STEFFENSEN–GRÜSS INEQUALITY

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Abstract. Two inequalities for the Jensen difference under Steffensen's conditions with Grüss type upper bounds are proved.

1. Introduction

Let $I \subset \mathbb{R}$ be an interval. It is well known that if a function $f : I \to \mathbb{R}$ is convex then

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

for all $n \in \mathbb{N}$, $x_1, \ldots, x_n \in I$ and $p_1, \ldots, p_n \geqslant 0$ such that $P_n = p_1 + \cdots + p_n > 0$. If f is strictly convex then (1.1) is strict unless all x_i are equal [7, p. 43]. This is the classical *Jensen inequality*, one of the most important inequalities in convex analysis, and it has various applications in mathematics, statistics, economics and engineering sciences.

It is also known that the assumption $p_1, \ldots, p_n \geqslant 0$ can be relaxed at the expense of restricting $x_1, \ldots, x_n$ more severely [8]. Namely, if $\boldsymbol{p} = (p_1, \ldots, p_n)$ is a real n-tuple such that

$$0 \leqslant P_k \leqslant P_n, \ k \in \{1, \ldots, n-1\} \text{ and } P_n > 0, \tag{1.2}$$

then for any monotonic n-tuple $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ (increasing or decreasing) we get

$$\overline{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \in I,$$

and for any function f convex on I inequality (1.1) still holds. Under such assumptions inequality (1.1) is called *the Jensen-Steffensen inequality* for convex functions and (1.2) are called *Steffensen's conditions* due to J. F. Steffensen. Again, for a strictly convex function f inequality (1.1) remains strict under certain additional assumptions on $\boldsymbol{x}$ and $\boldsymbol{p}$ [1].

Another important inequality in analysis is *the Grüss inequality*. It states that

$$\left| \frac{1}{b-a} \int_a^b f(s)g(s)\,ds - \frac{1}{b-a} \int_a^b f(s)\,ds \cdot \frac{1}{b-a} \int_a^b g(s)\,ds \right| \leqslant \frac{1}{4}(\Gamma - \gamma)(\Phi - \phi)$$

holds for integrable functions $f, g : [a, b] \to \mathbb{R}$ such that $\gamma \leqslant f(s) \leqslant \Gamma$ and $\phi \leqslant g(s) \leqslant \Phi$, for all $s \in [a, b]$, where $\gamma, \Gamma, \phi, \Phi \in \mathbb{R}$ [4, p. 296].

In [2, Theorem 1] Budimir and Pečarić proved an inequality which they named *the Jensen-Grüss inequality*: it gives an upper bound for the difference between the right hand side and the left hand side of the Jensen inequality in terms of the Grüss inequality. Their results were recently improved in [3]. The goal of this paper is to prove inequalities of the same type but now with weights satisfying Steffensen's conditions (1.2).

2. Main results

In this section we assume $n \in \mathbb{N} \setminus \{1\}$, and we denote $I = (\alpha, \beta) \subseteq \mathbb{R}$, $\alpha < \beta$,

$$P_k = p_1 + \cdots + p_k, \ \ \bar{P}_k = p_k + \cdots + p_n, \ k \in \{1, \ldots, n\}.$$

To prove our main result we need the following theorem (a modification of [6, Theorem 4]).

THEOREM 2.1. *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *and* $\boldsymbol{b} = (b_1, \ldots, b_n)$ *be two real n-tuples such that*

$$|a_{k+1} - a_k| \leqslant \delta, \ k \in \{1, \ldots, n-1\}.$$

Then for all real n-tuples $\boldsymbol{p} = (p_1, \ldots, p_n)$ *satisfying*

$$0 \leqslant P_k \leqslant P_n, \ k \in \{1, \ldots, n-1\},$$

the following inequalities hold

$$\left| \sum_{i=1}^n p_i \sum_{i=1}^n p_i a_i b_i - \sum_{i=1}^n p_i a_i \sum_{i=1}^n p_i b_i \right|$$

$$\leqslant (M-m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^n P_i \bar{P}_j \right) |a_{i+1} - a_i| \qquad (2.1)$$

$$\leqslant \delta (M-m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^n P_i \bar{P}_j \right),$$

where $m = \min \{b_1, \ldots, b_n\}$ *and* $M = \max \{b_1, \ldots, b_n\}$.

