Journal of Mathematical Inequalities
Volume 15, Number 2 (2021), 861–878 doi:10.7153/jmi-2021-15-60

QUANTITATIVE DUNKL ANALOGUE OF SZÁSZ–MIRAKYAN OPERATORS

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(*Communicated by M. Mursaleen*)

Abstract. The main object of this paper is to introduce a sequence of quantitative Dunkl analogue Szász-Mirakyan operators. Firstly, we have defined mentioned operators and have obtained test values and central moments for our operators. We have given weighted Korovkin theorem for these operators and then, have shed light on approximation properties of these operators with the help of the classical modulus of continuity, Peetre's K-functional, the second modulus of continuity, the modulus of weighted continuity defined by Holhos in [30] on some function space. Moreover, we have given Voronovskaya type theorems for our operators and basic operators defined by Sucu in [6]. Finally, graphics of these operators have been presented for some values of n.

1. Introduction

Recently, many results about various generelization of Szász-Mirakyan operators have been obtained. Some of mentioned generalizations that are important have given in [1], [2], [3], [4] and [5]. Sucu [6] has introduced Dunkl analogue of Szász-Mirakyan operators as follows

$$S_n^{\mu}(f;x) = \frac{1}{e_{\mu}(nx)} \sum_{k=0}^{\infty} \frac{(nx)^k}{\gamma_{\mu}(k)} f\left(\frac{k+2\mu\theta_k}{n}\right), \tag{1.1}$$

where $n \in \mathbb{N}$, $x \in [0,\infty)$, $\mu \geqslant 0$ and $f \in C[0,\infty)$. Moreover, $e_{\mu}(x)$ has been introduced by Rosenblum [7] in here as follows

$$e_{\mu}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\gamma_{\mu}(k)}, \tag{1.2}$$

where the coefficients $\gamma_{\mu}(k)$ is defined by

$$\gamma_{\mu}(2k) = \frac{2^{2k}k!\Gamma\left(k+\mu+\frac{1}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)} \text{ and } \gamma_{\mu}(2k+1) = \frac{2^{2k+1}k!\Gamma\left(k+\mu+\frac{3}{2}\right)}{\Gamma\left(\mu+\frac{1}{2}\right)} \tag{1.3}$$

Mathematics subject classification (2020): 41A25, 41A36.
Keywords and phrases: Dunkl exponential, Szász operators, Modulus of continuity, Voronovskaya type asymptotic formula.

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for $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $\mu > -\frac{1}{2}$.

We take note of the following recursion relation about γ_μ

$$\gamma_\mu(k+1) = (k+1+2\mu\theta_{k+1})\gamma_\mu(k), \; k = 0,1,2,\ldots, \tag{1.4}$$

where

$$\theta_k = \begin{cases} 1\;; & k \in 2\mathbb{N}_0+1 \\ 0\;; & k \in 2\mathbb{N}_0 \end{cases}.$$

Furthermore, many studies associated with Dunkl analogue of a linear positive operators have been carried out. Some of them are [8], [9], [10], [11], [12], [13], [14], [15] and [16].

In the other hand, Cárnedas-Morales et al. [17] has presented a new construction of Bernstein polynomials for $f \in C[0,1]$ as follows

$$B_n^\tau(f;x) = \sum_{k=0}^{\infty} \left(f \circ \tau^{-1}\right)\left(\frac{k}{n}\right)\binom{n}{k}\tau^k(x)\left(1-\tau(x)\right)^{n-k}, \tag{1.5}$$

where τ is a continuous infinite times differentiable function verfying the condition $\tau(0) = 0,\; \tau(1) = 1$ and $\tau'(x) > 0$ for $0 \leqslant x \leqslant 1$. The Korovkin test set is generalized from $\{1,t,t^2\}$ to $\{1,\tau(t),\tau^2(t)\}$ on this construction. A better degree of approximation depending on τ has been obtained for Bernstein operators on this study. Therefore, many of authors have carried out studies about this construction, likely [18], [19], [20] and [21].

At the present, assume that Ψ is the function which satisfies the following requirements:

G1) Ψ is a continuously differentiable function on interval $[0,\infty)$,

G2) $\Psi(0) = 0$ and $\inf\limits_{x \geqslant 0} \Psi'(x) \geqslant 1$.

By going on these conditions, Aral et al. [21] have defined a generalization as follows:

$$S_n^\Psi(f;x) = \exp(-n\Psi(x)) \sum_{k=0}^{\infty} \frac{(n\Psi(x))^k}{k!}\left(f \circ \Psi^{-1}\right)\left(\frac{k}{n}\right),$$

where $n \in \mathbb{N},\; f \in C[0,\infty),\; x \geqslant 0$.

Now, let Ψ be the function satisfying conditions G1 and G2. Then, we introduce a new generalization of Dunkl-Szász-Mirakyan operators as follows:

$$^{\Psi}S_n^\mu(f;x) = \frac{1}{e_\mu(n\Psi(x))} \sum_{k=0}^{\infty} \frac{(n\Psi(x))^k}{\gamma_\mu(k)}\left(f \circ \Psi^{-1}\right)\left(\frac{k+2\mu\theta_k}{n}\right), \tag{1.6}$$

where $n \in \mathbb{N},\; f \in C[0,\infty),\; x \geqslant 0,\; \mu \in \left(-\frac{1}{2},\frac{1}{2}\right),\; e_\mu(x)$ and γ_μ are defined by (1.2) and (1.3) in [7], respectively. If Ψ is the unit function defined on $[0,\infty)$, then $^{\Psi}S_n^\mu = S_n^\mu$ for $0 \leqslant \mu < 1/2$. It is $^{\Psi}S_n^\mu = S_n^\Psi$ for $\mu = 0$. Furthermore, It is clear that

$$^{\Psi}S_n^\mu(f;x) = S_n^\mu\left(f \circ \Psi^{-1};\Psi(x)\right). \tag{1.7}$$

Note that we take $\mu \in \left(-\frac{1}{2},\frac{1}{2}\right)$ differently from $\mu \geqslant 0$ in the study [6]. So, the operators defined by us are going to be more significant and have better approximation properties.

