmetic mean, geometric mean for operator

$$A\nabla_v B = (1-v)A + vB, \quad A\sharp_v B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^v A^{\frac{1}{2}}$$

where $A, B \in \mathscr{B}^{++}(\mathscr{H})$, $v \in [0,1]$. If $v = \frac{1}{2}$, we write $A\nabla B$ and $A\sharp B$ for brevity, respectively.

A linear map Φ: $\mathscr{B}(\mathscr{H}) \to \mathscr{B}(\mathscr{K})$, where $\mathscr{H}$ and $\mathscr{K}$ are complex Hilbert space, is called positive(strictly positive) if $\Phi(A) \geqslant 0$ $(\Phi(A) > 0)$ whenever $A \geqslant 0$ $(A > 0)$ and Φ is said to be unital or normalized if $\Phi(I_{\mathscr{H}}) = I_{\mathscr{K}}$. The relative operator entropy of A and B, where $A, B \in \mathscr{B}(\mathscr{H})$, is defined as $S(A|B) = A^{\frac{1}{2}}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$. The Kantorovich constants is defined by $K(h,2) = \frac{(h+1)^2}{4h}$, $h > 0$.

It's well known to us all that the classical Young inequality for scalars says that if $a, b \geqslant 0$ and $v \in [0,1]$, then

$$a^v b^{1-v} \leqslant va + (1-v)b \tag{1.1}$$

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* Corresponding author.

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with equality if and only if $a=b$. Simple as it is, what the inequality (1.1) conveys to us is not only interesting in itself but also meaningful in operator theory. For example, refining this inequality has taken the attention of many researchers, where adding one or two even many positive terms or multiplying a coefficient which is greater than the number 1 to the left side of the inequality becomes possible.

Recently, Kórus in [8] gave a new refinement of inequality (1.1) in the form that if $a,b>0$, $t\in[0,1]$, then

$$ta+(1-t)b \geqslant (1+\phi(t)(\log a-\log b)^2)a^t b^{1-t}, \tag{1.2}$$

where the function $\phi(t)$ is defined by

$$\phi(t)=\begin{cases} \frac{t^2}{2}(\frac{1-t}{t})^{2t} & \text{if } t\in(0,1),\\ 0 & \text{if } t=0,1. \end{cases} \tag{1.3}$$

In order to facilitate the calculation, we extend the function $\phi(t)$ to a periodic function whose period is one. So the function $\phi(t)$ has the following basic properties:

i) $\phi(t\pm 1)=\phi(t)$ for any $t\in[0,1]$.

ii) $\phi(t)=\phi(1-t)$ for any $t\in[0,1]$.

In the same paper, the operator version of (1.2) was obtained as well: if $A,B\in\mathscr{B}^+(\mathscr{H})$, $t\in[0,1]$, $K=\sqrt{\phi(t)}A^{-1}S(A|B)$, then

$$A\sharp_t B+K^*(A\sharp_t B)K \leqslant A\nabla_t B, \tag{1.4}$$

where $\phi(t)$ is defined by (1.3).

Furthermore, Yang et al. in [13] first presented a refinement of (1.2) and obtained the operator form of it as follows:

$$pa+(1-p)b \geqslant r(\sqrt{a}-\sqrt{b})^2+\left(1+\frac{\phi(2p)}{4}\left(\log\frac{a}{b}\right)^2\right)a^p b^{1-p} \tag{1.5}$$

$$A\sharp_p B+G^*(A\sharp_p B)G \leqslant A\nabla_p B-2r(A\nabla B-A\sharp B), \tag{1.6}$$

for $a,b>0$, $A,B\in\mathscr{B}^+(\mathscr{H})$, $G=\frac{\sqrt{\phi(2p)}}{2}A^{-1}S(A|B)$, $p\in[0,1]$, where $r=\min\{p,1-p\}$ and $\phi(p)$ is the form of (1.3).

On the other hand, Lin in [9] showed that the reverse-type of AG-GM inequality for operator can be squared: for $0<mI\leqslant A,B\leqslant MI$ with $h=\frac{M}{m}$, then for any unital positive linear map Φ,

$$\Phi^2\left(\frac{A+B}{2}\right) \leqslant K^2(h,2)\Phi^2(A\sharp B), \tag{1.7}$$

and

$$\Phi^2\left(\frac{A+B}{2}\right) \leqslant K^2(h,2)(\Phi(A)\sharp\Phi(B))^2, \tag{1.8}$$

where $K(h,2)=\frac{(h+1)^2}{4h}$ for $h>0$.

The results of (1.7) and (1.8) has been refined or generalized by a considerable number of researchers in different forms. And one of the recent work was presented by Yang and Lu in [14], who gave a better result which is both a refinement and a generalization: for $0 < mI \leqslant A \leqslant m'I < M'I \leqslant B \leqslant MI$ and $p \geqslant 1$, then for any unital positive linear map Φ,

$$\Phi^{2p}(A\nabla_t B) \leqslant \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}\left(1+\phi(t)\log\frac{M'}{m'}\right)^2}\right)^{2p}\Phi^{2p}(A\sharp_t B), \tag{1.9}$$

and

$$\Phi^{2p}(A\nabla_t B) \leqslant \left(\frac{K(h,2)}{4^{\frac{1}{p}-1}\left(1+\phi(t)\log\frac{M'}{m'}\right)^2}\right)^{2p}\Big(\Phi(A)\sharp_t\Phi(B)\Big)^{2p}, \tag{1.10}$$

where $t\in[0,1]$, $h=\frac{M}{m}$ and $\phi(t)$ is the form of (1.3). The other was obtained by Yang et al. as well in [13]: let $0<mI\leqslant A,B\leqslant MI$, Φ be a unital positive linear map on $\mathscr{B}(\mathscr{H})$, $p\in[0,1]$, $s>0$, then

$$\Phi^s\Big(A\nabla_p B+Mm\Big(G(A^{-1}\sharp_p B^{-1})G^*+2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\Big)\Big)\leqslant\alpha^s\Phi^s(A\sharp_p B). \tag{1.11}$$

$$\Phi^s\Big(A\nabla_p B+Mm\Big(G(A^{-1}\sharp_p B^{-1})G^*+2r(A^{-1}\nabla B^{-1}-A^{-1}\sharp B^{-1})\Big)\Big)\leqslant\alpha^s\Big(\Phi(A)\sharp_p\Phi(B)\Big)^s, \tag{1.12}$$

where $r=\min\{p,1-p\}$, $\alpha=\max\left\{\frac{(M+m)^2}{4Mm},\ \frac{(M+m)^2}{4^{\frac{2}{s}}Mm}\right\}$, $G=\frac{\sqrt{\phi(2p)}}{2}A^{-1}S(A|B)$ and $\phi(t)$ is the form of (1.3).