Proof. First note that since

$$0 \leqslant P_k = p_1 + \cdots + p_k \leqslant P_n = p_1 + \cdots + p_n, \ k \in \{1, \ldots, n\},$$

we know that

$$\bar{P}_k = p_k + \cdots + p_n \geqslant 0, \; k \in \{1, \ldots, n\}.$$

It can be easily proved (using summation by parts, sometimes called Abel's transformation) that for $k \in \{2, \ldots, n-1\}$

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{k-1} P_i (x_i - x_{i+1}) + P_k x_k + \bar{P}_{k+1} x_{k+1} + \sum_{i=k+2}^{n} \bar{P}_i (x_i - x_{i-1}), \qquad (2.2)$$

and in border cases $k = 1$ or $k = n$

$$\sum_{i=1}^{n} p_i x_i = \bar{P}_1 x_1 + \sum_{i=2}^{n} \bar{P}_i (x_i - x_{i-1})$$

$$\sum_{i=1}^{n} p_i x_i = P_n x_n - \sum_{i=1}^{n-1} P_i (x_{i+1} - x_i). \qquad (2.3)$$

In all of the abowe cases we assume

$$\sum_{i=k}^{l} x_i = 0, \text{ when } k > l.$$

The following identities hold (it could be checked directly)

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i$$

$$= \sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j b_i - \sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j b_j$$

$$= \sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j (b_i - b_j).$$

Using (2.3) with $x_i = a_i$ and weights $p_i \sum_{j=1}^{n} p_j (b_i - b_j)$ we get

$$\sum_{i=1}^{n} p_i a_i \sum_{j=1}^{n} p_j (b_i - b_j)$$

$$= a_n \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j (b_i - b_j) - \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j (b_k - b_j) \right) (a_{i+1} - a_i).$$

Since

$$\sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j (b_i - b_j) = \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j b_i - \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j b_j$$

$$= \sum_{i=1}^{n} p_i b_i \sum_{j=1}^{n} p_j - \sum_{i=1}^{n} p_i \sum_{j=1}^{n} p_j b_j = 0$$

we obtain

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i = \sum_{i=1}^{n-1} \left(\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k \right) \right) \left(a_{i+1} - a_i \right).$$

Using (2.3) with $x_i = \sum_{j=1}^{n} p_j \left(b_j - b_i \right)$ we obtain

$$\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k \right)$$

$$= P_i \sum_{j=1}^{n} p_j \left(b_j - b_i \right) - \sum_{k=1}^{i-1} P_k \left(\sum_{j=1}^{n} p_j \left(b_j - b_{k+1} \right) - \sum_{j=1}^{n} p_j \left(b_j - b_k \right) \right)$$

$$= P_i \left(\sum_{j=1}^{n} p_j b_j - P_n b_i \right) - P_n \sum_{k=1}^{i-1} P_k \left(b_k - b_{k+1} \right),$$

and next using (2.2) with $x_i = b_i$ we get

$$\sum_{k=1}^{i} p_k \sum_{j=1}^{n} p_j \left(b_j - b_k \right)$$

$$= P_i \left(\sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1} \right) + P_i b_i + \bar{P}_{i+1} b_{i+1} + \sum_{j=i+2}^{n} \bar{P}_j \left(b_j - b_{j-1} \right) - P_n b_i \right)$$

$$- P_n \sum_{k=1}^{i-1} P_k \left(b_k - b_{k+1} \right)$$

$$= P_i \left(\sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1} \right) - \bar{P}_{i+1} b_i + \bar{P}_{i+1} b_{i+1} + \sum_{j=i+2}^{n} \bar{P}_j \left(b_j - b_{j-1} \right) \right)$$

$$- P_n \sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1} \right)$$

$$= P_i \sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1} \right) + P_i \sum_{j=i+1}^{n} \bar{P}_j \left(b_j - b_{j-1} \right) - P_n \sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1} \right)$$

$$= P_i \sum_{j=i+1}^{n} \bar{P}_j \left(b_j - b_{j-1} \right) - \bar{P}_{i+1} \sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1} \right).$$

