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# A GENERALIZATION OF DARBO'S FIXED POINT THEOREM WITH AN APPLICATION TO FRACTIONAL INTEGRAL EQUATIONS

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*Abstract.* In this paper, we give a new generalization of Darbo's fixed point theorem of integral type. An application for the solvability of nonlinear fractional integral equation is given to illustrate our result.

## 1. Introduction

Kuratoski [15] introduced the concept of measure of noncompactness (MNC) which played an important role in fixed point theory. Gohberg et al. [12] gave another measure called Hausdorff measure of noncompactness. Later Darbo [11] used Kuratoski's measure of noncompactness to generalize the Schauder's fixed point theorem. After that many authors studied and solved some problems by using these MNCs in differential equations, integral equations and integro-differential equations.

In 1980, Banaś [8] gave an axiomatic definition of MNC which was used by many authors to study different problems, for instance see [1, 3, 4].

Aghajani et al. [3] gave an integral type generalization of Darbo's theorem and applied it in the existence of solution of functional integral equations. Jleli et.al. [13] gave another generalization to Darbo's theorem with an application. Recently, integral equations of fractional orders were studied in [14] and [17] via measures of noncompactness. For different types of integral equations, see [5], [19] and [20]. Cai et al. [10] used shifting distance functions to establish some new generalizations. Agarwal and Samet [2] studied solvability of fractional order via measure of compactness. Recently, solvability of infinite systems of differential equations are studied in [16], [18] and [22].

In the present paper, we give a generalization of Darbo's by using the concept of $\mathscr{C}$-class function and give an application of solvability to a nonlinear fractional integral equation in Banach space.

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## 2. Preliminaries

DEFINITION 2.1. [7] Let $E$ be a Banach space and $\mathbb{B}_E$ a collection of bounded subsets of $E$. A measure of noncompactness is a function $\mu : \mathbb{B}_E \to \mathbb{R}_+$ which satisfies the following conditions:

1. The set $ker\mu = \{X \in \mathbb{B}_{(E)} : \mu(X) = 0\}$ is no empty and $\ker\mu \subset \mathbb{B}_E$.
2. $X \subset Y$ implies $\mu(X) \subseteq \mu(Y)$.
3. $\mu(\overline{X}) = \mu(convX) = \mu(X)$
4. $\mu(\lambda X + (1-\lambda)X) \leqslant \lambda\mu(X) + (1-\lambda)\mu(X)$.
5. if $(X_n)$ is sequence of closed sets of $\mathbb{B}_E$ such $X_{n+1} \subseteq X_n$ and $\lim\limits_{n\to}\mu(X_n) = 0$. Then the set $X_\infty = \cap_{n=1}^{\infty} X_n$ is nonempty and $X_\infty$ is precompact.

REMARK 2.1. Since $\mu(X_\infty) = \mu(\bigcap\limits_{n=1}^{\infty} X_n) \leqslant \mu(X_n)$, $\mu(X_\infty) = 0$. So $X_\infty \in \ker\mu$.

THEOREM 2.1. [1] [Schauder] *Let $\Omega$ be a nonempty, bounded, closed and convex subset of a Banach space $E$. If $T : \Omega \to \Omega$ is continuous and compact self mapping on $\Omega$, then $T$ has at least a fixed point in $\Omega$.*

THEOREM 2.2. [11] [Darbo] *Let $\Omega$ be a nonempty, bounded, closed and convex subset of a Banach space $E$ and let $T : \Omega \to \Omega$ be a continuous self mapping on $\Omega$ satisfying for any subset $X$ of $\Omega$ the following inequality:*

$$\mu(TX) \leqslant k\mu(X),$$

*where $\mu$ is a measure of noncompactness on $E$ and $k \in [0,1)$. Then $T$ has a fixed point in $\Omega$.*

DEFINITION 2.2. [6] A continuous function $F : [0,\infty)^2 \to \mathbb{R}$ is called $C$*-class* function if it satisfies following axioms:

1. $F(s,t) \leqslant s$;
2. $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s,t \in [0,\infty)$.