Also, Lin in [10] established the following operator results

$$|\Phi(A^{-1})\Phi(A)+\Phi(A)\Phi(A^{-1})|\leqslant\frac{(M+m)^2}{2Mm}I, \tag{1.13}$$

and

$$\Phi(A^{-1})\Phi(A)+\Phi(A)\Phi(A^{-1})\leqslant\frac{(M+m)^2}{2Mm}I, \tag{1.14}$$

for $0<mI\leqslant A\leqslant MI$ and Φ is unital positive linear map.

Likewise, the recent research is showed by Yang et al., the reader can refer to [14] to get the detailed form.

Considering the above refinements and generalizations, now, we present some multiple-term refinements of (1.2), (1.4)–(1.6) and (1.9)–(1.14) in this paper.

2. Main results

In this section, we give a complete refinement of (1.2), (1.4)–(1.6) for scalars and operators. Moreover, the further improvement of (1.9)–(1.14) is also proved. In order to better connect with the former and the later of this article, we first give the case of (2.3) for $N=2$.

LEMMA 2.1. *For* $a,b>0$, $\tau\in[0,1]$ *and* $F(\tau)=\frac{\phi(4\tau)}{16}(\log\frac{a}{b})^2$ *with* $\phi(\tau)=\frac{\tau^2}{2}(\frac{1-\tau}{\tau})^{2\tau}$.

i) If $\tau\in[0,\frac{1}{2}]$, *then*

$$\tau a+(1-\tau)b\geqslant(1+F(\tau))a^{\tau}b^{1-\tau}+\tau(\sqrt{a}-\sqrt{b})^2+r_0(\sqrt[4]{ab}-\sqrt{b})^2, \tag{2.1}$$

where $r_0=\min\{2\tau,1-2\tau\}$.

ii) If $\tau\in[\frac{1}{2},1]$, *then*

$$\tau a+(1-\tau)b\geqslant(1+F(\tau))a^{\tau}b^{1-\tau}+(1-\tau)(\sqrt{a}-\sqrt{b})^2+r_0(\sqrt[4]{ab}-\sqrt{a})^2, \tag{2.2}$$

where $r_1=\min\{2\tau-1,2-2\tau\}$.

Proof. i) When $\tau\in[0,\frac{1}{4}]$, then $r_0=2\tau$. By simple calculation and (1.2), then

$$\begin{aligned}&\tau a+(1-\tau)b-\tau(\sqrt{a}-\sqrt{b})^2-2\tau(\sqrt[4]{ab}-\sqrt{b})^2\\ =&(1-4\tau)b+4\tau a^{\frac{1}{4}}b^{\frac{3}{4}}\\ \geqslant&\left[1+\phi(4\tau)\Big(\log(a^{\frac{1}{4}}b^{\frac{3}{4}})-\log b\Big)^2\right](a^{\frac{1}{4}}b^{\frac{3}{4}})^{4\tau}b^{1-4\tau}\\ =&\left[1+\frac{\phi(4\tau)}{16}\Big(\log\frac{a}{b}\Big)^2\right]a^{\tau}b^{1-\tau}\\ =&(1+F(\tau))a^{\tau}b^{1-\tau}.\end{aligned}$$

And when $\tau\in[\frac{1}{4},\frac{1}{2}]$, then $r_0=1-2\tau$. By the periodicity of $\phi(\tau)$, we have

$$\begin{aligned}&\tau a+(1-\tau)b-\tau(\sqrt{a}-\sqrt{b})^2-(1-2\tau)(\sqrt[4]{ab}-\sqrt{b})^2\\ =&(4\tau-1)\sqrt{ab}+(2-4\tau)a^{\frac{1}{4}}b^{\frac{3}{4}}\\ \geqslant&\left[1+\frac{\phi(4\tau-1)}{16}\Big(\log\frac{a}{b}\Big)^2\right]a^{\tau}b^{1-\tau}\\ =&\left[1+\frac{\phi(4\tau)}{16}\Big(\log\frac{a}{b}\Big)^2\right]a^{\tau}b^{1-\tau}\\ =&(1+F(\tau))a^{\tau}b^{1-\tau}.\end{aligned}$$

So (2.1) as desired.

ii) When $\tau\in[\frac{1}{2},\frac{3}{4}]$, then $r_1=2\tau-1$. By the periodicity of $\phi(\tau)$ and (1.2), we have

$$\begin{aligned}&\tau a+(1-\tau)b-(1-\tau)(\sqrt{a}-\sqrt{b})^2-(2\tau-1)(\sqrt[4]{ab}-\sqrt{a})^2\\ =&(3-4\tau)\sqrt{ab}+(4\tau-2)a^{\frac{3}{4}}b^{\frac{1}{4}}\\ \geqslant&\left[1+\frac{\phi(4\tau-2)}{16}\Big(\log\frac{a}{b}\Big)^2\right]a^{\tau}b^{1-\tau}\\ =&\left[1+\frac{\phi(4\tau)}{16}\Big(\log\frac{a}{b}\Big)^2\right]a^{\tau}b^{1-\tau}\\ =&(1+F(\tau))a^{\tau}b^{1-\tau}.\end{aligned}$$

Similarly, when $\tau \in [\frac{3}{4}, 1]$, then $r_1 = 2 - 2\tau$. So we have

$$
\begin{aligned}
& \tau a + (1-\tau)b - (1-\tau)(\sqrt{a} - \sqrt{b})^2 - (2-2\tau)(\sqrt[4]{ab} - \sqrt{a})^2 \\
= & (4\tau - 3)a + (4 - 4\tau)a^{\frac{3}{4}}b^{\frac{1}{4}} \\
\geqslant & \left[1 + \frac{\phi(4-4\tau)}{16}\left(\log\frac{b}{a}\right)^2\right]a^\tau b^{1-\tau} \\
= & \left[1 + \frac{\phi(4\tau)}{16}\left(\log\frac{a}{b}\right)^2\right]a^\tau b^{1-\tau} \\
= & (1 + F(\tau))a^\tau b^{1-\tau}.
\end{aligned}
$$

So (2.2) as desired.

This completes the proof. □

It's necessary for us to recall the lemma below for the sake of generalizing (1.5) and the result of Lemma 2.1 into the general form.