Hence

$$\sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i$$

$$= \sum_{i=1}^{n-1} \left(P_i \sum_{j=i+1}^{n} \bar{P}_j \left(b_j - b_{j-1} \right) - \bar{P}_{i+1} \sum_{j=1}^{i-1} P_j \left(b_j - b_{j+1} \right) \right) \left(a_{i+1} - a_i \right) \qquad (2.4)$$

$$= \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j \left(a_{i+1} - a_i \right) \left(b_{j+1} - b_j \right) + \sum_{j=i+1}^{n} P_i \bar{P}_j \left(a_{i+1} - a_i \right) \left(b_j - b_{j-1} \right) \right).$$

By the assumptions we have $|a_{i+1} - a_i| \leqslant \delta$ and $|b_{j+1} - b_j| \leqslant M - m$, and consequently

$$\left| \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right|$$

$$\leqslant (M-m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right) |a_{i+1} - a_i|$$

$$\leqslant \delta (M-m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right),$$

hence (2.1) is proved. $\square$

In [6, Theorem 4] the author considered some other conditions on weights, such as

$$0 \leqslant P_n \leqslant P_k, \ k \in \{1, \ldots, n-1\}$$

or

$$0 \leqslant P_n \leqslant \bar{P}_k, \ k \in \{2, \ldots, n\}.$$

It can be easily seen that if, for instance, the first assumption holds we get

$$\bar{P}_k \leqslant 0, \ k \in \{2, \ldots, n\},$$

and if the second holds we get

$$P_k \leqslant 0, \ k \in \{1, \ldots, n-1\},$$

so in all cases the products $\bar{P}_{i+1} P_j$ and $P_i \bar{P}_j$ in

$$\sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j (a_{i+1} - a_i)(b_{j+1} - b_j) + \sum_{j=i+1}^{n} P_i \bar{P}_j (a_{i+1} - a_i)(b_j - b_{j-1}) \right)$$

are not positive. From that we conclude

$$\left| \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i a_i b_i - \sum_{i=1}^{n} p_i a_i \sum_{i=1}^{n} p_i b_i \right|$$

$$\leqslant (M-m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \left| \bar{P}_{i+1} P_j \right| + \sum_{j=i+1}^{n} \left| P_i \bar{P}_j \right| \right) |a_{i+1} - a_i|$$

$$\leqslant \delta (M-m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \left| \bar{P}_{i+1} P_j \right| + \sum_{j=i+1}^{n} \left| P_i \bar{P}_j \right| \right)$$

$$= -\delta (M-m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right).$$

In the following we show how Theorem 2.1 can be used to prove the Jensen-Grüss inequality under Steffensen's conditions (1.2). Such inequality will be called the Jensen-Steffensen-Grüss inequality, or shorter, the *Steffensen-Grüss inequality*.

THEOREM 2.2. *Let $f : I \to \mathbb{R}$ be a differentiable function and suppose that there exist some $m, M \in \mathbb{R}$ such that*

$$m \leqslant f'(x) \leqslant M, \text{ for all } x \in I.$$

Let $\mathbf{x} = (x_1, \ldots, x_n) \in I^n$ be an n-tuple such that

$$|x_{k+1} - x_k| \leqslant \delta, \ k \in \{1, \ldots, n-1\}.$$

Then for all real n-tuples $\mathbf{p} = (p_1, \ldots, p_n)$ satisfying (1.2) such that $\bar{x} \in I$ the following inequalities hold

$$\left| f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) \right|$$

$$\leqslant \frac{(M-m)}{P_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}\bar{P}_{i+1}P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j\right)|x_{i+1}-x_i|$$

$$\leqslant \frac{\delta(M-m)}{P_n^2}\sum_{i=1}^{n-1}\left(\sum_{j=1}^{i-1}\bar{P}_{i+1}P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j\right). \tag{2.5}$$