Note that for some $F$ we have that $F(0,0) = 0$. We denote the set of $C$-class functions by $\mathscr{C}$.

EXAMPLE 2.3. [6] The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathscr{C}$. For all $s,t \in [0,\infty)$

(1) $F(s,t) = ks$, $0 < k < 1$, $F(s,t) = s \Rightarrow s = 0$;
(2) $F(s,t) = s - t$, $F(s,t) = s \Rightarrow t = 0$;

(3) $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(4) $F(s,t) = (s+l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$, $F(s,t) = s \Rightarrow t = 0$;

(5) $F(s,t) = \ln(1+a^s)/2$, $a > e$, $F(s,1) = s \Rightarrow s = 0$;

(6) $F(s,t) = \log(t+a^s)/(1+t)$, $a > 1$, $F(s,t) = s \Rightarrow s = 0$ or $t = 0$;

(7) $F(s,t) = \phi(s)$, $F(s,t) = s \Rightarrow s = 0$, where $\phi : [0,\infty) \to [0,\infty)$ is an upper semi continuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$;

(8) $F(s,t) = s\beta(s)$, where $\beta : [0,\infty) \to [0,1)$ is continuous function, $F(s,t) = s \Rightarrow s = 0$;

(9) $F(s,t) = sh(s,t)$, $F(s,t) = s \Rightarrow s = 0$, where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t,s > 0$.

Let $\Psi$ be the set of all continuous functions $\psi : [0,+\infty) \to [0,+\infty)$ satisfying the following conditions:

- $\psi(t) = 0$ if and only if $t = 0$,
- $\psi$ is non decreasing,
- $\psi(t) < t$, $\forall t > 0$.

Let $\Phi$ denote the set of all continuous functions $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\phi(0) = 0$.

## 3. Main results

THEOREM 3.1. *Let $\Omega$ be a nonempty, closed, bounded and convex subset of a Banach space $E$ and $T : \Omega \to \Omega$ a continuous mapping such that for any $X$ in $\mathbb{B}_E$*

$$\psi\Big(\int_0^{\mu(TX)} \varphi(t)dt\Big) \leqslant F\Big(\psi\Big(\int_0^{\mu(X)} \varphi(t)dt\Big), \phi\Big(\int_0^{\mu(X)} \varphi(t)dt\Big)\Big), \tag{1}$$

*where $\mu$ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable function, which is summable on each compact subset of $\mathbb{R}_+$ and satisfies*

$$\int_0^{\varepsilon} \varphi(t)dt > 0, \ \textit{for all} \ \varepsilon > 0.$$

*Then $T$ has at least one fixed point in $X$.*

*Proof.* Firstly, we will construct a nested sequence and by using properties of measure of noncompactness, we arrive to our main result. Define a sequence $\{\Omega_n\}$ as follows:

$$\Omega_0 = \Omega, \ \Omega_n = conv(T\Omega_{n-1}), \ \text{for all} \ n \geqslant 1,$$

where $conv(T\Omega_{n-1})$ is the convex hall of $T\Omega_{n-1}$.

If there exists $n_0$ such $\mu(\Omega_{n_0}) = 0$, so $\Omega_{n_0} \in \ker\mu$ and $\Omega_{n_0}$ is compact. Tince $T(\Omega_{n_0}) \subseteq \Omega_{n_0}$, from theorem 2.1, $T$ has a fixed point in $\Omega_{n_0}$.

