Journal of
Mathematical
Inequalities
Volume 15, Number 3 (2021), 899–910 doi:10.7153/jmi-2021-15-62

A COMPLETE CONVERGENCE THEOREM FOR WEIGHTED SUMS UNDER THE SUB–LINEAR EXPECTATIONS

FENGXIANG FENG* AND HAIWU HUANG

(*Communicated by X. Wang*)

Abstract. In this article, we study a complete convergence theorem for weighted sums in sub-linear expectations space. We establish a complete convergence theorem for weighted sums under the optimal moment conditions in sub-linear expectations space. Our result extends and improves the corresponding result of Cai (Metrika, 68:323-331, 2008) in some extent.

1. Introduction and notation

In the classical probability theory, probability and expectation are both additive. But the uncertainty phenomenon can not be modeled using additive probabilities or additive expectations in many areas of applications. Non-additive probabilities and non-additive expectations are useful tools for studying uncertainties in statistics, measures of risk, superhedging in finance and non-linear stochastic calculus [1–7]. Peng [6–8] introduced the general framework of the sub-linear expectation in a general function space by relaxing the linear property of the classical expectation to the sub-additivity and positive homogeneity (cf. Definition 1.1 below). Under Peng's sub-linear expectation framework, many limit theorems have been established recently, including the central limit theorem and weak law of large numbers [8–10], strong law of large numbers [11–15], the law of the iterated logarithm [16–17], Donsker's invariance principle and Chung's law of the iterated logarithm [18], the moment inequalities for the maximum partial sums and the Kolomogov strong law of large numbers [19], complete convergence theorems [20–22], self-normalized moderate deviation and law of the iterated logarithm [23], the asymptotic approximation of inverse moment [24], and so on. Because sub-linear expectation and capacity are not additive, the study of the limit theorems under sub-linear expectation becomes much more complex and challenging. Extending the limit theorems in the traditional probability space to the case of sub-linear expectation space is of great significance in the theory and application.

Complete convergence theorems are important limit theorems in probability theory. Many of related results have been obtained in the probability space. We refer the reader to [25–31]. Complete convergence for weighted sums are also important in sub-linear expectation space, which can be applied to nonparametric regression models [22].

Mathematics subject classification (2020): 60F15.
Keywords and phrases: Complete convergence, weighted sums, sub-linear expectations.
* Corresponding author.

Paper JMI-15-62

Feng et al. [20] and Zhong and Wu [21] established complete convergence theorems in sub-linear expectations space. We will establish a complete convergence theorem for weighted sums under the optimal moment conditions in sub-linear expectations space. Our complete convergence theorem is different from them. We prove our result by using capacity inequality under sub-linear expectations, fully combining the properties of sub-linear expectations, skillfully using local Lipschitz function, truncating the random variables and weights, and so on.

We use the framework and notations of Peng [8]. Let $(\Omega,\mathscr{F})$ be a given measurable space and let $\mathscr{H}$ be a linear space of real functions defined on $(\Omega,\mathscr{F})$ such that if $X_1,\cdots,X_n \in \mathscr{H}$ then $\varphi(X_1,\cdots,X_n) \in \mathscr{H}$ for each $\varphi \in C_{l.Lip}(\mathbb{R}^n)$, where $C_{l.Lip}(\mathbb{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(x)-\varphi(y)| \leqslant C(1+|x|^m+|y|^m)|x-y|, \ \forall x,y \in \mathbb{R}^n,$$

for some $C>0, m \in \mathbb{N}$ depending on φ. $\mathscr{H}$ is considered as a space of "random variables". If X is an element of $\mathscr{H}$, then we denote $X \in \mathscr{H}$.

DEFINITION 1.1. A sub-linear expectation $\widehat{\mathbb{E}}$ on $\mathscr{H}$ is a function $\widehat{\mathbb{E}}:\mathscr{H} \to \overline{\mathbb{R}}$ satisfying the following properties: for all $X,Y \in \mathscr{H}$, we have

(a) Monotonicity: If $X \geqslant Y$ then $\widehat{\mathbb{E}}[X] \geqslant \widehat{\mathbb{E}}[Y]$;

(b) Constant preserving: $\widehat{\mathbb{E}}[c]=c$;

(c) Sub-additivity: $\widehat{\mathbb{E}}[X+Y] \leqslant \widehat{\mathbb{E}}[X]+\widehat{\mathbb{E}}[Y]$ whenever $\widehat{\mathbb{E}}[X]+\widehat{\mathbb{E}}[Y]$ is not of the form $+\infty-\infty$ or $-\infty+\infty$;

(d) Positive homogeneity: $\widehat{\mathbb{E}}[\lambda X]=\lambda\widehat{\mathbb{E}}[X], \lambda>0$.