Proof. From the mean-value theorem we know that for any $x, y \in I$ there exist some ξ between them such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Choosing $x = \bar{x}$ and $y = x_i$ we get

$$f(x_i) - f(\bar{x}) = f'(\xi_i)(x_i - \bar{x}).$$

If we multiply the above equality by p_i, and then sum over i, we obtain

$$\sum_{i=1}^{n} p_i f(x_i) - P_n f(\bar{x})$$

$$= \sum_{i=1}^{n} p_i x_i f'(\xi_i) - \bar{x}\sum_{i=1}^{n} p_i f'(\xi_i)$$

$$= \sum_{i=1}^{n} p_i x_i f'(\xi_i) - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(\xi_i),$$

or written differently

$$P_n \sum_{i=1}^{n} p_i f(x_i) - P_n^2 f(\bar{x})$$

$$= \sum_{i=1}^{n} p_i \sum_{i=1}^{n} p_i x_i f'(\xi_i) - \sum_{i=1}^{n} p_i x_i \sum_{i=1}^{n} p_i f'(\xi_i).$$

Using (2.1) with $a_i = x_i$ and $b_i = f'(\xi_i)$ by the assumptions of this theorem we get

$$\left| P_n^2 f(\bar{x}) - P_n \sum_{i=1}^{n} p_i f(x_i) \right|$$

$$\leqslant \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j \left| x_{i+1} - x_i \right| \left| f'(\xi_{j+1}) - f'(\xi_j) \right| \right.$$

$$\left. + \sum_{j=i+1}^{n} P_i \bar{P}_j \left| x_{i+1} - x_i \right| \left| f'(\xi_j) - f'(\xi_{j-1}) \right| \right)$$

$$\leqslant (M - m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right) \left| x_{i+1} - x_i \right|$$

$$\leqslant \delta (M - m) \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right),$$

which after division by P_n^2 becomes (2.5).   □

REMARK 2.3. Note that if $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ is monotonic and $\boldsymbol{p} = (p_1, \ldots, p_n)$ satisfies (1.2) the condition $\bar{x} \in I$ becomes redundant.

Also, if in Theorem 2.2 instead of (1.2) we consider alternative conditions

$$0 \leqslant P_n \leqslant P_k, \ k \in \{1, \ldots, n-1\}$$

or

$$0 \leqslant P_n \leqslant \bar{P}_k, \ k \in \{2, \ldots, n\}.$$

we obtain

$$\left| f \left(\frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) - \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) \right|$$

$$\leqslant \frac{(M - m)}{P_n^2} \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \left| \bar{P}_{i+1} P_j \right| + \sum_{j=i+1}^{n} \left| P_i \bar{P}_j \right| \right) \left| x_{i+1} - x_i \right|$$

$$\leqslant -\frac{\delta (M - m)}{P_n^2} \sum_{i=1}^{n-1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1} P_j + \sum_{j=i+1}^{n} P_i \bar{P}_j \right).$$

3. Mercer type result

There is an easy way to obtain a Jensen-Mercer type inequality starting from some Jensen type inequality whenever we have weights $\boldsymbol{p} = (p_1, \ldots, p_n)$ satisfying (1.2). This can be seen in the following theorem.

THEOREM 3.1. *Let $f : I \to \mathbb{R}$ be a differentiable function and suppose that there exist some $m, M \in \mathbb{R}$ such that*

$$m \leqslant f'(x) \leqslant M, \text{ for all } x \in I.$$

Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in I^n$ and $a, b \in I$ be such that

$$|x_1 - a| \leqslant \delta, \ |x_{k+1} - x_k| \leqslant \delta, \ k \in \{1, \ldots, n-1\}, \ |b - x_n| \leqslant \delta. \qquad (3.1)$$

Then for all real n-tuples $\boldsymbol{p} = (p_1, \ldots, p_n)$ satisfying

$$0 \leqslant P_k \leqslant P_n, \ k \in \{1, \ldots, n-1\}, \ P_n > 0,$$

and

$$a + b - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \in I$$

the following inequalities hold

$$\left| f\left(a + b - \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) + \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f(a) - f(b) \right|$$

$$\leqslant \frac{M - m}{P_n^2} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} P_{i-1}\bar{P}_j + \sum_{j=i-1}^{n} \bar{P}_i P_j \right) |x_i - x_{i-1}|$$

$$+ \frac{M - m}{P_n} \left(|x_1 - a| \sum_{j=1}^{n} P_j + |b - x_n| \sum_{j=1}^{n} \bar{P}_j \right)$$

$$\leqslant \frac{\delta(M - m)}{P_n^2} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} P_{i-1}\bar{P}_j + \sum_{j=i-1}^{n} \bar{P}_i P_j \right) + \delta(M - m)(n + 1). \qquad (3.2)$$