Suppose now $\mu(\Omega_n) > 0$ for all $n \geqslant 0$. We claim $\{\int\limits_0^{\mu(\Omega_n)} \varphi(t)dt\}$ is a decreasing sequence. By using (1) we get

$$\begin{aligned}
\psi(\int\limits_0^{\mu(\Omega_{n+1})} \varphi(t)dt) &= \psi(\int\limits_0^{\mu(conv(T\Omega_n))} \varphi(t)dt \\
&\leqslant F(\psi(\int\limits_0^{\mu(\Omega_n)} \varphi(t)dt), \phi(\int\limits_0^{\mu(\Omega_n)} \varphi(t)dt)) \\
&\leqslant \psi(\int\limits_0^{\Omega_n} \varphi(t)dt),
\end{aligned}$$

since $\psi$ is non decreasing function, we get

$$\int\limits_0^{\mu(\Omega_{n+1})} \varphi(t)dt \leqslant \int_0^{\mu(\Omega_n)} \varphi(t)dt.$$

Then $\{\int\limits_0^{\mu(\Omega_n)} \varphi(t)dt\}$ is decreasing and bounded below, so it converges to $r = \inf\{\int_0^{\Omega_n} \varphi(t)dt\}$. Suppose $r > 0$. By using (1) we get

$$\begin{aligned}
\psi(\int\limits_0^{\Omega_{n+1}} \varphi(t)dt) &= \psi(\int\limits_0^{T(conv(\Omega_n))} \varphi(t)dt) \\
&\leqslant F(\psi(\int\limits_0^{\mu(\Omega_n)} \varphi(t)dt), \phi(\int\limits_0^{\mu(\Omega_n)} \varphi(t)dt)).
\end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$\psi(r) \leqslant F(\psi(r), \phi(r)) \leqslant \psi(r),$$

which implies

$$F(\psi(r), \phi(r)) = \psi(r),$$

from $(F_2)$ we get $\psi(r) = 0$, or, $\phi(r) = 0$. Hence $r = 0$.

Consequently and from the condition on $\varphi$, we obtain

$$\lim_{n\to\infty} \mu(\Omega_n) = 0.$$

Then from (5) of Definition 2.1 $\Omega_\infty = \cap_{n=0}^{\infty} \Omega_n$ is a nonempty, closed and convex subset of $\Omega$, moreover $\Omega_\infty \in \ker \mu$ which implies is compact. On other hand $T(\Omega_\infty) \subseteq \Omega_\infty$. Therefore from Schauder's theorem $T$ has a fixed point in $\Omega$. $\square$

If $\varphi(t) = 1$, we get the following corollary

COROLLARY 3.2. *Let $\Omega$ be a nonempty, closed, bounded and convex subset of a Banach space $E$ and let $T:\Omega\to\Omega$ be a continuous mapping such that for any $X$ in $\mathbb{B}_E$, we have*

$$\psi(\mu(TX))\leqslant F(\psi(\mu(X)),\phi(\mu(X))),$$

*where $\psi\in\Psi$ and $\phi\in\Phi$. Then $T$ has a fixed point in $X$.*

If $F(s,t)=s-t$, we get the following corollary.

COROLLARY 3.3. *Let $\Omega$ be a nonempty, closed, bounded and convex subset of a Banach space $E$ and let $T:\Omega\to\Omega$ be a continuous mapping such that for any $X$ in $\mathbb{B}_E$, we have*

$$\psi\Big(\int_0^{\mu(TX)}\varphi(t)dt\Big)\leqslant\psi\Big(\int_0^{\mu(X)}\varphi(t)dt\Big)-\phi\Big(\int_0^{\mu(X)}\varphi(t)dt\Big),$$

*where $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a Lebesgue-integrable function, which is summable on each compact subset of $\mathbb{R}_+$ and satisfies*

$$\int_0^{\varepsilon}\varphi(t)dt>0,\ \textit{for all}\ \varepsilon>0.$$

*Then $T$ has at least one fixed point in $X$.*

If we combine Theorem 3.1 with Example 7, we get the following corollary:

COROLLARY 3.4. *Let $\Omega$ be a nonempty, closed, bounded and convex subset of a Banach space $E$ and let $T:\Omega\to\Omega$ be a continuous mapping such for any $X$ in $\mathbb{B}_E$ we have*

$$\psi\Big(\int_0^{\mu(TX)}\varphi(t)dt\Big)\leqslant\phi\Big(\int_0^{\mu(X)}\varphi(t)dt\Big),$$