LEMMA 1. *The operators defined in* (1.6) *confirm the following values*

i. $\quad {}^{\Psi}S_n^{\mu}(1;x) = 1,$

ii. $\quad {}^{\Psi}S_n^{\mu}(\Psi(t);x) = \Psi(x),$

iii. $\quad {}^{\Psi}S_n^{\mu}\left(\Psi^2(t);x\right) = \Psi^2(x) + \left(1+2\mu\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi(x)}{n},$

iv.
$$
{}^{\Psi}S_n^{\mu}\left(\Psi^3(t);x\right) = \Psi^3(x) + \left(3-2\mu\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi^2(x)}{n} + \left(1+4\mu^2+4\mu\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi(x)}{n^2},
$$

v.
$$
\begin{aligned}
{}^{\Psi}S_n^{\mu}\left(\Psi^4(t);x\right) &= \Psi^4(x) + \left(6+4\mu\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi^3(x)}{n} \\
&+ \left(7+4\mu^2-8\mu\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi^2(x)}{n^2} \\
&+ \left(1+12\mu^2+2\mu\left(3+4\mu^2\right)\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi(x)}{n^3},
\end{aligned}
$$

vi.
$$
\begin{aligned}
{}^{\Psi}S_n^{\mu}\left(\Psi^5(t);x\right) &= \Psi^5(x) + \frac{\Psi^4(x)}{n}\left(10-4\mu\xi_{\Psi}^{\mu}(n,x)\right) \\
&+ \frac{\Psi^3(x)}{n^2}\left(25+12\mu^2+32\mu\xi_{\Psi}^{\mu}(n,x)\right) \\
&+ \frac{\Psi^2(x)}{n^3}\left(15+20\mu^2-\left(22\mu+8\mu^3\right)\xi_{\Psi}^{\mu}(n,x)\right) \\
&+ \frac{\Psi(x)}{n^4}\left(1+24\mu^2+16\mu^4+\left(8\mu+32\mu^3\right)\xi_{\Psi}^{\mu}(n,x)\right),
\end{aligned}
$$

vii.
$$
\begin{aligned}
{}^{\Psi}S_n^{\mu}\left(\Psi^6(t);x\right) &= \Psi^6(x) + \frac{\Psi^5(x)}{n}\left(15+6\mu\xi_{\Psi}^{\mu}(n,x)\right) \\
&+ \frac{\Psi^4(x)}{n^2}\left(65+12\mu^2-48\mu\xi_{\Psi}^{\mu}(n,x)\right) \\
&+ \frac{\Psi^3(x)}{n^3}\left(80+200\mu^2+138\mu\xi_{\Psi}^{\mu}(n,x)+192\mu^3\xi_{\Psi}^{\mu}(n,x)\right) \\
&+ \frac{\Psi^2(x)}{n^4}\left(51-16\mu^2+16\mu^4-\left(12\mu+208\mu^3\right)\xi_{\Psi}^{\mu}(n,x)\right) \\
&+ \frac{\Psi(x)}{n^5}\left(1+40\mu^2+80\mu^4+\left(10\mu+80\mu^3+32\mu^5\right)\xi_{\Psi}^{\mu}(n,x)\right),
\end{aligned}
$$

where $\xi_{\Psi}^{\mu}(n,x) := \frac{e_\mu(-n\Psi(x))}{e_\mu(n\Psi(x))}$.

Proof. From the equality (1.7), it can be seen that

$$
\begin{aligned}
{}^{\Psi}S_n^{\mu}\left(\Psi^k;x\right) &= S_n^{\mu}\left((\Psi)^k\circ\Psi^{-1};\Psi(x)\right) \\
&= S_n^{\mu}\left((\Psi\circ\Psi^{-1})^k;\Psi(x)\right) \\
&= S_n^{\mu}\left(e_k;\Psi(x)\right)
\end{aligned}
$$

where $e_k(t)=t^k,\ t\geqslant 0$ and $k\in\mathbb{N}_0$. We can calculate $S_n^{\mu}(e_5;x)$ and $S_n^{\mu}(e_6;x)$ for the operators S_n^{μ} given in [6] with help of similar operations Lemma 1 in [6], too. Then, the proof is completed by getting $x\to\Psi(x)$ in $S_n^{\mu}(e_k;x),\ k=0,1,\ldots,6.$ □

Now, we can give next lemma without the proof thanks to linearity of ${}^{\Psi}S_n^{\mu}$ and Lemma 1.

LEMMA 2. *The r-th central moment of operators ${}^{\Psi}S_n^{\mu}(f;x)$ is given by*

$$M_{n,r}(x)={}^{\Psi}S_n^{\mu}\left((\Psi(t)-\Psi(x))^r;x\right),\ r=0,1,2,\ldots$$

for $n\in\mathbb{N}$ and $x\in[0,\infty)$. Then, we obtain some central moments as follows

i. $M_{n,0}(x)=1,$

ii. $M_{n,1}(x)=0,$

iii. $M_{n,2}(x)=\left(1+2\mu\xi_{\Psi}^{\mu}(n,x)\right)\dfrac{\Psi(x)}{n},$

iv. $$M_{n,4}(x)=24\mu\xi_{\Psi}^{\mu}(n,x)\frac{\Psi^3(x)}{n}+\left(3-12\mu^2-24\mu\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi^2(x)}{n^2}+\left(1+12\mu^2+2\mu\left(3+4\mu^2\right)\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi(x)}{n^3},$$

v. $$\begin{aligned}M_{n,6}(x)&=160\mu\xi_{\Psi}^{\mu}(n,x)\frac{\Psi^5(x)}{n}-\left(80\mu^2+440\mu\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi^4(x)}{n^2}\\&+\left(5+260\mu^2+360\left(\mu+\mu^3\right)\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi^3(x)}{n^3}\\&-\left(160\mu^2+80\mu^4-45+\mu\left(60+400\mu^2\right)\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi^2(x)}{n^4}\\&+\left(1+40\mu^2\left(1+2\mu^2\right)+\left(10\mu+80\mu^3+32\mu^5\right)\xi_{\Psi}^{\mu}(n,x)\right)\frac{\Psi(x)}{n^5},\end{aligned}$$

where $\xi_{\Psi}^{\mu}(n,x):=\frac{e_{\mu}(-n\Psi(x))}{e_{\mu}(n\Psi(x))}$.

2. Weighted Korovkin type theorem

Korovkin's theorem [22] has played a important role in approximation theory. According to this study, if a sequence of linear positive operators approximates test functions defined on a bounded interval

$$e_k:e_k(x)=x^k,\qquad k=0,1,2 \tag{2.1}$$

then it approximates all continuous and bounded functions on this interval. Moreover, the test functions e_0,e_1 and e_2 can replace $\Psi^0,\ \Psi^1$ and Ψ^2. It is seen clearly that the operators defined by ${}^{\Psi}S_n^{\mu}(f;x)$ verify the following phase thanks to Lemma 1

$$\lim_{n\to\infty}{}^{\Psi}S_n^{\mu}\left(\Psi^k(t);x\right)=\Psi^k(x),\ \ k=0,1,2. \tag{2.2}$$

Korovkin theorem was extended under the name weighted Korovkin theorem to unbounded intervals by Gadzhiev [23]. Function Ψ, given in conditions G1 and G2, is a monotonous increased function and $\lim_{x\to\infty}\Psi(x)=\infty$ from condition G2. Therefore, $\rho(x)=1+\Psi^2(x)$ is a weighted function. So, we can define some function spaces associated with weighted Korovkin theorem by means of function ρ as follows:

$$\begin{aligned}
B_\rho[0,\infty) &:= \left\{f:[0,\infty)\longrightarrow\mathbb{R} \mid \ |f(x)|\leqslant M_f\left(1+\Psi^2(x)\right)\right\}\\
C_\rho[0,\infty) &:= \left\{f\in B_\rho[0,\infty) \mid f \text{ is continuous on } [0,\infty)\right\}\\
U_\rho[0,\infty) &:= \left\{f\in B_\rho[0,\infty) \mid \frac{f}{\rho} \text{ is uniformly continuous on } [0,\infty)\right\}\\
C_\rho^k[0,\infty) &:= \left\{f\in C_\rho[0,\infty) \mid \lim_{x\to\infty}\frac{|f(x)|}{1+\Psi^2(x)}=k_f<\infty\right\}.
\end{aligned}$$

$B_\rho[0,\infty)$ is a normed space with $\|\cdot\|_\rho$ defined as follows:

$$\|f\|_\rho=\sup_{x\geqslant 0}\frac{|f(x)|}{1+\Psi^2(x)}$$

where $f\in B_\rho[0,\infty)$. Furhermore, it's clear that

$$C_\rho^k[0,\infty)\subset C_\rho[0,\infty)\subset B_\rho[0,\infty).$$

Now, we remind some features of linear positive operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ in the following Lemma 3 and Theorem 1 using the special cases of the definitions of the function spaces given above for $\Psi(x)=x$.