Proof. Using $n + 2$ instead of n points, and making some substitutions, the assertion immediately follows from Theorem 2.2. Namely, we define

$$\xi_1 = a, \quad \xi_2 = x_1, \quad \xi_3 = x_2, \quad \cdots, \quad \xi_{n+1} = x_n, \quad \xi_{n+2} = b, \qquad (3.3)$$

$$p_1' = 1, \quad p_2' = -\frac{p_1}{P_n}, \quad p_3' = -\frac{p_2}{P_n}, \quad \cdots, \quad p_{n+1}' = -\frac{p_n}{P_n}, \quad p_{n+2}' = 1.$$

Then the $n + 2$-tuple $\boldsymbol{p}'$ satisfies the conditions

$$0 \leqslant P_k' \leqslant P_{n+2}', \ k \in \{1, \ldots, n+1\} \text{ and } P_{n+2}' = 1 > 0,$$

and by Theorem 2.2 we have

$$\left| f\left(\frac{1}{P_{n+2}'}\sum_{i=1}^{n+2} p_i' \xi_i\right) - \frac{1}{P_{n+2}'}\sum_{i=1}^{n+2} p_i' f(\xi_i) \right|$$

$$\leqslant \frac{(M - m)}{P_{n+2}'^2} \sum_{i=1}^{n+1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1}' P_j' + \sum_{j=i+1}^{n+2} P_i' \bar{P}_j' \right) |\xi_{i+1} - \xi_i|$$

$$\leqslant \frac{\delta(M - m)}{P_{n+2}'^2} \sum_{i=1}^{n+1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1}' P_j' + \sum_{j=i+1}^{n+2} P_i' \bar{P}_j' \right).$$

The left hand side of (3.2) follows from

$$f\left(\frac{1}{P'_{n+2}}\sum_{i=1}^{n+2}p'_i\xi_i\right) - \frac{1}{P'_{n+2}}\sum_{i=1}^{n+2}p'_i f(\xi_i) \tag{3.4}$$

$$= f\left(a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_i x_i\right) + \frac{1}{P_n}\sum_{i=1}^{n}p_i f(x_i) - f(a) - f(b).$$

It can be easily seen that

$$P'_1 = P'_{n+2} = 1, \ P'_{n+1} = 0,$$

$$P'_i = 1 - \frac{p_1}{P_n} - \cdots - \frac{p_{i-1}}{P_n} = \frac{\bar{P}_i}{P_n}, \ i \in \{2,\ldots,n\},$$

$$\bar{P}'_1 = \bar{P}'_{n+2} = 1, \ \bar{P}'_2 = 0,$$

$$\bar{P}'_i = 1 - \frac{p_{i-1}}{P_n} - \cdots - \frac{p_n}{P_n} = \frac{P_{i-2}}{P_n}, \ i \in \{3,\ldots,n+1\},$$

hence for $i \in \{2,\ldots,n\}$

$$\sum_{j=1}^{i-1}\bar{P}'_{i+1}P'_j + \sum_{j=i+1}^{n+2}P'_i\bar{P}'_j$$

$$= \frac{P_{i-1}}{P_n}\left(1 + \frac{\bar{P}_2}{P_n} + \cdots + \frac{\bar{P}_{i-1}}{P_n}\right) + \frac{\bar{P}_i}{P_n}\left(\frac{P_{i-1}}{P_n} + \cdots + \frac{P_{n-1}}{P_n} + 1\right)$$

$$= \frac{P_{i-1}}{P_n}\left(\frac{\bar{P}_1}{P_n} + \frac{\bar{P}_2}{P_n} + \cdots + \frac{\bar{P}_{i-1}}{P_n}\right) + \frac{\bar{P}_i}{P_n}\left(\frac{P_{i-1}}{P_n} + \cdots + \frac{P_{n-1}}{P_n} + \frac{P_n}{P_n}\right)$$

$$= \frac{1}{P_n^2}\left(\sum_{j=1}^{i-1}P_{i-1}\bar{P}_j + \sum_{j=i-1}^{n}\bar{P}_i P_j\right).$$