*where $\psi\in\Psi$ and $\phi:\mathbb{R}_+\to\mathbb{R}_+$ is a upper semi continuous function such $\phi(0)=0$, $\phi(t)<t$ for all $t>0$ and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a Lebesgue-integrable function, which is summable on each compact subset of $\mathbb{R}_+$ and satisfies*

$$\int_0^{\varepsilon}\varphi(t)dt>0,\ \textit{for all}\ \varepsilon>0.$$

*Then $T$ has a fixed point in $X$.*

Combining Theorem3.1 with Example 8, we get the following corollary:

COROLLARY 3.5. *Let $\Omega$ be a nonempty, closed, bounded and convex subset of a Banach space $E$ and let $T:\Omega\to\Omega$ be a continuous mapping such that for any $X$ in $\mathbb{B}_E$ we have:*

$$\psi\Big(\int_0^{\mu(TX)}\varphi(t)dt\Big)\leqslant\psi\Big(\int_0^{\mu(X)}\varphi(t)dt\Big)\beta\Big(\psi\Big(\int_0^{\mu(X)}\varphi(t)dt\Big)\Big),$$

*where $\psi\in\Psi$, $\beta:[0,\infty)\to[0,1)$ is a continuous function and $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a Lebesgue-integrable function, which is summable on each compact subset of $\mathbb{R}_+$ and satisfies*

$$\int_0^{\varepsilon}\varphi(t)dt>0,\ \textit{for all}\ \varepsilon>0.$$

*Then $T$ has a fixed point in $X$.*

REMARK 3.1.

1. If $\varphi(t)=1$, $\psi(t)=t$ and $F(s,t)=ks$, $k\in[0,1)$, we get theorem 2.2 (Darbo's theorem).
2. Corollary 3.3 improves and generalizes Theorem 2.7 of [10].
3. Corollary 3.4 improves and generalizes Theorem 2.1 of Aghajani et.al [4].

## 4. Application

In this section, we demonstrate an application of our results for the existence of solution of fractional integral equation in Banach space.

Consider the integral equation:

$$x(t)=f(t,x(t))+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{u(t,x(s))}{(t-s)^{1-\alpha}}ds, \tag{2}$$

where $t\in[0,T]$, $0\leqslant\alpha<1$ and $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ is continuous function. Assume the following assumptions hold:

1. There exists an upper semi continuous function $\phi:[0,\infty)\to[0,\infty)$ such $\phi(0)=0$, $\phi(t)<t$, for all $t\in[0,T]$ and

$$\int_0^{|f(t,x(t))-f(t,y(t))}\varphi(\tau)d\tau\leqslant\phi\Big(\int_0^{|x-y|}\varphi(\tau)d\tau\Big),$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Lebesgue-integrable function, which is summable on each compact subset of $\mathbb{R}_+$ and satisfies

$$\int_0^{\varepsilon} \varphi(\tau)d\tau > 0, \quad \text{for all } \varepsilon > 0$$

and for positive numbers $a,b,$ we have

$$\int_0^{a+b} \varphi(\tau)d\tau \leqslant \int_0^{a} \varphi(\tau)d\tau + \int_0^{b} \varphi(\tau)d\tau.$$

2. The function $u : [0,T] \times [0,\infty) \to [0,\infty)$ is continuous and there exists a nondecreasing function $\theta ; [0,\infty) \to [0,\infty)$ satisfying

$$|u(t,x)| \leqslant \theta(|x|), \quad (t,x) \in [0,T] \times [0,\infty).$$

3. There exists $r_0 > 0$ such

$$\phi(\int_0^{r_0} \varphi(\tau)d\tau) + \int_0^{M+\frac{\theta(r_0)T^{\alpha}}{\Gamma(\alpha+1)}} \varphi(\tau)d\tau \leqslant \int_0^{r_0} \varphi(\tau)d\tau,$$

where $M = \max\{|f(t,0)|, t \in [0,T]\}$.