LEMMA 3. [23] *The linear operators L_n, $n\geqslant 1$, act from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ if and only if*

$$|L_n(\rho;x)|\leqslant K_\rho(\rho(x))$$

where $\rho(x)=1+x^2$, $x\in[0,\infty)$ and K_ρ is a positive constant.

THEOREM 1. [23] *Let the sequence of linear positive operators $\{L_n\}_{n\geqslant 1}$ acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ and satisfying the condition*

$$\lim_{n\to\infty}\|L_n(e_k;\cdot)-e_k\|_\rho=0,\qquad k=0,1,2.$$

Then, for any function $f\in C_\rho^k[0,\infty)$,

$$\lim_{n\to\infty}\|L_n(f;\cdot)-f\|_\rho=0.$$

THEOREM 2. *${}^{\Psi}S_n^{\mu}$ defined in (1.6) fulfill the following equality*

$$\lim_{n\to\infty}\left\|{}^{\Psi}S_n^{\mu}(f;\cdot)-f\right\|_\rho=0 \tag{2.3}$$

where each function $f\in C_\rho^k[0,\infty)$ and $\rho:\rho(x)=1+\Psi^2(x)$.

Proof. Firstly, it's clear that we can change Korovkin test system from $\{e_0, e_1, e_2\}$ to $\{\Psi^0, \Psi^1, \Psi^2\}$ in Lemma 3 and Theorem 1. ${}^{\Psi}S_n^{\mu}$ are linear positive operators acting from $C_\rho[0,\infty)$ to $B_\rho[0,\infty)$ with help of Lemma 1 and Lemma 3. Moreover, it is obvious that $\left\|{}^{\Psi}S_n^{\mu}\left(\Psi^0;\cdot\right) - \Psi^0\right\|_\rho = 0$, $\left\|{}^{\Psi}S_n^{\mu}\left(\Psi^1;\cdot\right) - \Psi^1\right\|_\rho = 0$. We can see $\left|\xi_\Psi^\mu(n,x)\right| = \left|\frac{e_\mu(-n\Psi(x))}{e_\mu(n\Psi(x))}\right| \leqslant 1$ [24] and from Lemma 1, we obtain

$$\begin{aligned}
\sup_{x\geqslant 0}\frac{\left|{}^{\Psi}S_n^{\mu}\left(\Psi^2(t);x\right) - \Psi^2(x)\right|}{1+\Psi^2(x)} &= \sup_{x\geqslant 0}\frac{\left|\Psi^2(x) + \left(1+2\mu\xi_\Psi^\mu(n,x)\right)\frac{\Psi(x)}{n} - \Psi^2(x)\right|}{1+\Psi^2(x)} \\
&\leqslant \frac{1+2\mu}{n}\sup_{x\geqslant 0}\frac{\Psi(x)}{1+\Psi^2(x)} \\
&\leqslant \frac{1+2\mu}{2n}.
\end{aligned}$$

By above inequality, we get

$$\lim_{n\to\infty}\left\|{}^{\Psi}S_n^{\mu}\left(\Psi^2;\cdot\right) - \Psi^2\right\|_\rho = 0.$$

Consequently, we have

$$\lim_{n\to\infty}\left\|{}^{\Psi}S_n^{\mu}\left(\Psi^k;\cdot\right) - \Psi^k\right\|_\rho = 0, \quad k = 0,1,2. \tag{2.4}$$

We obtain the desired result from equality (2.4). □

Now, we can get approximation properties for these operators in the next section.

3. Approximation properties

Firstly, let's remind the definitions of some criteria to find rate convergence of operators ${}^{\Psi}S_n^{\mu}$.

The classical modulus of continuity of $f \in C_B[0,\infty)$ is defined as follows

$$\omega(f;\delta) := \sup_{|h|\leqslant\delta}\{|f(x+h) - f(x)| : x \in [0,\infty)\} \tag{3.1}$$

where $\delta > 0$ [25].

We define the second-order modulus of smoothness of function $f \in C_B[0,\infty)$ by following equality

$$\omega_2(f;\delta) := \sup_{0<h\leqslant\delta}\{|f(x+2h) - 2f(x+h) + f(x)| : x \in [0,\infty)\} \tag{3.2}$$

for $\delta > 0$ [25].

Peetre's K-functional of the function $f \in C_B[0,\infty)$ can be defined by

$$K(f;\delta) := \inf_{h\in C_B^2[0,\infty)}\left\{\|f-h\|_{C_B[0,\infty)} + \delta\|h\|_{C_B^2[0,\infty)}\right\} \tag{3.3}$$

for $\delta > 0$ [26]. In here, $C_B^2[0,\infty) := \{f \in C_B[0,\infty) : f', f'' \in C_B[0,\infty)\}$ is a normed space with norm $\|\cdot\|_{C_B^2[0,\infty)}$ defined by

$$\|f\|_{C_B^2[0,\infty)} = \|f\|_{C_B[0,\infty)} + \|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)}$$

for every $f \in C_B^2[0,\infty)$. Moreover, $\|\cdot\|_{C_B[0,\infty)}$ is defined as follows

$$\|f\|_{C_B[0,\infty)} := \sup\{|f(x)| : x \in [0,\infty)\}$$

for $f \in C_B[0,\infty)$.

Furthermore, we know that there is the relation between second order modulus of smoothness ω_2 and Peetre's K-functional $K(f;\delta)$ of the function $f \in C_B[0,\infty)$ as follows

$$K(f;\delta) \leqslant M\left\{\omega_2\left(f, \sqrt{\delta}\right) + \min(1,\delta)\|f\|_{C_B[0,\infty)}\right\} \tag{3.4}$$

for all $\delta > 0$ in [27] and where M is a positive constant.

Now, we obtain some approximation theorems for operators ${}^{\Psi}S_n^{\mu}$ in some function spaces.