Here $\overline{\mathbb{R}}=[-\infty,+\infty]$. The triple $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$ is called a sub-linear expectation space. Given a sub-linear expectation $\widehat{\mathbb{E}}$, let us denote the conjugate expectation $\widehat{\mathscr{E}}$ of $\widehat{\mathbb{E}}$ by

$$\widehat{\mathscr{E}}[X]:=-\widehat{\mathbb{E}}[-X], \ \forall X \in \mathscr{H}.$$

From the definition, we can easily get that $\widehat{\mathscr{E}}[X] \leqslant \widehat{\mathbb{E}}[X]$, $\widehat{\mathbb{E}}[X+c]=\widehat{\mathbb{E}}[X]+c$, $\widehat{\mathbb{E}}[X-Y] \geqslant \widehat{\mathbb{E}}[X]-\widehat{\mathbb{E}}[Y]$ and $|\widehat{\mathbb{E}}[X]-\widehat{\mathbb{E}}[Y]| \leqslant \widehat{\mathbb{E}}[|X-Y|]$. Further, if $\widehat{\mathbb{E}}[|X|]$ is finite, then $\widehat{\mathscr{E}}[X]$ and $\widehat{\mathbb{E}}[X]$ are both finite.

DEFINITION 1.2. (See[8]). (i) (Identical distribution) Let X_1 and X_2 be two n-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1,\mathscr{H}_1,\widehat{\mathbb{E}}_1)$ and $(\Omega_2,\mathscr{H}_2,\widehat{\mathbb{E}}_2)$. They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$, if $\widehat{\mathbb{E}}_1[\varphi(X_1)]=\widehat{\mathbb{E}}_2[\varphi(X_2)], \quad \forall \varphi \in C_{l.Lip}(\mathbb{R}^n)$, whenever the sub-expectations are finite.

(ii) (Independence) In a sub-linear expectation space $(\Omega,\mathscr{H},\widehat{\mathbb{E}})$, a random vector $Y=(Y_1,Y_2,\cdots,Y_n)$, $Y_i \in \mathscr{H}$ is said to be independent to another random vector $X=(X_1,X_2,\cdots,X_m)$, $X_i \in \mathscr{H}$ under $\widehat{\mathbb{E}}$ if for each test function $\varphi \in C_{l.Lip}(\mathbb{R}^m \times \mathbb{R}^n)$ we have $\widehat{\mathbb{E}}[\varphi(X,Y)]=\widehat{\mathbb{E}}[\widehat{\mathbb{E}}[\varphi(x,Y)]|_{x=X}]$, whenever $\bar{\varphi}(x):=\widehat{\mathbb{E}}[|\varphi(x,Y)|]<\infty$ for all x and $\widehat{\mathbb{E}}[|\bar{\varphi}(X)|]<\infty$.

(iii) (IID random variables) A sequence of random variables $\{X_n;n \geqslant 1\}$ is said to be independent if X_{i+1} is independent to $(X_1,X_2,\cdots,X_i)$ for each $i \geqslant 1$, and it is said to be identically distributed if $X_i \overset{d}{=} X_1$, for each $i \geqslant 1$.

We omit the definitions of extended independence and Negative dependence. For these definitions, please refer to [8, 32, 17]. In view of the definition of identically distribution, if $\{X, X_n; n \geqslant 1\}$ is a sequence of identically distributed random variables in the sub-linear expectation space $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, then $\widehat{\mathbb{E}}[\varphi(X_n)] = \widehat{\mathbb{E}}[\varphi(X)]$, $\forall \varphi \in C_{l.Lip}(\mathbb{R}), n \geqslant 1$. It can be showed that the independence implies the extended independence [32].

Next, we introduce the capacities corresponding to the sub-linear expectations. Let $\mathscr{G} \subset \mathscr{F}$. A function $V : \mathscr{G} \to [0,1]$ is called a capacity if

$$V(\phi) = 0,\ V(\Omega) = 1,\ \text{ and } V(A) \leqslant V(B)\ \ \forall A \subset B, A, B \in \mathscr{G}.$$

It is called to be sub-additive if $V(A \cup B) \leqslant V(A) + V(B)$ for all $A, B \in \mathscr{G}$ with $A \cup B \in \mathscr{G}$.