LEMMA 2.2. ([12]) *Let $a, b > 0$ and $\tau \in [0,1]$. Given $N \in \mathbb{N}$, consider the integers $k_m(\tau) = [2^{m-1}\tau]$ and $r_m(\tau) = [2^m\tau]$, $m = 1, 2, \ldots, N$. Then*

$$
\begin{aligned}
& \tau a + (1-\tau)b - \sum_{m=1}^{N} s_m(\tau)\left(\sqrt[2^m]{b^{2^{m-1}-k_m(\tau)}a^{k_m(\tau)}} - \sqrt[2^m]{a^{k_m(\tau)+1}b^{2^{m-1}-k_m(\tau)-1}}\right)^2, \\
= & ([2^N\tau] + 1 - 2^N\tau)\sqrt[2^N]{a^{[2^N\tau]}b^{2^N-[2^N\tau]}} + (2^N\tau - [2^N\tau])\sqrt[2^N]{a^{[2^N\tau]+1}b^{2^N-[2^N\tau]-1}},
\end{aligned}
$$

where $s_m(\tau) = (-1)^{r_m(\tau)}2^{m-1}\tau + (-1)^{r_m(\tau)+1}[\frac{r_m(\tau)+1}{2}]$.

Now, we give a complete refinement of (1.2) and (1.5) for any $N \in \mathbb{N}$.

LEMMA 2.3. *Let $a, b > 0$, $\tau \in [0,1]$ and $N \in \mathbb{N}$, then*

$$
\begin{aligned}
\tau a + (1-\tau)b \geqslant & \left[1 + \frac{\phi(2^N\tau)}{4^N}\left(\log\frac{a}{b}\right)^2\right]a^\tau b^{1-\tau} \\
& + \sum_{m=1}^{N} s_m(\tau)\left(\sqrt[2^m]{b^{2^{m-1}-k_m(\tau)}a^{k_m(\tau)}} - \sqrt[2^m]{a^{k_m(\tau)+1}b^{2^{m-1}-k_m(\tau)-1}}\right)^2,
\end{aligned}
\tag{2.3}
$$

where $s_m(\tau) = (-1)^{r_m(\tau)}2^{m-1}\tau + (-1)^{r_m(\tau)+1}[\frac{r_m(\tau)+1}{2}]$, $r_m(\tau) = [2^m\tau]$, $k_m(\tau) = [2^{m-1}\tau]$, $m = 1, 2, \ldots, N$. $[x]$ is the greatest integer less than or equal to x and $\phi(t)$ is defined by (1.3).

Proof. By the periodicity of $\phi(t)$, Lemma 2.2 and (1.2), then it follows that

$$
\begin{aligned}
&\tau a+(1-\tau)b-\sum_{m=1}^{N} s_m(\tau)\left(\sqrt[2^m]{b^{2^{m-1}-k_m(\tau)}a^{k_m(\tau)}}-\sqrt[2^m]{a^{k_m(\tau)+1}b^{2^{m-1}-k_m(\tau)-1}}\right)^2\\
&=(2^N\tau-[2^N\tau])\sqrt[2^N]{a^{[2^N\tau]+1}b^{2^N-[2^N\tau]-1}}+([2^N\tau]+1-2^N\tau)\sqrt[2^N]{a^{[2^N\tau]}b^{2^N-[2^N\tau]}}\\
&\geqslant\left[1+\phi(2^N\tau-[2^N\tau])\left(\log\left(a^{\frac{[2^N\tau]+1}{2^N}}b^{\frac{2^N-[2^N\tau]-1}{2^N}}\right)-\log\left(a^{\frac{[2^N\tau]}{2^N}}b^{\frac{2^N-[2^N\tau]}{2^N}}\right)\right)^2\right]\\
&\quad\times\left(\sqrt[2^N]{a^{[2^N\tau]+1}b^{2^N-[2^N\tau]-1}}\right)^{2^N\tau-[2^N\tau]}\left(\sqrt[2^N]{a^{[2^N\tau]}b^{2^N-[2^N\tau]}}\right)^{[2^N\tau]+1-2^N\tau}\\
&=\left[1+\frac{\phi(2^N\tau)}{4^N}\left(\log\frac{a}{b}\right)^2\right]a^\tau b^{1-\tau}.
\end{aligned}
$$

So (2.3) holds. This completes the proof. □

REMARK 2.4. For one thing, it's obvious that (1.2) and (1.5) are two special cases of the inequality (2.3) if $N=0$ and $N=1$, respectively, which implies that (2.3) is a generalization of those in the literature. For another, the inequality (2.3) becomes an equality as $N\to\infty$, which indicates that (2.3) is a complete refinement of (1.2) and (1.5).

LEMMA 2.5. ([4]) *For $X\in\mathscr{B}(\mathscr{H})$ be self-adjoint and f, g be continuous real functions such that $f(t)\leqslant g(t)$ for all $t\in Sp(X)$ (the Spectrum of X). Then $f(X)\leqslant g(X)$.*

Next, the operator version of (2.3) are gained as well.

THEOREM 2.6. *Assume that $A,B\in\mathscr{B}^{++}(\mathscr{H})$, $\tau\in[0,1]$ and*

$$J=\frac{\sqrt{\phi(2^N\tau)}}{2^N}A^{-1}S(A|B).$$

Then

$$
\begin{aligned}
&A\sharp_\tau B+J^*(A\sharp_\tau B)J\\
&\leqslant A\nabla_\tau B-\sum_{m=1}^{N} s_m(\tau)\left[A\sharp_{\alpha_m(\tau)}B+A\sharp_{2^{1-m}+\alpha_m(\tau)}B-2(A\sharp_{2^{-m}+\alpha_m(\tau)}B)\right],
\end{aligned}\tag{2.4}
$$

where $s_m(\tau)=(-1)^{r_m(\tau)}2^{m-1}\tau+(-1)^{r_m(\tau)+1}[\frac{r_m(\tau)+1}{2}]$, $r_m(\tau)=[2^m\tau]$, $k_m(\tau)=[2^{m-1}\tau]$, $m=1,2,\ldots,N$. $[x]$ is the greatest integer less than or equal to x, $\alpha_m(\tau)=\frac{k_m(\tau)}{2^{m-1}}$ and $\phi(t)$ is defined by (1.3).

Proof. From (2.3), we get

$$\tau t+1-\tau\geqslant t^\tau+\frac{\phi(2^N\tau)}{4^N}(\log t)t^\tau(\log t)+\sum_{m=1}^{N} s_m(\tau)\left(t^{\frac{k_m(\tau)}{2^{m-1}}}+t^{\frac{k_m(\tau)+1}{2^{m-1}}}-2t^{\frac{2k_m(\tau)+1}{2^m}}\right),$$

for any $t > 0$.