PROPOSITION 1. *For $f \in C_B[0,\infty)$, we have*

$$\left\|{}^{\Psi}S_n^{\mu}(f;\cdot)\right\|_{C_B[0,\infty)} \leqslant \|f\|_{C_B[0,\infty)}. \tag{3.5}$$

Proof. From (1.7), the triangle inequality and $S_n^{\mu}\left(\Psi^0;x\right) = 1$ (see [6]), we obtain

$$\begin{aligned}\left|{}^{\Psi}S_n^{\mu}(f;x)\right| &\leqslant \left|S_n^{\mu}\left(f\circ\Psi^{-1};\Psi(x)\right)\right| \\ &\leqslant S_n^{\mu}\left(\left|f\circ\Psi^{-1}\right|;\Psi(x)\right) \\ &\leqslant \left\|f\circ\Psi^{-1}\right\|_{C_B[0,\infty)} S_n^{\mu}\left(\Psi^0;\Psi(x)\right) \\ &\leqslant \left\|f\circ\Psi^{-1}\right\|_{C_B[0,\infty)} \\ &\leqslant \|f\|_{C_B[0,\infty)}.\end{aligned}$$

Then, we have the desired result. □

LEMMA 4. [28] *We have the following equalities*

$$i.\quad \left(f\circ\Psi^{-1}\right)'(\Psi(x)) = \frac{f'(x)}{\Psi'(x)},$$

$$ii.\quad \left(f\circ\Psi^{-1}\right)''(\Psi(x)) = \frac{f''(x)}{\{\Psi'(x)\}^2} - \frac{f'(x)\Psi''(x)}{\{\Psi'(x)\}^3}.$$

THEOREM 3. *The operators given by ${}^{\Psi}S_n^{\mu}$ in (1.6) confirm the following inequality*

$$\left|{}^{\Psi}S_n^{\mu}(f;x) - f(x)\right| \leqslant 2\sqrt{M_{n,2}(x)}\,\omega\left(\left(f\circ\Psi^{-1}\right)'; \sqrt{M_{n,2}(x)}\right) \tag{3.6}$$

where $f \in C_B^1[0,\infty) := \{f \in C_B[0,\infty) : f' \text{exists in } C_B[0,\infty)\}$ *and* $M_{n,2}(x)$ *is given in Lemma 2.*

Proof. The classical modulus of continuity of function $f \in C_B^1[0,\infty)$ verifies the following inequality

$$|f(t)-f(x)| \leqslant \left(\frac{|t-x|}{\delta}+1\right)\omega(f;\delta) \quad [25]. \tag{3.7}$$

Moreover, we can clearly write

$$\begin{aligned} f(t)-f(x) &= \left(f\circ\Psi^{-1}\right)(\Psi(t)) - \left(f\circ\Psi^{-1}\right)(\Psi(x)) \\ &= (\Psi(t)-\Psi(x))\left(f\circ\Psi^{-1}\right)'(\Psi(x)) \\ &\quad + \int\limits_{\Psi(x)}^{\Psi(t)} \left\{\left(f\circ\Psi^{-1}\right)'(s) - \left(f\circ\Psi^{-1}\right)'(\Psi(x))\right\} ds \end{aligned} \tag{3.8}$$

for $x,t \in [0,\infty)$. Moreover, we can write

$$\begin{aligned} &\left| \int\limits_{\Psi(x)}^{\Psi(t)} \left\{\left(f\circ\Psi^{-1}\right)'(s) - \left(f\circ\Psi^{-1}\right)'(\Psi(x))\right\} ds \right| \\ &\leqslant \omega\left(\left(f\circ\Psi^{-1}\right)';\delta\right)\left\{\frac{(\Psi(t)-\Psi(x))^2}{\delta} + |\Psi(t)-\Psi(x)|\right\} \end{aligned} \tag{3.9}$$

from inequality (3.7). Now, if we apply ${}^{\Psi}S_n^{\mu}$ to the equality (3.8) and then absolute value to the both sides of this inequality, we obtain

$$\begin{aligned} \left|{}^{\Psi}S_n^{\mu}(f;x) - f(x)\right| &\leqslant \left|\left(f\circ\Psi^{-1}\right)'(\Psi(x))\right| |M_{n,1}(x)| \\ &\quad + \omega\left(\left(f\circ\Psi^{-1}\right)';\delta\right)\left\{\frac{M_{n,2}(x)}{\delta} + {}^{\Psi}S_n^{\mu}(|\Psi(t)-\Psi(x)|;x)\right\} \end{aligned} \tag{3.10}$$

from inequality (3.9) and the triangle inequality. Using Cauchy-Schwarz and Lemma 2 in inequality (3.11), we have

$$\left|{}^{\Psi}S_n^{\mu}(f;x) - f(x)\right| \leqslant \omega\left(\left(f\circ\Psi^{-1}\right)';\delta\right)\left\{\frac{M_{n,2}(x)}{\delta} + \sqrt{M_{n,2}(x)}\right\}.$$

Finally, the proof is completed by choosing $\delta := \delta_n = \sqrt{M_{n,2}(x)}$ in the last inequality. □

LEMMA 5. *The operators defined by* ${}^{\Psi}S_n^{\mu}$ *verify the following inequality about Peetre's K-functional*

$$\left|{}^{\Psi}S_n^{\mu}(f;x) - f(x)\right| \leqslant 2K\left(f; \frac{M_{n,2}(x)\max\left\{1, \|\Psi''\|_{C_B[0,\infty)}\right\}}{4}\right) \tag{3.11}$$

where $f \in C_B[0,\infty)$ and $M_{n,2}(x)$ is given in Lemma 2.

Proof. Let $h \in C_B^2[0,\infty)$. By using Taylor's formula at point $\Psi(x) \in [0,\infty)$, we get

$$\left(h\circ\Psi^{-1}\right)(\Psi(t)) = \left(h\circ\Psi^{-1}\right)(\Psi(x)) + (\Psi(t)-\Psi(x))\left(h\circ\Psi^{-1}\right)'(\Psi(x)) + \frac{((\Psi(t)-\Psi(x)))^2}{2!}\left(h\circ\Psi^{-1}\right)''(\Psi(c)).$$

From Lemma 4, we obtain

$$h(t)-h(x) = \frac{h'(x)}{\Psi'(x)}(\Psi(t)-\Psi(x)) + \frac{((\Psi(t)-\Psi(x)))^2}{2}\left(\frac{h''(c)}{\{\Psi'(c)\}^2} - \frac{h'(c)\Psi''(c)}{\{\Psi'(c)\}^3}\right) \tag{3.12}$$

where c between x and t. By applying ${}^{\Psi}S_n^{\mu}$ to the equality (3.12) and then taking absolute value, we obtain

$$\begin{aligned}\left|{}^{\Psi}S_n^{\mu}(h;x)-h(x)\right| &\leqslant M_{n,1}(x)\frac{h'(x)}{\Psi'(x)} + \frac{M_{n,2}(x)}{2}\left(\frac{h''(c)}{\{\Psi'(c)\}^2} - \frac{h'(c)\Psi''(c)}{\{\Psi'(c)\}^3}\right)\\ &\leqslant \frac{M_{n,2}(x)}{2}\left(\|h''\|_{C_B[0,\infty)} + \|h'\|_{C_B[0,\infty)}\|\Psi''\|_{C_B[0,\infty)} + \|h\|_{C_B[0,\infty)}\right)\\ &\leqslant \frac{M_{n,2}(x)}{2}\max\left\{1,\|\Psi''\|_{C_B[0,\infty)}\right\}\|h\|_{C_B^2[0,\infty)} \end{aligned} \tag{3.13}$$