Let $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$ be a sub-linear space, and $\widehat{\mathscr{E}}$ be the conjugate expectation of $\widehat{\mathbb{E}}$. We denote a pair $(\mathbb{V}, \mathscr{V})$ of capacities by

$$\mathbb{V}(A) := \inf\{\widehat{\mathbb{E}}[\xi] : I_A \leqslant \xi, \xi \in \mathscr{H}\},\ \mathscr{V}(A) := 1 - \mathbb{V}(A^c),\ \forall A \in \mathscr{F},$$

where A^c is the complement set of A. It is obvious that $\mathbb{V}$ is sub-additive and

$$\widehat{\mathbb{E}}[f] \leqslant \mathbb{V}(A) \leqslant \widehat{\mathbb{E}}[g],\ \widehat{\mathscr{E}}[f] \leqslant \mathscr{V}(A) \leqslant \widehat{\mathscr{E}}[g],\ \text{if } f \leqslant I_A \leqslant g, f, g \in \mathscr{H}. \tag{1.1}$$

This implies Markov inequality: $\forall X \in \mathscr{H}$,

$$\mathbb{V}(|X| \geqslant x) \leqslant \widehat{\mathbb{E}}[|X|^p]/x^p,\ \ \forall x > 0, p > 0$$

from $I(|X| \geqslant x) \leqslant |X|^p/x^p \in \mathscr{H}$. By Lemma 4.1 of [17], we have Hölder inequality: $\forall X, Y \in \mathscr{H}, p, q > 1$, satisfying $p^{-1} + q^{-1} = 1$,

$$\widehat{\mathbb{E}}[|XY|] \leqslant (\widehat{\mathbb{E}}[|X|^p])^{\frac{1}{p}} (\widehat{\mathbb{E}}[|Y|^q])^{\frac{1}{q}}.$$

Particularly, Jensen inequality:

$$(\widehat{\mathbb{E}}[|X|^r])^{\frac{1}{r}} \leqslant (\widehat{\mathbb{E}}[|X|^s])^{\frac{1}{s}},\ \text{ for } 0 < r \leqslant s.$$

We define the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathscr{V}})$ by

$$C_V[X] := \int_0^\infty V(X \geqslant x)dx + \int_{-\infty}^0 (V(X \geqslant x) - 1)dx$$

with V being replaced by $\mathbb{V}$ and $\mathscr{V}$, respectively. If $\lim_{c\to\infty} \widehat{\mathbb{E}}[(|X| - c)^+] = 0$, then $\widehat{\mathbb{E}}[|X|] \leqslant C_{\mathbb{V}}[|X|]$. (see Lemma 4.5(iii) of [17])

Throughout this paper, C stands for a positive constant which may differ from one place to another. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leqslant cb_n$ for sufficiently large n, $I(.)$ denote an indicator function.

2. Main results

THEOREM 2.1. *Let* $\{X, X_n; n \geqslant 1\}$ *be a sequence of independent identically distributed random variables in the sub-linear expectation space* $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$. *Let* $1 < \alpha < 2$ *and* $\alpha < \gamma$. *Set* $b_n = n^{1/\alpha}(\log n)^{1/\gamma}$. *Assume that* $\{a_{ni}; 1 \leqslant i \leqslant n,\ n \geqslant 1\}$ *is an array of real numbers satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n). \tag{2.1}$$

If $\widehat{\mathbb{E}}[|X|^{\gamma}] \leqslant C_{\mathbb{V}}[|X|^{\gamma}]$, $\sum_{k=1}^{\infty} k^{\gamma} \mathbb{V}(k < |X| \leqslant k+1) < \infty$ *and* $\widehat{\mathbb{E}}[X] = \widehat{\mathscr{E}}[X] = 0$, *then for any* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(|\sum_{i=1}^{n} a_{ni} X_i| > \varepsilon b_n \right) < \infty. \tag{2.2}$$

Conversely, if (2.2) holds for any positive array $\{a_{ni}\}$ *satisfying (2.1), then* $C_{\mathbb{V}}[|X|^{\gamma}] < \infty$.

REMARK 2.1. In the classical probability space $\sum_{k=1}^{\infty} k^{\gamma} \mathbb{V}(k < |X| \leqslant k+1) < \infty \Leftrightarrow E[|X|^{\gamma}] < \infty$. We give the condition $\sum_{k=1}^{\infty} k^{\gamma} \mathbb{V}(k < |X| \leqslant k+1) < \infty$ in sub-linear expectation space is equality to the moment condition in the classical probability space. By $\sum_{k=1}^{\infty} k^{\gamma} \mathbb{V}(k < |X| \leqslant k+1) < \infty$, we have $C_{\mathbb{V}}[|X|^{\gamma}] < \infty$. But $C_{\mathbb{V}}[|X|^{\gamma}] < \infty$ dose not imply $\sum_{k=1}^{\infty} k^{\gamma} \mathbb{V}(k < |X| \leqslant k+1) < \infty$ in sub-linear expectation space.