For the operator $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ has a positive spectrum, I be the identity operator, then it can be deduced from Lemma 2.5 and the above inequality that

$$\begin{aligned}
&\tau A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + (1-\tau)I \\
&\geqslant (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\tau} + \frac{\sqrt{\phi(2^N\tau)}}{2^N}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\tau}\frac{\sqrt{\phi(2^N\tau)}}{2^N}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) \\
&\quad + \sum_{m=1}^{N} s_m(\tau)\left[(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{k_m(\tau)}{2^{m-1}}} + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{k_m(\tau)+1}{2^{m-1}}} - 2(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{2k_m(\tau)+1}{2^m}}\right].
\end{aligned} \tag{2.5}$$

Finally, by multiplying $A^{\frac{1}{2}}$ on both sides of the inequality (2.5), then it follows that

$$\begin{aligned}
A\nabla_\tau B \geqslant A\sharp_\tau B &+ \frac{\sqrt{\phi(2^N\tau)}}{2^N}S(A|B)A^{-1}(A\sharp_\tau B)\frac{\sqrt{\phi(2^N\tau)}}{2^N}A^{-1}S(A|B) \\
&+ \sum_{m=1}^{N} s_m(\tau)\left[A\sharp_{\frac{k_m(\tau)}{2^{m-1}}}B + A\sharp_{\frac{k_m(\tau)+1}{2^{m-1}}}B - 2(A\sharp_{\frac{2k_m(\tau)+1}{2^m}}B)\right],
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&A\nabla_\tau B - \sum_{m=1}^{N} s_m(\tau)\left[A\sharp_{\alpha_m(\tau)}B + A\sharp_{2^{1-m}+\alpha_m(\tau)}B - 2(A\sharp_{2^{-m}+\alpha_m(\tau)}B)\right] \\
&\geqslant A\sharp_\tau B + J^*(A\sharp_\tau B)J.
\end{aligned}$$

So (2.4) holds. Here the proof is completed. □

REMARK 2.7. Clearly, the result (2.4) is not only a generalization but also a complete refinement of (1.4) and (1.6).

In order to get further results for operator, it's imperative for us to recall the following lemma.

LEMMA 2.8. ([15]) *If A, B are positive operators on a Hilbert space and $\tau, \omega \in [0,1]$, then*

$$A\nabla_\tau(A\sharp_\omega B) = A\nabla_{\tau\omega}B - \tau(A\nabla_\omega B - A\sharp_\omega B).$$

THEOREM 2.9. *Suppose that $A, B \in \mathscr{B}^{++}(\mathscr{H})$, $\tau \in [0,1]$ and*

$$K = \frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}A^{-1}S(A|B).$$

i) If $\tau \in [0, \frac{1}{2}]$, then

$$\begin{aligned}
&A\sharp_\tau B + K^*(A\sharp_\tau B)K + \sum_{m=1}^{N} s_m(2\tau)\left[A\sharp_{\beta_m(\tau)}B + A\sharp_{2^{-m}+\beta_m(\tau)}B - 2(A\sharp_{2^{-m-1}+\beta_m(\tau)}B)\right] \\
&\leqslant A\nabla_\tau B - 2r(A\nabla B - A\sharp B).
\end{aligned} \tag{2.6}$$

ii) If $\tau \in [\frac{1}{2}, 1]$, *then*

$$A\sharp_\tau B + K^*(A\sharp_\tau B)K + \sum_{m=1}^{N} s_m(2-2\tau)\Big[A\sharp_{\gamma_m(\tau)}B + A\sharp_{\gamma_m(\tau)-2^{-m}}B - 2(A\sharp_{\gamma_m(\tau)-2^{-m-1}}B)\Big]$$
$$\leqslant A\nabla_\tau B - 2r(A\nabla B - A\sharp B) \tag{2.7}$$

where $\beta_m(\tau) = 2^{-m}k_m(2\tau),\ \gamma_m(\tau) = 1-\beta_m(1-\tau),\ r = \min\{\tau, 1-\tau\}$.

Proof. If $\tau \in [0, \frac{1}{2}]$, then $2\tau \in [0,1]$ and $r = \tau$. By substituting B by $A\sharp B$ and τ by 2τ in (2.4), then it becomes that

$$A\sharp_{2\tau}(A\sharp B) + \frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}S(A|B)A^{-1}(A\sharp_{2\tau}(A\sharp B))\frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}A^{-1}S(A|B)$$
$$\leqslant A\nabla_{2\tau}(A\sharp B) - \sum_{m=1}^{N} s_m(2\tau)\Big[A\sharp_{\frac{k_m(2\tau)}{2^{m-1}}}(A\sharp B) + A\sharp_{\frac{k_m(2\tau)+1}{2^{m-1}}}(A\sharp B) - 2(A\sharp_{\frac{2k_m(2\tau)+1}{2^m}}(A\sharp B))\Big].$$

On account of Lemma 2.8, we have

$$A\sharp_\tau B + K^*(A\sharp_\tau B)K + \sum_{m=1}^{N} s_m(2\tau)\Big[A\sharp_{\frac{k_m(2\tau)}{2^m}}B + A\sharp_{\frac{k_m(2\tau)+1}{2^m}}B - 2(A\sharp_{\frac{2k_m(2\tau)+1}{2^{m+1}}}B)\Big]$$
$$\leqslant A\nabla_\tau B - 2\tau(A\nabla B - A\sharp B),$$

which is equivalent to

$$A\sharp_\tau B + K^*(A\sharp_\tau B)K + \sum_{m=1}^{N} s_m(2\tau)\Big[A\sharp_{\beta_m(\tau)}B + A\sharp_{2^{-m}+\beta_m(\tau)}B - 2(A\sharp_{2^{-m-1}+\beta_m(\tau)}B)\Big]$$
$$\leqslant A\nabla_\tau B - 2r(A\nabla B - A\sharp B).$$

So (2.6) holds.