Using the above equalities and substitutions (3.3) we get

$$\frac{(M-m)}{P'^2_{n+2}}\sum_{i=1}^{n+1}\left(\sum_{j=1}^{i-1}\bar{P}'_{i+1}P'_j + \sum_{j=i+1}^{n+2}P'_i\bar{P}'_j\right)|\xi_{i+1}-\xi_i|$$

$$= (M-m)\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1}\bar{P}'_{i+1}P'_j + \sum_{j=i+1}^{n+2}P'_i\bar{P}'_j\right)|x_i-x_{i-1}|$$

$$+ (M-m)\left[|x_1-a| + |b-x_n| + |x_1-a|\sum_{j=3}^{n+1}\bar{P}'_j + |b-x_n|\sum_{j=2}^{n}P'_j\right]$$

$$= \frac{M-m}{P_n^2}\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1}P_{i-1}\bar{P}_j + \sum_{j=i-1}^{n}\bar{P}_i P_j\right)|x_i-x_{i-1}|$$

$$+ (M-m)\left[|x_1-a| + |b-x_n| + \frac{|x_1-a|}{P_n}\sum_{j=3}^{n+1}P_{j-2} + \frac{|b-x_n|}{P_n}\sum_{j=2}^{n}\bar{P}_j\right].$$

Obviously

$$|x_1 - a| + |b - x_n| + \frac{|x_1 - a|}{P_n} \sum_{j=3}^{n+1} P_{j-2} + \frac{|b - x_n|}{P_n} \sum_{j=2}^{n} \bar{P}_j$$

$$= \frac{|x_1 - a|}{P_n} \sum_{j=3}^{n+2} P_{j-2} + \frac{|b - x_n|}{P_n} \sum_{j=1}^{n} \bar{P}_j$$

$$= \frac{|x_1 - a|}{P_n} \sum_{j=1}^{n} P_j + \frac{|b - x_n|}{P_n} \sum_{j=1}^{n} \bar{P}_j,$$

hence

$$\frac{(M-m)}{P_{n+2}'^2} \sum_{i=1}^{n+1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1}' P_j' + \sum_{j=i+1}^{n+2} P_i' \bar{P}_j' \right) |\xi_{i+1} - \xi_i|$$

$$\leqslant \frac{M-m}{P_n^2} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} P_{i-1} \bar{P}_j + \sum_{j=i-1}^{n} \bar{P}_i P_j \right) |x_i - x_{i-1}|$$

$$+ \frac{M-m}{P_n} \left(|x_1 - a| \sum_{j=1}^{n} P_j + |b - x_n| \sum_{j=1}^{n} \bar{P}_j \right).$$

Applying (3.1) we obtain

$$\frac{(M-m)}{P_{n+2}'^2} \sum_{i=1}^{n+1} \left(\sum_{j=1}^{i-1} \bar{P}_{i+1}' P_j' + \sum_{j=i+1}^{n+2} P_i' \bar{P}_j' \right) |\xi_{i+1} - \xi_i|$$

$$\leqslant \frac{M-m}{P_n^2} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} P_{i-1} \bar{P}_j + \sum_{j=i-1}^{n} \bar{P}_i P_j \right) |x_i - x_{i-1}|$$

$$+ \frac{M-m}{P_n} \left(|x_1 - a| \sum_{j=1}^{n} P_j + |b - x_n| \sum_{j=1}^{n} \bar{P}_j \right)$$

$$\leqslant \frac{\delta (M-m)}{P_n^2} \left[\sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} P_{i-1} \bar{P}_j + \sum_{j=i-1}^{n} \bar{P}_i P_j \right) + \sum_{j=1}^{n} P_n P_j + \sum_{j=1}^{n} P_n \bar{P}_j \right]$$

$$= \frac{\delta (M-m)}{P_n^2} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} P_{i-1} \bar{P}_j + \sum_{j=i-1}^{n} \bar{P}_i P_j \right) + \delta (M-m)(n+1),$$