REMARK 2.2. Cai [34] obtained analogous result of (2.2) under much stronger moment condition $E\exp(h|X|^{\gamma}) < \infty$ in the classical probability space. Theorem 2.1 is established under the optimal moment conditions. Our Theorem 2.1 extends and improves the corresponding result of Cai [34] in some extent.

3. Proofs of main results

In order to prove our results, we need the following lemmas.

LEMMA 3.1. ([17]) *Let* $\{X_n; n \geqslant 1\}$ *be a sequence of negatively dependent random variables in* $(\Omega, \mathscr{H}, \widehat{\mathbb{E}})$, *with* $\widehat{\mathbb{E}}[X_n] \leqslant 0$. *Let* $S_n = \sum_{i=1}^{n} X_i$, $B_n = \sum_{i=1}^{n} \widehat{\mathbb{E}}[X_i^2]$. *Then for any* $q \geqslant 2$, *there exists a constant* $C_q \geqslant 1$ *such that for all* $x > 0$ *and* $0 < \delta \leqslant 1$

$$\mathbb{V}(S_n \geqslant x) \leqslant C_q \delta^{-2q} \frac{\sum_{i=1}^{n} \widehat{\mathbb{E}}[|X_i|^q]}{x^q} + \exp\left(-\frac{x^2}{2B_n(1+\delta)} \right).$$

REMARK 3.1. By the fact if Y is independent to X, then Y is negatively dependent to X [19], obviously Lemma 3.1 holds for independent random variables sequence.

LEMMA 3.2. *Under the conditions of Theorem 2.1 , we have*

$$I := \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{V}(|a_{ni} X| > b_n) < \infty.$$

Proof. (i) When $|a_{ni}| \leqslant 1$, we have

$$\begin{aligned} I &\leqslant \sum_{n=1}^{\infty} \frac{1}{n} \sum_{i=1}^{n} \mathbb{V}(|X| > b_n) \\ &\leqslant \sum_{n=1}^{\infty} b_n^{-\gamma} \widehat{\mathbb{E}}[|X|^\gamma] \\ &= \sum_{n=1}^{\infty} n^{-\gamma/\alpha} (\log n)^{-1} \widehat{\mathbb{E}}[|X|^\gamma] < \infty. \end{aligned}$$

(ii) When $|a_{ni}| > 1$, similar to the proof of Lemma 2.3 (replace P by $\mathbb{V}$) of [33], we have

$$\begin{aligned} I &\leqslant \sum_{k=1}^{\infty} k^\gamma \mathbb{V}(k < |X| \leqslant k+1) + \sum_{n=1}^{\infty} \sum_{k=[n^{1/\alpha}(\log n)^{1/\gamma}]+1}^{\infty} \mathbb{V}(k < |X| \leqslant k+1) \\ &=: I_1 + I_2. \end{aligned}$$

By the condition of Theorem 2.1, we have $I_1 < \infty$. Note that

$$\begin{aligned} I_2 &\leqslant C \sum_{k=1}^{\infty} k^\alpha / (\log k)^{\alpha/\gamma} \mathbb{V}(k < |X| \leqslant k+1) \\ &\leqslant C \sum_{k=1}^{\infty} k^\alpha \mathbb{V}(k < |X| \leqslant k+1) < \sum_{k=1}^{\infty} k^\gamma \mathbb{V}(k < |X| \leqslant k+1) < \infty. \end{aligned}$$

We complete the proof of Lemma 3.2. □

Proof of Theorem 2.1. We may assume that $\sum_{i=1}^{n} |a_{ni}|^\alpha \leqslant n$. Since $a_{ni} = a_{ni}^+ - a_{ni}^-$, we also assume that $a_{ni} > 0$. We just need to prove

$$\sum_{n=1}^{\infty} \frac{1}{n} \mathbb{V}\left(\sum_{i=1}^{n} a_{ni} X_i > \varepsilon b_n \right) < \infty \tag{3.1}$$

because of considering $\{-X_i; i \geqslant 1\}$ instead of $\{X_i; i \geqslant 1\}$ in (3.1), we can obtain (2.2). For $1 \leqslant i \leqslant n$ and $n \geqslant 1$, let