ii) If $\tau \in [\frac{1}{2}, 1]$, then $1-\tau \in [0, \frac{1}{2}]$ and $r = 1-\tau$. By interchanging A for B and replacing τ with $1-\tau$ in the above inequality, then we have

$$B\sharp_{1-\tau}A + \hat{K}^*(B\sharp_{1-\tau}A)\hat{K} + \sum_{m=1}^{N} s_m(2-2\tau)\Big[B\sharp_{\beta_m(1-\tau)}A + B\sharp_{2^{-m}+\beta_m(1-\tau)}A$$
$$- 2(B\sharp_{2^{-m-1}+\beta_m(1-\tau)}A)\Big]$$
$$\leqslant B\nabla_{1-\tau}A - 2r(B\nabla A - B\sharp A),$$

where $\hat{K} = \frac{\sqrt{\phi(2^{N+1}(1-\tau))}}{2^{N+1}}B^{-1}S(B|A)$. By the properties of the function $\phi(t)$, we get $\phi(2^{N+1}(1-\tau)) = \phi(2^{N+1}\tau + 1 - 2^{N+1}) = \phi(2^{N+1}\tau)$. And by simple calculation, it can be deduced that

$$\begin{aligned}\hat{K} &= \frac{\sqrt{\phi(2^{N+1}(1-\tau))}}{2^{N+1}}B^{-1}S(B|A) = \frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}B^{-\frac{1}{2}}\log(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})B^{\frac{1}{2}}\\ &= \frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}\log(B^{-1}A) = -\frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}\log(A^{-1}B)\\ &= -\frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}A^{-1}A^{\frac{1}{2}}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}\\ &= -\frac{\sqrt{\phi(2^{N+1}\tau)}}{2^{N+1}}A^{-1}S(A|B) = -K.\end{aligned}$$

And then, it is shown in [4] that $B\sharp_{1-\tau}A = A\sharp_\tau B$. So the above inequality becomes

$$\begin{aligned}&A\sharp_\tau B + K^*(A\sharp_\tau B)K + \sum_{m=1}^{N} s_m(2-2\tau)\Big[A\sharp_{1-\beta_m(1-\tau)}B + A\sharp_{1-2^{-m}-\beta_m(1-\tau)}B\\ &\quad -2(A\sharp_{1-2^{-m-1}-\beta_m(1-\tau)}B)\Big]\\ &\leqslant A\nabla_\tau B - 2r(A\nabla B - A\sharp B),\end{aligned}$$

which implies the desired result (2.7). Here the proof is completed. □

Now we are at the position to state our main results for positive linear map. Before these, some lemmas are needed.

LEMMA 2.10. *i)* ([6]) *Let* Φ *be a unital positive linear map,* $\omega \in [0,1]$ *and* A,B *be two positive operator, then*

$$\Phi(A\sharp_\omega B) \leqslant \Phi(A)\sharp_\omega \Phi(B). \tag{2.8}$$

ii) ([4]) (Choi inequality) *For* Φ *be a unital positive linear map and* $A>0$, *then*

$$\Phi^{-1}(A) \leqslant \Phi(A^{-1}). \tag{2.9}$$

LEMMA 2.11. *i)* ([2]) *Let* $A,B \geqslant 0$, *then the following norm inequality holds:*

$$||AB|| \leqslant \frac{1}{4}||A+B||^2. \tag{2.10}$$

ii) ([3]) *For* $A,B \geqslant 0$, *then for* $1 \leqslant k < +\infty$,

$$||A^k + B^k|| \leqslant ||(A+B)^k||. \tag{2.11}$$

LEMMA 2.12. ([4]) (L-H inequality) *If* $0 \leqslant A \leqslant B$ *and* $k \in [0,1]$, *then*

$$A^k \leqslant B^k.$$

THEOREM 2.13. *Suppose that* $A,B \in \mathscr{B}^+(\mathscr{H})$, $\tau \in [0,1]$ *and* z,Z *are constants with* $h = \frac{Z}{z}$ *such that* $0 < zI \leqslant A,B \leqslant ZI$. *If* $\lambda > 0$, *then for any positive unital linear map* Φ *on* $\mathscr{B}(\mathscr{H})$, *it holds*

i)

$$\Phi^{\lambda}\left(A\nabla_{\tau}B + zZ(J(A^{-1}\sharp_{\tau}B^{-1})J^{*} + S_N(A^{-1},B^{-1}))\right) \leqslant \zeta^{\lambda}\Phi^{\lambda}(A\sharp_{\tau}B), \tag{2.12}$$

ii)

$$\Phi^{\lambda}\left(A\nabla_{\tau}B + zZ(J(A^{-1}\sharp_{\tau}B^{-1})J^{*} + S_N(A^{-1},B^{-1}))\right) \leqslant \zeta^{\lambda}(\Phi(A)\sharp_{\tau}\Phi(B))^{\lambda}, \tag{2.13}$$

where $\zeta = \max\{K(h,2), \frac{K(h,2)}{4^{\frac{2}{\lambda}-1}}\}$, $K(.,2)$ *is Kantorovich constants,* $S_N(A^{-1},B^{-1}) = \sum_{m=1}^{N} s_m(\tau)\left[A^{-1}\sharp_{\alpha_m(\tau)}B^{-1} + A^{-1}\sharp_{2^{1-m}+\alpha_m(\tau)}B^{-1} - 2(A^{-1}\sharp_{2^{-m}+\alpha_m(\tau)}B^{-1})\right]$, $\alpha_m(\tau) = \frac{k_m(\tau)}{2^{m-1}}$, $J = \frac{\sqrt{\phi(2^N\tau)}}{2^N}A^{-1}S(A|B)$ *and* $\phi(t)$ *is defined by* (1.3).

Proof. Under the condition $0 < zI \leqslant A,B \leqslant ZI$, we have

$$(A - zI)(ZI - A)A^{-1} \geqslant 0, \quad (B - zI)(ZI - B)B^{-1} \geqslant 0,$$

which imply that

$$(1-\tau)A + zZ(1-\tau)A^{-1} \leqslant (1-\tau)(z+Z)I, \quad \tau B + zZ\tau B^{-1} \leqslant \tau(z+Z)I,$$

for $\tau \in [0,1]$.