thanks to $M_{n,1}(x)=0$ and $\inf_{x\geqslant 0}\Psi'(x)\geqslant 1$. Now, consider $f \in C_B[0,\infty)$. Thus, we have

$$\begin{aligned}\left|{}^{\Psi}S_n^{\mu}(f;x)-f(x)\right| &\leqslant {}^{\Psi}S_n^{\mu}(|f-h|;x) + |f(x)-h(x)| + \left|{}^{\Psi}S_n^{\mu}(h;x)-h(x)\right|\\ &\leqslant 2\|f-h\|_{C_B[0,\infty)} + \frac{M_{n,2}(x)}{2}\max\left\{1,\|\Psi''\|_{C_B[0,\infty)}\right\}\|h\|_{C_B^2[0,\infty)}\end{aligned} \tag{3.14}$$

from the triangle inequality and (3.13). The desired result is obtained by taking the infimum over all $h \in C_B^2[0,\infty)$ in (3.14). So, the proof is done. □

At the present, we give the next theorem without the proof thanks to (3.4) and Lemma 5.

THEOREM 4. *We have the following inequality*

$$\left|{}^{\Psi}S_n^{\mu}(f;x)-f(x)\right| \leqslant C\left\{\omega_2\left(f,\sqrt{\xi_n}\right) + \min(1,\xi_n)\|f\|_{C_B[0,\infty)}\right\} \tag{3.15}$$

where $f \in C_B[0,\infty)$, C is a positive constant that is independent of n and

$$\xi_n = \frac{M_{n,2}(x)\max\left\{1,\|\Psi''\|_{C_B[0,\infty)}\right\}}{4}.$$

At the present, we recall that it is given $\lim_{n\to\infty}\xi_n^{\mu}(x)=0$ in [24] and furthermore, we calculate that $\lim_{n\to\infty} n\xi_n^{\mu}(x)=\frac{\mu}{2x}$ thanks to Maple where $\xi_n^{\mu}(x)=\frac{e_\mu(-nx)}{e_\mu(nx)}$, $x\in(0,\infty)$ and $\mu\in\left(\frac{-1}{2},\frac{1}{2}\right)$. We can give the following lemma thanks to Lemma 2 and values of these limits.

LEMMA 6. *It is*

i. $\lim_{n\to\infty} nS_n^{\mu}((t-x);x)=0,$

ii. $\lim_{n\to\infty} nS_n^{\mu}\left((t-x)^2;x\right)=x,$

iii. $\lim_{n\to\infty} n^2S_n^{\mu}\left((t-x)^4;x\right)=3x^2,$

iv. $\lim_{n\to\infty} nS_n^{\mu}\left((t-x)^6;x\right)=0,$

v. $\lim_{n\to\infty} n\ {}^{\Psi}S_n^{\mu}((\Psi(t)-\Psi(x));x)=0,$

vi. $\lim_{n\to\infty} n\ {}^{\Psi}S_n^{\mu}\left((\Psi(t)-\Psi(x))^2;x\right)=\Psi(x),$

vii. $\lim_{n\to\infty} n^2\ {}^{\Psi}S_n^{\mu}\left((\Psi(t)-\Psi(x))^4;x\right)=3\Psi^2(x),$

viii. $\lim_{n\to\infty} n\ {}^{\Psi}S_n^{\mu}\left((\Psi(t)-\Psi(x))^6;x\right)=0,$

where $\mu\in\left(\frac{-1}{2},\frac{1}{2}\right)$, $x,t\in[0,\infty)$, *the operators* S_n^{μ} *and* ${}^{\Psi}S_n^{\mu}$ *are respectively defined in* (1.1) *and* (1.6)*.*

Now, we give Voronovskaya type asymptotic formula for the operators S_n^{μ} defined in (1.1) in Lemma 7. Then, we compare our operators given in (1.6) with S_n^{μ} by using this formula in Theorem 5.

LEMMA 7. *Let* $x\in[0,\infty)$ *be fixed point and let* $f\in U_B[0,\infty)$*. If* f *is of the class* $C^1[0,\infty)$ *in a particular neighbourhood of a point* x *and* $f''(x)$ *exists, then*

$$\lim_{n\to\infty} n\left\{S_n^{\mu}(f;x)-f(x)\right\}=\frac{xf''(x)}{2} \tag{3.16}$$

where $\mu\in\left(\frac{-1}{2},\frac{1}{2}\right)$, S_n^{μ} *is the operators defined in* (1.1) *and* $U_B[0,\infty)=\{f:\ f$ *is an uniformly continuous and bounded on* $[0,\infty)\}$.

Proof. From Taylor formula of f at $x\in[0,\infty)$ that is a fixed point, we can write

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+\lambda_x(t)(t-x)^2 \tag{3.17}$$

where $\lambda_x(t)$ that is given by

$$\lambda_x(t)=\begin{cases}\frac{f(t)-f(x)-(t-x)f'(x)-\frac{(t-x)^2}{2}f''(x)}{(t-x)^2} & ;\ t\neq x\\ 0 & ;\ t=x\end{cases}.$$

With the help of L'Hôpital's Rule, we get

$$\lim_{t\to x}\lambda_x(t)=0.$$

Therefore, $\lambda_x(t)$ is the uniformly continuous and bounded function on $[0,\infty)$. So, It's can be seen that

$$\lim_{n\to\infty} S_n^{\mu}\left((\lambda_x(t))^2;x\right)=(\lambda_x(x))^2=0 \tag{3.18}$$

from Theorem 4 in [6]. By applying operators S_n^{μ} to equality (3.17) and from Lemma 6, we obtain

$$\lim_{n\to\infty} n\left\{S_n^{\mu}(f;x)-f(x)\right\}=\frac{xf''(x)}{2}+n\lim_{n\to\infty}S_n^{\mu}\left(\lambda_x(t)(t-x)^2;x\right). \tag{3.19}$$

Moreover, from Cauchy-Schwarz inequality, we have

$$nS_n^{\mu}\left(\lambda_x(t)(t-x)^2;x\right)\leqslant\sqrt{S_n^{\mu}\left((\lambda_x(t))^2;x\right)}\sqrt{n^2S_n^{\mu}\left((t-x)^4;x\right)}. \tag{3.20}$$

Using Lemma 6 and the equality (3.18) in (3.20), we have

$$\lim_{n\to\infty} nS_n^{\mu}\left(\lambda_x(t)(t-x)^2;x\right)=0. \tag{3.21}$$

Desired result is obtained by writting (3.21) in (3.19). So, the proof is done. □

THEOREM 5. *Let $x\in[0,\infty)$ be fixed point and let $f\in U_B[0,\infty)$. If f is of the class $C^1[0,\infty)$ in a specific neighbourhood of a point x and $f''(x)$ exists, then*

$$\lim_{n\to\infty} n\left\{{}^{\Psi}S_n^{\mu}(f;x)-f(x)\right\}=\frac{\Psi(x)}{2}\left(\frac{f''(x)}{\{\Psi'(x)\}^2}-\frac{f'(x)\Psi''(x)}{\{\Psi'(x)\}^3}\right) \tag{3.22}$$

where ${}^{\Psi}S_n^{\mu}$ is the operators defined in (1.6).