$$\begin{aligned} X_{ni}^{(1)} &= -b_n (\log n)^{-\beta} I(a_{ni} X_i < -b_n (\log n)^{-\beta}) + a_{ni} X_i I(|a_{ni} X_i| \leqslant b_n (\log n)^{-\beta}) \\ &\quad + b_n (\log n)^{-\beta} I(a_{ni} X_i > b_n (\log n)^{-\beta}), \\ X_{ni}^{(2)} &= (a_{ni} X_i - b_n (\log n)^{-\beta}) I(b_n (\log n)^{-\beta} < a_{ni} X_i \leqslant \varepsilon b_n / (4N)), \\ X_{ni}^{(3)} &= (a_{ni} X_i + b_n (\log n)^{-\beta}) I(-\varepsilon b_n / (4N) \leqslant a_{ni} X_i < -b_n (\log n)^{-\beta}), \\ X_{ni}^{(4)} &= (a_{ni} X_i - b_n (\log n)^{-\beta}) I(a_{ni} X_i > \varepsilon b_n / (4N)) + (a_{ni} X_i + b_n (\log n)^{-\beta}) I(a_{ni} X_i \\ &\quad < -\varepsilon b_n / (4N)), \end{aligned}$$

where $0<\beta<1/\gamma$ and N is large enough. Then $a_{ni}X_i = X_{ni}^{(1)} + X_{ni}^{(2)} + X_{ni}^{(3)} + X_{ni}^{(4)}$ and $\{X_{ni}^{(1)}, 1\leqslant i\leqslant n, n\geqslant 1\}$ is a sequence of independent random variables. It follows that

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\sum_{i=1}^{n}a_{ni}X_i>\varepsilon b_n\right)\\
\leqslant&\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\sum_{i=1}^{n}X_{ni}^{(1)}>\varepsilon b_n/4\right)+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\sum_{i=1}^{n}X_{ni}^{(2)}>\varepsilon b_n/4\right)\\
&+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\sum_{i=1}^{n}X_{ni}^{(3)}>\varepsilon b_n/4\right)+\sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\sum_{i=1}^{n}X_{ni}^{(4)}>\varepsilon b_n/4\right)\\
:=&J_1+J_2+J_3+J_4.
\end{aligned}$$

In order to prove $J_1<\infty$, we first show that

$$\frac{1}{b_n}\left|\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)}]\right|\to 0,\ n\to\infty. \tag{3.2}$$

For $0<\mu<1$, let $g(x)\in C_{l.Lip}(\mathbb{R})$, $0\leqslant g(x)\leqslant 1$ for all x, $g(x)=1$ if $|x|\leqslant\mu$, $g(x)=0$ if $|x|>1$ and $g(x)$ is non-increasing function when $x>0$. Then

$$I(|x|\leqslant\mu)\leqslant g(x)\leqslant I(|x|\leqslant 1),\ I(|x|>1)\leqslant 1-g(x)\leqslant I(|x|>\mu). \tag{3.3}$$

In view of $\widehat{\mathbb{E}}[X_i]=0$, we have that

$$\begin{aligned}
&{b_n}^{-1}\left|\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)}]\right|\\
&\leqslant {b_n}^{-1}\sum_{i=1}^{n}|\widehat{\mathbb{E}}[X_{ni}^{(1)}]|\\
&= {b_n}^{-1}\sum_{i=1}^{n}|\widehat{\mathbb{E}}[a_{ni}X_i]-\widehat{\mathbb{E}}[X_{ni}^{(1)}]|\\
&\leqslant {b_n}^{-1}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|a_{ni}X_i-X_{ni}^{(1)}|]\\
&= {b_n}^{-1}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|(a_{ni}X_i+b_n(\log n)^{-\beta})I(a_{ni}X_i<-b_n(\log n)^{-\beta})\\
&\quad+(a_{ni}X_i-b_n(\log n)^{-\beta})I(a_{ni}X_i>b_n(\log n)^{-\beta})|]\\
&\leqslant 2{b_n}^{-1}\sum_{i=1}^{n}|a_{ni}|\widehat{\mathbb{E}}|X|\left(1-g\left(\frac{a_{ni}(\log n)^{\beta}X}{b_n}\right)\right)\\
&\leqslant 2{b_n}^{-\alpha}(\log n)^{\beta(\alpha-1)}\sum_{i=1}^{n}|a_{ni}|^{\alpha}\widehat{\mathbb{E}}|X|^{\alpha}\left(1-g\left(\frac{a_{ni}(\log n)^{\beta}X}{b_n}\right)\right)\\
&= C\widehat{\mathbb{E}}[|X|^{\alpha}](\log n)^{\beta(\alpha-1)-\alpha/\gamma}\to 0
\end{aligned}$$

as $n\to\infty$, since $0<\beta<1/\gamma$ and $\widehat{\mathbb{E}}|X|^{\alpha}\leqslant(\widehat{\mathbb{E}}[|X|^{\gamma}])^{\alpha/\gamma}<\infty$. In order to prove that $J_1<\infty$, it is enough to show that

$$J_1' := \sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(\sum_{i=1}^{n}(X_{ni}^{(1)}-\widehat{\mathbb{E}}[X_{ni}^{(1)}])>\varepsilon b_n/8\right)<\infty. \tag{3.4}$$