By summing up the two operator inequalities above, one can have

$$A\nabla_{\tau}B + zZ(A^{-1}\nabla_{\tau}B^{-1}) \leqslant z + Z$$

Then for any positive unital linear map Φ on $\mathscr{B}(\mathscr{H})$, it follows from the properties of positive unital linear map and the above inequality that

$$\Phi(A\nabla_{\tau}B) + zZ\Phi(A^{-1}\nabla_{\tau}B^{-1}) \leqslant z + Z. \tag{2.14}$$

And then, by direct calculation, we have

$$\begin{aligned}
\frac{\sqrt{\phi(2^N\tau)}}{2^N}AS(A^{-1}|B^{-1}) &= \frac{\sqrt{\phi(2^N\tau)}}{2^N}A^{\frac{1}{2}}\log(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})A^{-\frac{1}{2}} \\
&= \frac{\sqrt{\phi(2^N\tau)}}{2^N}\log(AB^{-1}) = -\frac{\sqrt{\phi(2^N\tau)}}{2^N}\log(BA^{-1}) \\
&= -\frac{\sqrt{\phi(2^N\tau)}}{2^N}A^{\frac{1}{2}}\log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}A^{-1} \\
&= -\frac{\sqrt{\phi(2^N\tau)}}{2^N}S(A|B)A^{-1} = -J^{*}.
\end{aligned}$$

Hence on account of (2.4), we have

$$A^{-1}\sharp_\tau B^{-1} + J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}) \leqslant A^{-1}\nabla_\tau B^{-1}, \tag{2.15}$$

where $\sum_{m=1}^{N} s_m(\tau)\Big[A^{-1}\sharp_{\alpha_m(\tau)}B^{-1} + A^{-1}\sharp_{2^{1-m}+\alpha_m(\tau)}B^{-1} - 2(A^{-1}\sharp_{2^{-m}+\alpha_m(\tau)}B^{-1})\Big]$ is denoted by $S_N(A^{-1},B^{-1})$.

i) If $0<\lambda\leqslant 2$, now, by taking (2.14) and (2.15) into consideration together, one can have

$$\begin{aligned}
&||\Phi\Big(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\Big)zZ\Phi^{-1}(A\sharp_\tau B)||\\
&\leqslant \frac{1}{4}||\Phi\Big(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\Big) + zZ\Phi^{-1}(A\sharp_\tau B)||^2\\
&\leqslant \frac{1}{4}||\Phi\Big(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\Big) + zZ\Phi((A\sharp_\tau B)^{-1})||^2\\
&= \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\sharp_\tau B^{-1} + J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))||^2\\
&\leqslant \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\nabla_\tau B^{-1})||^2\\
&\leqslant \frac{(z+Z)^2}{4},
\end{aligned}$$

where the first inequality is by (2.10), the second one is due to (2.9), the third one is by (2.15), and the last one follows from (2.14). Namely,

$$||\Phi\Big(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\Big)\Phi^{-1}(A\sharp_\tau B)|| \leqslant \frac{(z+Z)^2}{4zZ}.$$

Therefore,

$$\Phi^2\Big(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\Big) \leqslant \left(\frac{(z+Z)^2}{4zZ}\right)^2 \Phi^2(A\sharp_\tau B).$$

Since $0<\lambda\leqslant 2$, so $0<\frac{\lambda}{2}\leqslant 1$. Then the above inequality can be deduced from Lemma 2.12 (L-H inequality) that

$$\begin{aligned}
&\Phi^\lambda\Big(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\Big)\\
&\leqslant \left(\frac{(z+Z)^2}{4zZ}\right)^\lambda \Phi^\lambda(A\sharp_\tau B) = (K(h,2))^\lambda \Phi^\lambda(A\sharp_\tau B)
\end{aligned}$$

for any $0<\lambda\leqslant 2$.

If $\lambda \geqslant 2$, by simple calculation, one can have

$$\begin{aligned}
&||\Phi^{\frac{\lambda}{2}}(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1})))(zZ)^{\frac{\lambda}{2}}\Phi^{-\frac{\lambda}{2}}(A\sharp_\tau B)|| \\
&\leqslant \frac{1}{4}||\Phi^{\frac{\lambda}{2}}\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + (zZ)^{\frac{\lambda}{2}}\Phi^{-\frac{\lambda}{2}}(A\sharp_\tau B)||^2 \\
&\leqslant \frac{1}{4}||\left(\Phi(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))) + zZ\Phi^{-1}(A\sharp_\tau B)\right)^{\frac{\lambda}{2}}||^2 \\
&\leqslant \frac{1}{4}||\Phi\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + zZ\Phi((A\sharp_\tau B)^{-1})||^\lambda \\
&= \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\sharp_\tau B^{-1} + J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))||^\lambda \\
&\leqslant \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\nabla_\tau B^{-1})||^\lambda \\
&\leqslant \frac{(z+Z)^\lambda}{4},
\end{aligned}$$

where the first inequality is by (2.10), the second one is deduced by (2.11), the third one is due to (2.9), the forth one is by (2.15), and the last one follows from (2.14). That is,

$$||\Phi^{\frac{\lambda}{2}}(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1})))\Phi^{-\frac{\lambda}{2}}(A\sharp_\tau B)|| \leqslant \frac{(z+Z)^\lambda}{4(zZ)^{\frac{\lambda}{2}}}$$

Therefore, for any $\lambda \geqslant 2$, we have

$$\begin{aligned}
&\Phi^\lambda(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))) \\
&\leqslant 4^{\lambda-2}\left(\frac{(z+Z)^2}{4zZ}\right)^\lambda \Phi^\lambda(A\sharp_\tau B) = \left(\frac{K(h,2)}{4^{\frac{2}{\lambda}-1}}\right)^\lambda \Phi^\lambda(A\sharp_\tau B).
\end{aligned}$$

In summary, we can come to the conclusion that (2.12) holds for any $\lambda > 0$.

ii) If $0 < \lambda \leqslant 2$, utilizing the same method presented in i), we have

$$\begin{aligned}
&||\Phi(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1})))zZ(\Phi(A)\sharp_\tau\Phi(B))^{-1}|| \\
&\leqslant \frac{1}{4}||\Phi\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + zZ(\Phi(A)\sharp_\tau\Phi(B))^{-1}||^2 \\
&\leqslant \frac{1}{4}||\Phi\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + zZ\Phi^{-1}(A\sharp_\tau B)||^2 \\
&\leqslant \frac{1}{4}||\Phi\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + zZ\Phi(A^{-1}\sharp_\tau B^{-1})||^2 \\
&= \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\sharp_\tau B^{-1} + J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))||^2 \\
&\leqslant \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\nabla_\tau B^{-1})||^2 \\
&\leqslant \frac{(z+Z)^2}{4},
\end{aligned}$$

where the first inequality is by (2.10), the second one is due to (2.8), the third one is deduced by (2.9), the forth one is by (2.15), and the last one follows from (2.14). So by Lemma 2.12, for any $0 < \lambda \leqslant 2$, we have

$$\Phi^{\lambda}(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))) \leqslant (K(h,2))^{\lambda}\left(\Phi(A)\sharp_\tau\Phi(B)\right)^{\lambda}.$$