Proof. We have the following equality

$$\begin{aligned}\lim_{n\to\infty} n\left\{{}^{\Psi}S_n^{\mu}(f;x)-f(x)\right\} &= \lim_{n\to\infty} n\left\{S_n^{\mu}\left(f\circ\Psi^{-1};\Psi(x)\right)-\left(f\circ\Psi^{-1}\right)(\Psi(x))\right\}\\ &=\frac{\Psi(x)\left(f\circ\Psi^{-1}\right)''(\Psi(x))}{2}\\ &=\frac{\Psi(x)}{2}\left(\frac{f''(x)}{\{\Psi'(x)\}^2}-\frac{f'(x)\Psi''(x)}{\{\Psi'(x)\}^3}\right)\end{aligned}$$

by means of (1.7), Lemmas 4 and 7. So, the proof is done. □

At the present, we remember weighted modulus of continuity that is defined by İspir in [29] as follows

$$\Omega(f;\delta):=\sup_{|h|\leqslant\delta}\left\{\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}:\ x\in[0,\infty)\right\} \tag{3.23}$$

where $f\in C_{e_0+e_2}[0,\infty)=\left\{f\in C_{\rho}[0,\infty):\rho(x)=1+x^2\right\}$ and $\delta>0$. Moreover, modulus of continuity is defined by Holhos in [30] following that

$$\omega_{\Psi}(f;\delta):=\sup_{|\Psi(y)-\Psi(x)|\leqslant\delta}\left\{\frac{|f(y)-f(x)|}{(1+\Psi^2(y))+(1+\Psi^2(x))}:\ x,y\in[0,\infty)\right\} \tag{3.24}$$

where $f\in C_{\rho}[0,\infty)$ and $\delta>0$. It is clearly that $\omega_{\Psi}(f;\delta)\leqslant\Omega(f;\delta)$ for $\Psi(x)=x$ and $\delta\leqslant\frac{1}{\sqrt{2}}$. Moreover, the properties of modulus of continuity ω_{Ψ} is given by Holhos in [30] as follows:

i. For every $f \in U_\rho[0,\infty)$, it's $\lim_{\delta \to 0^+} \omega_\Psi(f;\delta) = 0$.
ii. For every $f \in C_\rho[0,\infty)$, we have

$$|f(y) - f(x)| \leqslant (\rho(x) + \rho(y)) \left(2 + \frac{|\Psi(y) - \Psi(x)|}{\delta}\right) \omega_\Psi(f;\delta) \tag{3.25}$$

where $x, y \geqslant 0$ and $\delta > 0$.

THEOREM 6. *Let Ψ be satisfies conditions G1 and G2, then for every $\delta_n(x) = \sqrt[4]{nM_{n,6}(x)}$ verified $\delta_n(x) < 1$, we have*

$$\begin{aligned} &\left| n\left\{{}^{\Psi}S_n^\mu(f;x) - f(x)\right\} - \frac{\left(1 + 2\mu\xi_\Psi^\mu(n,x)\right)\Psi(x)}{2}\left(f \circ \Psi^{-1}\right)''(\Psi(x)) \right| \\ &\leqslant 6\left\{\Psi^2(x) + \Psi(x) + 2\right\}\left[\left(1 + 2\mu\xi_\Psi^\mu(n,x)\right)\Psi(x) + 1\right] \\ &\quad \times \left(\omega_\Psi\left(\frac{f''}{(\Psi')^2};\delta_n(x)\right) + \omega_\Psi\left(\frac{f'\Psi''}{(\Psi')^3};\delta_n(x)\right)\right) \end{aligned}$$

where $\frac{f''}{(\Psi')^2}, \frac{f'\Psi''}{(\Psi')^3} \in C_\rho[0,\infty)$ and $x \geqslant 0$.

Proof. We prove this theorem making similar expansions in [18] and [20]. By Taylor formula of $f \circ \Psi^{-1}$ at $\Psi(x) \in [0,\infty)$, we have

$$\begin{aligned} f(t) &= \left(f \circ \Psi^{-1}\right)(\Psi(x)) + (\Psi(t) - \Psi(x))\left(f \circ \Psi^{-1}\right)'(\Psi(x)) \\ &\quad + \frac{(\Psi(t) - \Psi(x))^2}{2}\left(f \circ \Psi^{-1}\right)''(\Psi(x)) + \mu(t,x)(\Psi(t) - \Psi(x))^2 \end{aligned} \tag{3.26}$$

where $\mu(t,x)$ is given by

$$\mu(t,x) = \frac{\left(f \circ \Psi^{-1}\right)''(\Psi(c)) - \left(f \circ \Psi^{-1}\right)''(\Psi(x))}{2}$$

for c between x and t. Applying operator ${}^{\Psi}S_n^\mu$ to equation (3.26) and with help of Lemma 2, we obtain

$$\begin{aligned} &\left| {}^{\Psi}S_n^\mu(f;x) - f(x) - \frac{\left(1+2\mu\xi_\Psi^\mu(n,x)\right)\Psi(x)}{2n}\left(f \circ \Psi^{-1}\right)''(\Psi(x)) \right| \\ &\qquad \leqslant {}^{\Psi}S_n^\mu\left(|\mu(t,x)|(\Psi(t) - \Psi(x))^2;x\right). \end{aligned} \tag{3.27}$$

From (3.25) and Lemma 4, we can write that

$$\begin{aligned} \mu(t,x) &= \frac{\left(f \circ \Psi^{-1}\right)''(\Psi(c)) - \left(f \circ \Psi^{-1}\right)''(\Psi(x))}{2} \\ &= \frac{1}{2}\left[\frac{f''(c)}{(\Psi'(c))^2} - \frac{f''(x)}{(\Psi'(x))^2} + \frac{f'(x)\Psi''(x)}{(\Psi'(x))^3} - \frac{f'(c)\Psi''(c)}{(\Psi'(c))^3}\right] \end{aligned}$$

$$\leqslant (\rho(t)+\rho(x))\left(2+\frac{|\Psi(t)-\Psi(x)|}{\delta}\right)$$
$$\times\left(\omega_\Psi\left(\frac{f''}{(\Psi')^2};\delta\right)+\omega_\Psi\left(\frac{f'\Psi''}{(\Psi')^3};\delta\right)\right).$$