Note that for any $m>0$, by C_r inequality and (3.3), we have

$$\begin{aligned}|X_{ni}^{(1)}|^m &\ll |a_{ni}|^m|X_i|^m I(|a_{ni}X_i|\leqslant b_n(\log n)^{-\beta})+b_n^m(\log n)^{-m\beta}I(|a_{ni}X_i|>b_n(\log n)^{-\beta})\\ &\leqslant |a_{ni}|^m|X_i|^m g\left(\frac{\mu a_{ni}(\log n)^{\beta}X_i}{b_n}\right)+b_n^m(\log n)^{-m\beta}\left(1-g\left(\frac{a_{ni}(\log n)^{\beta}X_i}{b_n}\right)\right),\end{aligned}$$

thus

$$\begin{aligned}\widehat{\mathbb{E}}[|X_{ni}^{(1)}|^m] &\ll |a_{ni}|^m\widehat{\mathbb{E}}\left[|X_i|^m g\left(\frac{\mu a_{ni}(\log n)^{\beta}X_i}{b_n}\right)\right]\\ &\quad + b_n^m(\log n)^{-m\beta}\mathbb{V}(|a_{ni}X_i|>\mu b_n(\log n)^{-\beta}).\end{aligned} \tag{3.5}$$

We will prove $J_1'<\infty$ in two cases ($\gamma<2$ and $\gamma\geqslant 2$). When $\alpha<\gamma<2$, by Markov's inequality, Lemma 3.1 and (3.5), we have

$$\begin{aligned}J_1' &\leqslant C\sum_{n=1}^{\infty}n^{-1}b_n^{-2}\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)}-\widehat{\mathbb{E}}[X_{ni}^{(1)}]]^2+C\sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{Cb_n^2}{\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)}-\widehat{\mathbb{E}}[X_{ni}^{(1)}]]^2}\right)\\
&\leqslant C\sum_{n=1}^{\infty}n^{-1}b_n^{-2}\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)2}]+C\sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{Cb_n^2}{\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)}]^2}\right)\\
&\leqslant C\sum_{n=1}^{\infty}n^{-1}b_n^{-2}\sum_{i=1}^{n}\left[a_{ni}^2\widehat{\mathbb{E}}\left[X_i^2 g\left(\frac{\mu a_{ni}(\log n)^{\beta}X_i}{b_n}\right)\right]\right.\\
&\qquad\left.+b_n^2(\log n)^{-2\beta}\mathbb{V}(|a_{ni}X_i|>\mu b_n(\log n)^{-\beta})\right]\\
&\qquad+C\sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{Cb_n^2}{\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)}]^2}\right)\\
&\leqslant C\sum_{n=1}^{\infty}n^{-1}b_n^{-2}\sum_{i=1}^{n}|a_{ni}|^{\gamma}\widehat{\mathbb{E}}|X|^{\gamma}(b_n(\log n)^{-\beta})^{2-\gamma}\\
&\qquad+C\sum_{n=1}^{\infty}n^{-1}b_n^{-2}b_n^2(\log n)^{-2\beta}(b_n(\log n)^{-\beta})^{-\gamma}\sum_{i=1}^{n}|a_{ni}|^{\gamma}\widehat{\mathbb{E}}|X|^{\gamma}\\
&\qquad+C\sum_{n=1}^{\infty}n^{-1}\exp\left(-\frac{Cb_n^2}{\sum_{i=1}^{n}\widehat{\mathbb{E}}[X_{ni}^{(1)}]^2}\right)\\
&\leqslant C\sum_{n=1}^{\infty}n^{-1}b_n^{-2}(b_n(\log n)^{-\beta})^{2-\gamma}\sum_{i=1}^{n}|a_{ni}|^{\gamma}\widehat{\mathbb{E}}|X|^{\gamma}\end{aligned}$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty} n^{-1}\exp\left(-\frac{Cb_n^2}{(b_n(\log n)^{-\beta})^{2-\gamma}\sum_{i=1}^n |a_{ni}|^\gamma \widehat{\mathbb{E}}|X|^\gamma}\right)\\
&= C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-\beta(2-\gamma)})(b_n)^{-\gamma}(\sum_{i=1}^{n}|a_{ni}|^{\alpha})^{\gamma/\alpha}\widehat{\mathbb{E}}|X|^\gamma\\
&+C\sum_{n=1}^{\infty} n^{-1}\exp\left(-\frac{Cb_n^2}{(b_n(\log n)^{-\beta})^{2-\gamma}\sum_{i=1}^n |a_{ni}|^\gamma \widehat{\mathbb{E}}|X|^\gamma}\right)\\
&\leqslant C\sum_{n=1}^{\infty}\frac{1}{n}(\log n)^{-\beta(2-\gamma)-1} + C\sum_{n=1}^{\infty} n^{-1}\exp\left(-C(\log n)^{1+\beta(2-\gamma)}\right)\\
&<\infty.
\end{aligned}
\tag{3.6}
$$