If $\lambda \geqslant 2$, by direct calculation, we have

$$\begin{aligned}
&||\Phi^{\frac{\lambda}{2}}(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1})))(zZ)^{\frac{\lambda}{2}}\left(\Phi(A)\sharp_\tau\Phi(B)\right)^{-\frac{\lambda}{2}}|| \\
&\leqslant \frac{1}{4}||\Phi^{\frac{\lambda}{2}}\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + (zZ)^{\frac{\lambda}{2}}\left(\Phi(A)\sharp_\tau\Phi(B)\right)^{-\frac{\lambda}{2}}||^2 \\
&\leqslant \frac{1}{4}||\left(\Phi(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))) + zZ\left(\Phi(A)\sharp_\tau\Phi(B)\right)^{-1}\right)^{\frac{\lambda}{2}}||^2 \\
&\leqslant \frac{1}{4}||\Phi\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + zZ\Phi^{-1}(A\sharp_\tau B)||^{\lambda} \\
&\leqslant \frac{1}{4}||\Phi\left(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))\right) + zZ\Phi((A\sharp_\tau B)^{-1})||^{\lambda} \\
&= \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\sharp_\tau B^{-1} + J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))||^{\lambda} \\
&\leqslant \frac{1}{4}||\Phi(A\nabla_\tau B) + zZ\Phi(A^{-1}\nabla_\tau B^{-1})||^{\lambda} \\
&\leqslant \frac{(z+Z)^{\lambda}}{4},
\end{aligned}$$

where the first inequality is due to (2.10), the second one is by (2.11), the third one is deduced by (2.8), the forth one is by (2.9), the fifth one and the last one are by (2.15) and (2.14), respectively. Thus for $\lambda \geqslant 2$, we have

$$\Phi^{\lambda}(A\nabla_\tau B + zZ(J(A^{-1}\sharp_\tau B^{-1})J^* + S_N(A^{-1},B^{-1}))) \leqslant \left(\frac{K(h,2)}{4^{\frac{2}{\lambda}-1}}\right)^{\lambda}\left(\Phi(A)\sharp_\tau\Phi(B)\right)^{\lambda}.$$

So (2.13) holds for any $\lambda > 0$.

Here the proof is completed. □

REMARK 2.14. On the one hand, (2.12) and (2.13) are better than (1.7) and (1.8) if $N = 0$, $\tau = \frac{1}{2}$, $\lambda = 2$. On the other hand, (2.12) reduces to (1.11) and (2.13) gives to (1.12) when $N = 1$. And when $N > 1$, they are multiple-term refinements of those any in the literature.

LEMMA 2.15. *Let $A,B \in \mathscr{B}^+(\mathscr{H})$, $\tau \in [0,1]$ and z,Z are constants such that $0 < zI \leqslant B \leqslant z'I < Z'I \leqslant A \leqslant ZI$, then*

$$A\nabla_\tau B + zZS_N(A^{-1},B^{-1}) + zZ\left[1 + \frac{\phi(2^N\tau)}{4^N}(\log h')^2\right](A\sharp_\tau B)^{-1} \leqslant (z+Z)I, \quad (2.16)$$

where $h=\frac{Z}{z}$, $h'=\frac{Z'}{z'}$, $S_N(A^{-1},B^{-1})$ *are as in Theorem* 2.13*,* $N\in\mathbb{N}$ *and* $\phi(t)$ *is defined by* (1.3)*.*

Proof. From the proof of Theorem 2.13, we have

$$A\nabla_\tau B+zZ(A^{-1}\nabla_\tau B^{-1})\leqslant z+Z. \tag{2.17}$$

By (2.3), we get

$$\tau t+1-\tau\geqslant\Big[1+\frac{\phi(2^N\tau)}{4^N}(\log t)^2\Big]t^\tau+\sum_{m=1}^N s_m(\tau)\Big(t^{\frac{k_m(\tau)}{2^{m-1}}}+t^{\frac{k_m(\tau)+1}{2^{m-1}}}-2t^{\frac{2k_m(\tau)+1}{2^m}}\Big),$$

for any $t>0$.

For $X\in\mathscr{B}^+(\mathscr{H})$ such that $0<\alpha I\leqslant X\leqslant\beta I$. Then it can be deduced from Lemma 2.5 that

$$\begin{aligned}\tau X+(1-\tau)I\geqslant &\min_{\alpha\leqslant t\leqslant\beta}\Big[1+\frac{\phi(2^N\tau)}{4^N}(\log t)^2\Big]X^\tau\\ &+\sum_{m=1}^N s_m(\tau)\Big(X^{\frac{k_m(\tau)}{2^{m-1}}}+X^{\frac{k_m(\tau)+1}{2^{m-1}}}-2X^{\frac{2k_m(\tau)+1}{2^m}}\Big).\end{aligned}$$

By the condition $0<zI\leqslant B\leqslant z'I<Z'I\leqslant A\leqslant ZI$, we have $I<h'I=\frac{Z'}{z'}I\leqslant A^{\frac12}B^{-1}A^{\frac12}\leqslant\frac{Z}{z}I=hI$. Here we put $X=A^{\frac12}B^{-1}A^{\frac12}$, then the above inequality becomes that

$$\begin{aligned}&\tau(A^{\frac12}B^{-1}A^{\frac12})+(1-\tau)I\geqslant\min_{h'\leqslant t\leqslant h}\Big[1+\frac{\phi(2^N\tau)}{4^N}(\log t)^2\Big](A^{\frac12}B^{-1}A^{\frac12})^\tau\\ &+\sum_{m=1}^N s_m(\tau)\Big[(A^{\frac12}B^{-1}A^{\frac12})^{\frac{k_m(\tau)}{2^{m-1}}}+(A^{\frac12}B^{-1}A^{\frac12})^{\frac{k_m(\tau)+1}{2^{m-1}}}-2(A^{\frac12}B^{-1}A^{\frac12})^{\frac{2k_m(\tau)+1}{2^m}}\Big].\end{aligned}$$

Now, by applying the monotonicity of logarithmic function and multiplying $A^{-\frac12}$ on both sides of the operator inequality nearby, then it can be deduced that

$$A^{-1}\nabla_\tau B^{-1}\geqslant\Big[1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\Big](A^{-1}\sharp_\tau B^{-1})+S_N(A^{-1},B^{-1}), \tag{2.18}$$

where $\sum_{m=1}^N s_m(\tau)\Big[A^{-1}\sharp_{\alpha_m(\tau)}B^{-1}+A^{-1}\sharp_{2^{1-m}+\alpha_m(\tau)}B^{-1}-2(A^{-1}\sharp_{2^{-m}+\alpha_m(\tau)}B^{-1})\Big]$ is denoted by $S_N(A^{-1},B^{-1})$ and $\alpha_m=\frac{k_m(\tau)}{2^{m-1}}$.