Furthermore, in the event of $|\Psi(y)-\Psi(x)|\leqslant\delta$ since

$$\rho(t)+\rho(x)\leqslant\delta^2+2\delta\Psi(x)+2\Psi^2(x)+2,$$

we get

$$|\mu(t,x)|\leqslant 3\left(\delta^2+2\delta\Psi(x)+2\Psi^2(x)+2\right)$$
$$\times\left(\omega_\Psi\left(\frac{f''}{(\Psi')^2};\delta\right)+\omega_\Psi\left(\frac{f'\Psi''}{(\Psi')^3};\delta\right)\right) \tag{3.28}$$

and whenever $|\Psi(t)-\Psi(x)|>\delta$ since

$$\rho(t)+\rho(x)\leqslant\left(\frac{\Psi(t)-\Psi(x)}{\delta}\right)^2\left\{\delta^2+2\Psi(x)\delta+2\Psi^2(x)+2\right\},$$

we have

$$|\mu(t,x)|\leqslant 3\left(\frac{\Psi(t)-\Psi(x)}{\delta}\right)^4\left\{\delta^2+2\Psi(x)\delta+2\Psi^2(x)+2\right\}$$
$$\times\left(\omega_\Psi\left(\frac{f''}{(\Psi')^2};\delta\right)+\omega_\Psi\left(\frac{f'\Psi''}{(\Psi')^3};\delta\right)\right). \tag{3.29}$$

By choosing $\delta<1$ and combining (3.28) and (3.29), we obtain

$$|\mu(t,x)|\leqslant 6\left\{\Psi^2(x)+\Psi(x)+2\right\}\left(\left(\frac{\Psi(t)-\Psi(x)}{\delta}\right)^4+1\right)$$
$$\times\left(\omega_\Psi\left(\frac{f''}{(\Psi')^2};\delta\right)+\omega_\Psi\left(\frac{f'\Psi''}{(\Psi')^3};\delta\right)\right). \tag{3.30}$$

It is clearly that ${}^{\Psi}S_n^{\mu}(f;x)$ operators are linear and positive. So, these operators are monoton increased. Using monotonicity of these operators, writting inequality (3.30) in (3.27) and multiplying with n to both sides of inequality (3.27), we have

$$\left|n\left\{{}^{\Psi}S_n^{\mu}(f;x)-f(x)\right\}-\frac{\left(1+2\mu\xi_\Psi^{\mu}(n,x)\right)\Psi(x)}{2}\left(f\circ\Psi^{-1}\right)''(\Psi(x))\right|$$
$$\leqslant 6n\left\{\Psi^2(x)+\Psi(x)+2\right\}\left(\frac{M_{n,6}(x)}{\delta^4}+M_{n,2}(x)\right)$$
$$\times\left(\omega_\Psi\left(\frac{f''}{(\Psi')^2};\delta\right)+\omega_\Psi\left(\frac{f'\Psi''}{(\Psi')^3};\delta\right)\right)$$

$$= 6\left\{\Psi^2(x)+\Psi(x)+2\right\}\left[\left(1+2\mu\xi_\Psi^\mu(n,x)\right)\Psi(x)+n\frac{M_{n,6}(x)}{\delta^4}\right]$$
$$\times\left(\omega_\Psi\left(\frac{f''}{(\Psi')^2};\delta\right)+\omega_\Psi\left(\frac{f'\Psi''}{(\Psi')^3};\delta\right)\right).$$

By choosing $\delta := \delta_n(x) = \sqrt[4]{nM_{n,6}(x)}$ in the above inequality, we finish the proof. □

COROLLARY 1. *The followings holds:*

i. Let $f'' \in C_{e_0+e_2}[0,\infty)$. By choosing $\Psi(x)=x$ in Theorem 6, we obtain the quantative Voronovskaya theorem for Dunkl analogue of Szász operators defined by Sucu in [6] *following that*

$$\left|n\left\{S_n^\mu(f;x)-f(x)\right\}-f''(x)\frac{\left(1+2\mu\xi_n^\mu(x)\right)x}{2}\right|$$
$$\leqslant 24(1+x)^3\,\Omega\left(f'';\delta_n^{e_1}(x)\right)$$

for $\delta_n^{e_1}(x)$ verified $\delta_n^{e_1}(x)\leqslant\frac{1}{\sqrt{2}}$ where $\xi_n^\mu(x)=\frac{e_\mu(-nx)}{e_\mu(nx)}$,

$$\delta_n^{e_1}(x) = \sqrt[4]{nS_n^\mu\left((t-x)^6;x\right)}$$
$$= \sqrt[4]{\begin{array}{l}160\mu\xi_n^\mu(x)x^5-\left(80\mu^2+440\mu\xi_n^\mu(x)\right)\frac{x^4}{n}\\ +\left(5+260\mu^2+360\left(\mu+\mu^3\right)\xi_n^\mu(x)\right)\frac{x^3}{n^2}\\ -\left(160\mu^2+80\mu^4-45+\mu\left(60+400\mu^2\right)\xi_n^\mu(x)\right)\frac{x^2}{n^3}\\ +\left(1+40\mu^2\left(1+2\mu^2\right)+\left(10\mu+80\mu^3+32\mu^5\right)\xi_n^\mu(x)\right)\frac{x}{n^4}\end{array}}$$

and $\delta_n^{e_1}(x)\to 0$ as $n\to\infty$ from Lemma 6.

ii. Let $\frac{f''}{(\Psi')^2},\frac{f'\Psi''}{(\Psi')^3}\in U_\rho[0,\infty)$. By taking limit for $n\to\infty$ in Theorem 6 and from Lemma 6, we have the Voronovskaya theorem for our operators defined by (1.6) *as follows*

$$\lim_{n\to\infty} n\left\{{}^\Psi S_n^\mu(f;x)-f(x)\right\} = \frac{\Psi(x)}{2}\left(f\circ\Psi^{-1}\right)''(\Psi(x))$$
$$= \frac{\Psi(x)}{2}\left(\frac{f''(x)}{\{\Psi'(x)\}^2}-\frac{f'(x)\Psi''(x)}{\{\Psi'(x)\}^3}\right).$$

iii. Let $f''\in U_\rho[0,\infty)$. By taking limit for $n\to\infty$ and choosing $\Psi(x)=x$ in Theorem 6, we have the Voronovskaya theorem for Dunkl analogue of Szász operators defined by Sucu in [6] *following that $\lim_{n\to\infty} n\left\{S_n^\mu(f;x)-f(x)\right\}=\frac{x}{2}f''(x)$.*

4. Numerical results

In this section, we resolve the theoretical results given in the previous sections by numerical examples. Now, we select Ψ satisfying G1 and G2 as follows

$$\Psi(x) = (x+1)^2 - 1.$$

By regarding to Ψ, we observe the following example.