When $\gamma \geqslant 2$, taking $p>\gamma$, by Lemma 3.1, we have

$$
\begin{aligned}
J_1^{'} &\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-p}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|X_{ni}^{(1)}-\widehat{\mathbb{E}}[X_{ni}^{(1)}]|^p]\\
&\quad + C\sum_{n=1}^{\infty} n^{-1}\exp\left(-\frac{Cb_n^2}{\sum_{i=1}^n \widehat{\mathbb{E}}[X_{ni}^{(1)}-\widehat{\mathbb{E}}[X_{ni}^{(1)}]]^2}\right)\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-p}\sum_{i=1}^{n}\widehat{\mathbb{E}}[|X_{ni}^{(1)}|^p] + C\sum_{n=1}^{\infty} n^{-1}\exp\left(-\frac{Cb_n^2}{\sum_{i=1}^n \widehat{\mathbb{E}}[X_{ni}^{(1)}]^2}\right)\\
&:= J_{11}^{'} + J_{12}^{'}.
\end{aligned}
\tag{3.7}
$$

From the prove of (3.6), we have $J_{12}^{'}<\infty$. By (3.5), we have that

$$
\begin{aligned}
J_{11}^{'} &\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-p}\sum_{i=1}^{n}\left(\widehat{\mathbb{E}}\left[|a_{ni}|^p|X_i|^p g\left(\frac{\mu a_{ni}(\log n)^\beta X_i}{b_n}\right)\right]\right.\\
&\qquad \left. + b_n^p(\log n)^{-p\beta}\mathbb{V}(|a_{ni}X_i|>\mu b_n(\log n)^{-\beta})\right)\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1} b_n^{-p}(b_n(\log n)^{-\beta})^{p-\gamma}\sum_{i=1}^{n}|a_{ni}|^\gamma\widehat{\mathbb{E}}|X|^\gamma\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-\beta(p-\gamma)})(b_n)^{-\gamma}(\sum_{i=1}^{n}|a_{ni}|^{\alpha})^{\gamma/\alpha}\widehat{\mathbb{E}}|X|^\gamma\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1}(\log n)^{-\beta(p-\gamma)-1}<\infty.
\end{aligned}
\tag{3.8}
$$

Hence, we have proved $J_1<\infty$.

Now we prove $J_2<\infty$. We should note that the identical distribution is defined under $\widehat{\mathbb{E}}$, not under $\mathbb{V}$ (see Definition 2.2 of [17]). X_i identical distribution implies $\widehat{\mathbb{E}}[f(X_i)] = \widehat{\mathbb{E}}[f(X_1)]$ for $f(.)\in C_{l.Lip}(\mathbb{R})$, but does not imply $\mathbb{V}(f(X_i)\in A) = \mathbb{V}(f(X_1)\in A)$. Therefore, in the calculation of $\mathbb{V}(f(X_i)\in A)$, we need to convert $\mathbb{V}$ to

$\widehat{\mathbb{E}}$. As to J_2, by the definition of $X_{ni}^{(2)}$, the definition of independent, (3.3) and Markov inequality we have

$$\begin{aligned}
&\mathbb{V}\left(\sum_{i=1}^{n} X_{ni}^{(2)} > \varepsilon b_n/4\right)\\
&\leqslant \mathbb{V}(\text{there exist at least } N \text{ indices } i \text{ such that } a_{ni}X_i > b_n(\log n)^{-\beta})\\
&\leqslant \sum_{1\leqslant i_1<i_2<\cdots<i_N\leqslant n} \mathbb{V}\left(a_{ni_1}X_{i_1} > b_n(\log n)^{-\beta}), \cdots, a_{ni_N}X_{i_N} > b_n(\log n)^{-\beta})\right)\\
&\leqslant \sum_{1\leqslant i_1<i_2<\cdots<i_N\leqslant n} \widehat{\mathbb{E}}\left[\left(1-g\left(\frac{a_{ni_1}(\log n)^{\beta}X_{i_1}}{b_n}\right)\right)\cdots\left(1-g\left(\frac{a_{ni_N}(\log n)^{\beta}X_{i_N}}{b_n}\right)\right)\right]\\
&= \sum_{1\leqslant i_1<i_2<\cdots<i_N\leqslant n} \widehat{\mathbb{E}}\left[1-g\left(\frac{a_{ni_1}(\log n)^{\beta}X_{i_1}}{b_n}\right)\right]\cdots\widehat{\mathbb{E}}\left[1-g\left(\frac{a_{ni_N}(\log n)^{\beta}X_{i_N}}{b_n}\right)\right]\\
&\leqslant \sum_{1\leqslant i_1<i_2<\cdots<i_N\leqslant n} \mathbb{V}(a_{ni_1}X_{i_1} > \mu b_n(\log n)^{-\beta})\cdots\mathbb{V}(a_{ni_N}X_{i_N} > \mu b_n(\log n)^{-\beta})\\
&\leqslant \left(\sum_{i=1}^{n}\mathbb{V}(a_{ni}X_i > \mu b_n(\log n)^{-\beta})\right)^N\\
&\leqslant C\left(\widehat{\mathbb{E}}|X|^{\gamma}b_n^{-\gamma}(\log n)^{\beta\gamma}\sum_{i=1}^{n}|a_{ni}|^{\gamma}\right)^N\\
&\leqslant C(\widehat{\mathbb{E}}|X|^{\gamma})^N(\log n)^{(-1+\beta\gamma)N},
\end{aligned}$$