and consequently, with (3.4),

$$\left| f \left(a + b - \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i \right) + \frac{1}{P_n} \sum_{i=1}^{n} p_i f(x_i) - f(a) - f(b) \right|$$

$$\leqslant \frac{M-m}{P_n^2} \sum_{i=2}^{n} \left(\sum_{j=1}^{i-1} P_{i-1} \bar{P}_j + \sum_{j=i-1}^{n} \bar{P}_i P_j \right) |x_i - x_{i-1}|$$

$$+ \frac{M-m}{P_n} \left(|x_1 - a| \sum_{j=1}^{n} P_j + |b - x_n| \sum_{j=1}^{n} \bar{P}_j \right)$$

$$\leqslant \frac{\delta\,(M-m)}{P_n^2}\left[\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1}P_{i-1}\bar{P}_j+\sum_{j=i-1}^{n}\bar{P}_iP_j\right)+P_n\left(\sum_{j=1}^{n}P_j+\sum_{j=1}^{n}\bar{P}_j\right)\right]$$

$$=\frac{\delta\,(M-m)}{P_n^2}\sum_{i=2}^{n}\left(\sum_{j=1}^{i-1}P_{i-1}\bar{P}_j+\sum_{j=i-1}^{n}\bar{P}_iP_j\right)+\delta\,(M-m)\,(n+1),$$

which is the desired result. $\square$

REMARK 3.2. Note that if $\boldsymbol{x}=(x_1,\ldots,x_n)\in I^n$ is monotonic and $\boldsymbol{p}=(p_1,\ldots,p_n)$ satisfies (1.2) any choice of $a,b\in I$ such that

$$\min\{x_1,\ldots,x_n\}-a\leqslant\delta,\quad b-\max\{x_1,\ldots,x_n\}\leqslant\delta$$

makes the condition

$$a+b-\frac{1}{P_n}\sum_{i=1}^{n}p_ix_i\in I \tag{3.5}$$

redundant (see for instance [1]).

Also, if $\boldsymbol{p}=(p_1,\ldots,p_n)$ is positive then (1.2) holds for every permutation of the components of $\boldsymbol{x}=(x_1,\ldots,x_n)$ which means that (3.5) is true even if $\boldsymbol{x}$ is not monotonic.

REMARK 3.3. In [6] the author investigated some generalizations of the integral Ostrowski inequality [5]. The obtained results included those involving Steffensen type weights, and one of them is strongly related to (2.1). Namely, in [6, Theorem 2] the author proved that if f and g are two differentiable functions on $[a,b]\subset\mathbb{R}$ monotonic in the same sense, p an integrable function on $[a,b]$ such that

$$0\leqslant P(x)\leqslant P(b),\text{ for all }x\in[a,b],\ P(x)=\int_a^x p(t)\,\mathrm{d}t, \tag{3.6}$$

and M,N two real numbers satisfying

$$\left|f'(x)\right|\leqslant M,\ \left|g'(x)\right|\leqslant N,\text{ for all }x\in[a,b],$$

then

$$|T(f,g;p)|\leqslant MNT(x-a,x-a;p), \tag{3.7}$$

where

$$T(f,g;p)=\int_a^b p(x)\,\mathrm{d}x\int_a^b p(x)f(x)g(x)\,\mathrm{d}x-\int_a^b p(x)f(x)\,\mathrm{d}x\int_a^b p(x)g(x)\,\mathrm{d}x.$$

Note that (3.6) is an integral variant of (1.2), and the main tool used in the proof of (3.7) is the identity

$$T(f,g;p)=\int_a^b \bar{P}(x)\int_a^x P(t)\,\mathrm{d}g(t)\,\mathrm{d}f(x)+\int_a^b P(x)\int_x^b \bar{P}(t)\,\mathrm{d}g(t)\,\mathrm{d}f(x)$$

which is an integral variant of (2.4). Nevertheless, it is not possible to obtain integral version of (2.1) employing similar reasoning as in the proof of Theorem 2.2 and using (3.7) instead of (2.1).

In the same paper [6] the author proved several integral Chebyshev type inequalities but the problem of finding results of this type that can be used to obtain the integral Steffensen-Grüss inequality remains open.

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