EXAMPLE 1. Let $\mu = \frac{1}{3}$, $f(x) = \frac{2(x+1)^2}{(x+1)^2+1}$ and $\varepsilon_{n,\mu}^{\Psi}(f;x) = \left|{}^{\Psi}S_n^{\mu}(f;x) - f(x)\right|$ be the error of approximation to $f(x)$ of ${}^{\Psi}S_n^{\mu}(f;x)$. The graphs of $f(x)$ and ${}^{\Psi}S_n^{\mu}(f;x)$ for $n = 1,2,4,12$ on interval $[0,2]$ are given, respectively in Figure 1. Moreover, the graphs of $\varepsilon_{n,\mu}^{\Psi}(f;x)$ for these values n on interval $[0,10]$ are given in Figure 2. These graphs show that if n is increased, the approximation to $f(x)$ of ${}^{\Psi}S_n^{\mu}(f;x)$ increase and error $\varepsilon_{n,\mu}^{\Psi}(f;x)$ decrease. Note that operators ${}^{\Psi}S_n^{\mu}(f;x)$ verify Proposition 1 thanks to $\left|{}^{\Psi}S_n^{\mu}(f;x)\right| \leqslant |f(x)|$ on Figure 1.

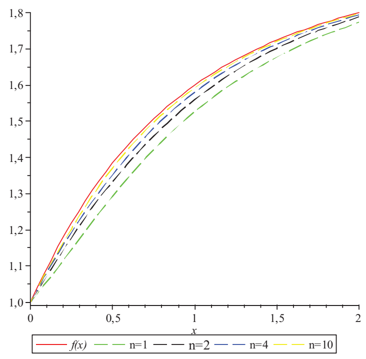

Figure 1: *Approximation Processes*

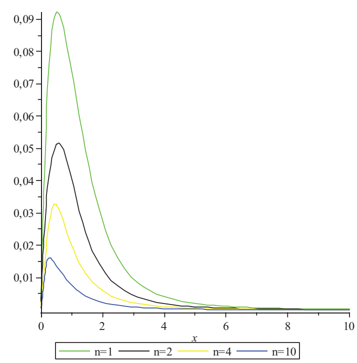

Figure 2: *Approximation Errors*

EXAMPLE 2. Let $\mu = \frac{1}{3}$, $f(x) = \sin((x+1)^2)$ and $\varepsilon_{n,\mu}^{\Psi}(f;x) = \left|{}^{\Psi}S_n^{\mu}(f;x) - f(x)\right|$ be the error of approximation to $f(x)$ of ${}^{\Psi}S_n^{\mu}(f;x)$. The graphs of $f(x)$ and ${}^{\Psi}S_n^{\mu}(f;x)$ for $n = 6,9,12,15,25,50$ on interval $[0,2]$ are given, respectively in Figure 3. Moreover, the graphs of $\varepsilon_{n,\mu}^{\Psi}(f;x)$ for these values n on this interval are given in Figure 4. These graphs show that if n is increased, the approximation to $f(x)$ of ${}^{\Psi}S_n^{\mu}(f;x)$ increase and error $\varepsilon_{n,\mu}^{\Psi}(f;x)$ decrease. Furthermore, note that operators ${}^{\Psi}S_n^{\mu}(f;x)$ satisfy Proposition 1 thanks to $\left|{}^{\Psi}S_n^{\mu}(f;x)\right| \leqslant |f(x)|$ on Figure 3.

EXAMPLE 3. Let $\mu = \frac{1}{5}$, $f(x) = \frac{2(x+1)^2}{(x+1)^2+1}$. In Figures 5 and 6, we compare the approximation to $f(x)$ of ${}^{\Psi}S_n^{\mu}(f;x)$ with $S_n^{\mu}(f;x)$, original operator, for $n = 5$. Then, we compare with these operators for $n = 10$ in Figures 7 and 8. Consequently, we obtain better approximation to $f(x)$ for ${}^{\Psi}S_n^{\mu}(f;x)$ than $S_n^{\mu}(f;x)$.

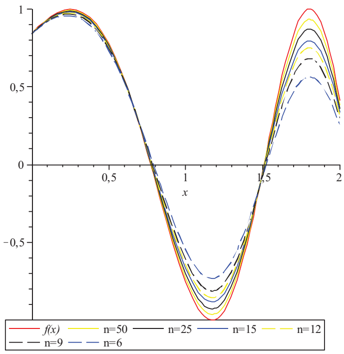

Figure 3: *Approximation Processes*

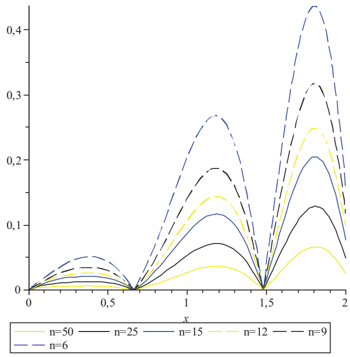

Figure 4: *Approximation Errors*

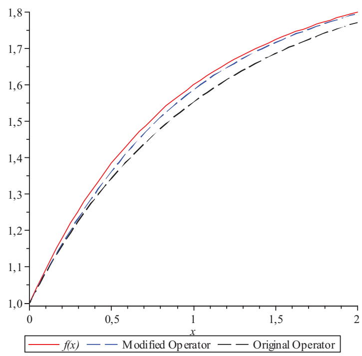

Figure 5: *Comparison of Approximation for* $n = 5$ *and* $\mu = 1/5$

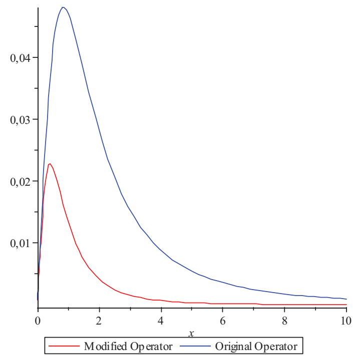

Figure 6: *Comparison of Error for* $n = 5$ *and* $\mu = 1/5$

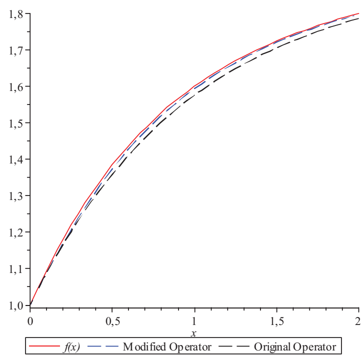

Figure 7: *Comparison of Approximation for* $n = 10$ *and* $\mu = 1/5$

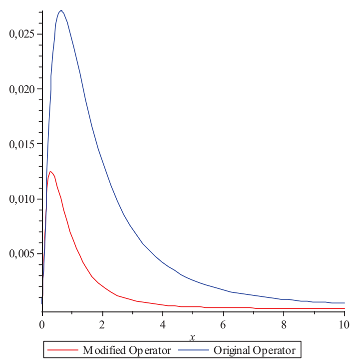

Figure 8: *Comparison of Error for* $n = 10$ *and* $\mu = 1/5$

Acknowledgement. This work is supported by the Natural Science Foundation of Fujian Province of China (Grant No. 2020J01783), the Project for High-level Talent Innovation and Entrepreneurship of Quanzhou (Grant No. 2018C087R) and the Program for New Century Excellent Talents in Fujian Province University. We also thank Fujian Provincial Key Laboratory of Data-Intensive Computing, Fujian University Laboratory of Intelligent Computing and Information Processing and Fujian Provincial Big Data Research Institute of Intelligent Manufacturing of China.

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(Received September 8, 2020)

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