which implies that $J_2 < \infty$ for large enough N such that $(1-\beta\gamma)N > 1$. Similarly, we can have $J_3 < \infty$. By Lemma 3.2, we can have

$$J_4 \leqslant \sum_{n=1}^{\infty}\frac{1}{n}\sum_{i=1}^{n}\mathbb{V}(|a_{ni}X_i| > \varepsilon b_n/4) < \infty.$$

Therefore (2.2) holds.

Conversely, suppose that (2.2) holds for any array $\{a_{ni}\}$ satisfying (2.1). For each $n \geqslant 1$, we take $a_{n1} = n^{1/\alpha}$ and $a_{ni} = 0$ for $2 \leqslant i \leqslant n$. Then $\{a_{ni}\}$ obviously satisfies (2.1). By the assumption, we get that for any $\varepsilon > 0$,

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty}\frac{1}{n}\mathbb{V}\left(|X_1| > \varepsilon(\log n)^{1/\gamma}\right)\\
&= \sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{n}\mathbb{V}\left(|X_1| > \varepsilon(\log n)^{1/\gamma}\right)\\
&\geqslant \sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}\frac{1}{2^{k+1}}\mathbb{V}\left(|X_1| > \varepsilon(\log 2^{k+1})^{1/\gamma}\right)
\end{aligned}$$

$$= \sum_{k=0}^{\infty} 2^k \frac{1}{2^{k+1}} \mathbb{V}\left(|X_1| > \varepsilon((k+1)\log 2)^{1/\gamma}\right)$$
$$\geqslant \sum_{k=0}^{\infty} \frac{1}{2} \mathbb{V}(|X_1| > Ck^{1/\gamma}).$$

Note that for any $c > 0$

$$C_{\mathbb{V}}[|X_1|^\gamma/c] = \int_0^\infty \mathbb{V}(|X_1|^\gamma \geqslant cx)dx < \infty \Leftrightarrow \sum_{n=1}^{\infty} \mathbb{V}(|X_1|^\gamma \geqslant cn) < \infty.$$

Hence, we have $C_{\mathbb{V}}[|X_1|^\gamma] < \infty$. We complete the proof of Theorem 2.1. □

Availability of data and material. All of the availability of data and material about this article are in the article. There aren't other data and material about this article.

Competing interests. The authors declare that they have no competing interests.

Authors' contributions. Fengxiang Feng conceived of the study and drafted, complete the manuscript. Haiwu Huang participated in the discussion of the manuscript. Fengxiang Feng and Haiwu Huang read and approved the final manuscript.

Acknowledgements. We would like to thank the editors and anonymous reviewers for their careful reading and detailed comments, which have led to significant improvements of our paper.

Funding. Fengxiang Feng's research work is supported by the National Natural Science Foundation of China (11961015), Natural Science Foundation of Guangxi (2019GXNSFAA245031 and 2018GXNSFBA281019) and supported by Foundation of Guilin University of Technology (GUTQDJJ2004044).

Haiwu Huang's research work is supported by the Scientific Research Fund of Hunan Provincial Education Department (18C0660), the National Statistical Science Research Project of China (2018LY05) and the State Scholarship Fund of China Scholarship Council (No. 201908430242).

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(Received August 10, 2019)

Fengxiang Feng
College of Science
Guilin University of Technology
Guilin, Guangxi 541004, P. R. China
e-mail: fengfengxiang2013@163.com

Haiwu Huang
College of Mathematics and Statistics
Hengyang Normal University
Hengyang 421002, P. R. China
e-mail: haiwuhuang@126.com