Finally, by simple calculation, we have

$$\begin{aligned}&A\nabla_\tau B+zZS_N(A^{-1},B^{-1})+zZ\left[1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\right](A\sharp_\tau B)^{-1}\\ &=A\nabla_\tau B+zZ\Big(\Big(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\Big)(A^{-1}\sharp_\tau B^{-1})+S_N(A^{-1},B^{-1})\Big)\\ &\leqslant A\nabla_\tau B+zZ(A^{-1}\nabla_\tau B^{-1}) \qquad \text{by } (2.18)\\ &\leqslant (z+Z)I. \qquad \text{by } (2.17)\end{aligned}$$

This completes the proof. □

Applying Lemma 2.15 and the same method presented by Theorem 2.13, we obtain the following theorem.

THEOREM 2.16. *Assume that* $A,B\in\mathscr{B}^+(\mathscr{H})$, $\tau\in[0,1]$, Φ *be a positive unital linear map on* $\mathscr{B}(\mathscr{H})$ *and positive real numbers* z,z',Z,Z' *satisfy* $0<zI\leqslant B\leqslant z'I<Z'I\leqslant A\leqslant ZI$ *with* $h=\frac{Z}{z}$, $h'=\frac{Z'}{z'}$. *If* $s>0$, *then*

i)

$$\Phi^s\left(A\nabla_\tau B+zZS_N(A^{-1},B^{-1})\right)\leqslant\xi^s\Phi^s(A\sharp_\tau B), \tag{2.19}$$

ii)

$$\Phi^s\left(A\nabla_\tau B+zZS_N(A^{-1},B^{-1})\right)\leqslant\xi^s(\Phi(A)\sharp_\tau\Phi(B))^s, \tag{2.20}$$

where $\xi=\max\left\{\frac{K(h,2)}{1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2},\frac{K(h,2)}{4^{\frac{2}{s}-1}\left(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\right)}\right\}$, $N\in\mathbb{N}$, $K(.,2)$ *is Kantorovich constants,* $S_N(A^{-1},B^{-1})$ *are as in Theorem* 2.13 *and* $\phi(t)$ *is defined by* (1.3).

REMARK 2.17. Firstly, (1.9) and (1.10) are special cases of (2.19) and (2.20) if s is even with $s\geqslant 2$, $N=0$. Secondly, (2.19) and (2.20) are further refinements of the corresponding results if s is even with $s\geqslant 2$, $N\geqslant 1$.

LEMMA 2.18. ([5]) *For any bounded operator* X,

$$|X|\leqslant tI\Leftrightarrow||X||\leqslant t\Leftrightarrow\begin{bmatrix}tI & X\\ X^* & tI\end{bmatrix}\geqslant 0\quad(t\geqslant 0).$$

THEOREM 2.19. *For* $0<zI\leqslant B\leqslant z'I<Z'I\leqslant A\leqslant ZI$ *with* $h=\frac{Z}{z},h'=\frac{Z'}{z'}$ *and* $s\geqslant 1$, *then for any positive unital linear map* Φ, *it holds*

i)

$$\begin{aligned}&|\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))\Phi^s((A\sharp_\tau B)^{-1})\\&+\Phi^s((A\sharp_\tau B)^{-1})\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))|\\&\leqslant 2\left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^s I,\end{aligned} \tag{2.21}$$

ii)

$$\begin{aligned}&\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))\Phi^s((A\sharp_\tau B)^{-1})\\&+\Phi^s((A\sharp_\tau B)^{-1})\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))\\&\leqslant 2\left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^s I,\end{aligned} \tag{2.22}$$

where $S_N(A^{-1},B^{-1})$ are as in Theorem 2.13, $K(.,2)$ is Kantorovich constants, $N\in\mathbb{N}$ and $\phi(t)$ is defined by (1.3).

Proof. By simple calculation, we have

$$\begin{aligned}
&||\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))(zZ)^s\Big(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\Big)^s\Phi^s((A\sharp_\tau B)^{-1})||\\
&\leqslant\frac{1}{4}||\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))+(zZ)^s\Big(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\Big)^s\Phi^s((A\sharp_\tau B)^{-1})||^2\\
&\leqslant\frac{1}{4}||\Big(\Phi(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))+zZ\Big(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\Big)\Phi((A\sharp_\tau B)^{-1})\Big)^s||^2\\
&=\frac{1}{4}||\Phi(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))+zZ\Big(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2\Big)\Phi((A\sharp_\tau B)^{-1})||^{2s}\\
&\leqslant\frac{(z+Z)^{2s}}{4},
\end{aligned}$$

where the first inequality is by (2.10), the second one is due to (2.11), and the last one follows from (2.16). Therefore,

$$||\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))\Phi^s((A\sharp_\tau B)^{-1})||\leqslant\left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^s.$$

And then, it can be deduced from Lemma 2.18 that

$$\begin{bmatrix}\left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^sI & X_1\\ X_1^* & \left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^sI\end{bmatrix}\geqslant 0,$$

where $X_1=\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))\Phi^s((A\sharp_\tau B)^{-1})$.

Similarly, we also have

$$\begin{bmatrix}\left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^sI & X_2\\ X_2^* & \left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^sI\end{bmatrix}\geqslant 0,$$

where $X_2=\Phi^s((A\sharp_\tau B)^{-1})\Phi^s(A\nabla_\tau B+zZS_N(A^{-1},B^{-1}))$.

Let $X=X_1+X_2$, then it's clear that the operator X is self-adjoint. Summing up the two operator matrices above, we have

$$\begin{bmatrix}2\left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^sI & X\\ X^* & 2\left(\frac{K(h,2)}{4^{\frac{1}{s}-1}(1+\frac{\phi(2^N\tau)}{4^N}(\log h')^2)}\right)^sI\end{bmatrix}\geqslant 0.$$

By utilizing Lemma 2.18 again, finally, we get (2.21) and (2.22). □

REMARK 2.21. For one thing, (2.21) and (2.22) give to (2.5) and (2.6) in [14] when $N = 0$, respectively. For another, (2.21) and (2.22) are multiple-term refinements of them when $N \geqslant 1$.

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Changsen Yang
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007, China
e-mail: yangchangsen0991@sina.com

Yu Li
College of Mathematics and Information Science
Henan Normal University
Xinxiang, Henan, 453007, China
e-mail: mathyuli8102@163.com