Let $E$ be the space of bounded and continuous functions, which is a Banach space with the norm

$$\|x\| = \sup_{0 \leqslant t \leqslant T} |x(t)|, \quad x \in E$$

Let $\mathbb{B}_E$ be set of non empty and bounded subsets of $E$ and let $X \subseteq \mathbb{B}_E$. For $x \in X$ and two arbitrary numbers $\varepsilon > 0$ and $T > 0$, set

$$\omega^T(x,\varepsilon) = \sup\{|x(s) - x(t)| : \ t,s \in [[0,1], |s-t| \leqslant \varepsilon\}$$

$$\omega^T(X,\varepsilon) = \sup\{\omega^T(x,\varepsilon), x \in X\}$$

$$\omega_0^T(X) = \lim_{\varepsilon \to 0} \omega^T(X,\varepsilon), \quad \omega_0(X) = \lim_{T \to 0} \omega^T(X).$$

Banas et.al [7] proved that $\omega_0 : \mathbb{B}_E \to \mathbb{R}_+$ is a measure of noncompactness.

THEOREM 4.1. *Under the assumptions* $(1)$–$(4)$, *equation* 2 *has at least a solution in* $E$.

*Proof.* Consider a mapping:

$$Hx(t) = f(t,x(t)) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(t,x(s))}{(t-s)^{1-\alpha}} ds.$$

We show that $T$ maps $E$ into itself. Let $\{x_n\}$ be a sequence in $E$, which is convergent to $x$, so we have:

$$\int_0^{|Hx_n(t)-Hx(t)|} \varphi(\tau)d\tau \leqslant \int_0^{|f(t,x_n)-f(t,x)|+\frac{1}{\Gamma(\alpha)}\int_0 \frac{|u(t,x_n)-u(t,x)|}{(t-s)^{1-\alpha}}ds} \varphi(\tau)d\tau,$$

letting $n \to \infty$, and since $f$ and $u$ are continuous, we get

$$\int_0^{|H(x_n(t),H(x(t)))|} \varphi(\tau)d\tau \to 0,$$

from condition on $\varphi$ this yield that $H$ is continuous.

Let $B_{r_0}$ be the closed ball of the rayon $r_0$ centered at the origin, i.e,

$$B_{r_0} = \{x \in E, \|x(t)\| \leqslant r_0\}.$$

We claim $H$ maps continuously $B_{r_0}$ into itself, in fact for $x \in B_{r_0}$ and $t \in [0,T]$ we have:

$$\begin{aligned}
\int_0^{|Hx(t)|} \varphi(\tau)d\tau &= \int_0^{|f(t,x(t))+\frac{1}{\Gamma(\alpha)}\int_0^t \frac{u(t,x(s))}{(t-s)^{1-\alpha}}ds|} \varphi(\tau)d\tau \\
&\leqslant \int_0^{|f(t,x(t))-f(t,0)+f(t,0)|} \varphi(\tau)d\tau + \int_0^{\frac{1}{\Gamma(\alpha)}\int_0^t \frac{|u(t,x(s))|}{(t-s)^{1-\alpha}}ds} \varphi(\tau)d\tau \\
&\leqslant \phi(\int_0^{|x(t)|} \varphi(\tau)d\tau) + \int_0^{M+\frac{\theta(|x|)}{\Gamma(\alpha+1)}t^{\alpha}} \varphi(\tau)d\tau \\
&\leqslant \leqslant \phi(\int_0^{r_0} \varphi(\tau)d\tau) + \int_0^{M+\frac{\theta(r_0)}{\Gamma(\alpha+1)}T^{\alpha}} \varphi(\tau)d\tau \\
&\leqslant \int_0^{r_0} \varphi(\tau)d\tau.
\end{aligned}$$

Hence $H$ maps $B_{r_0}$ into itself. Now we will show that's $H$ is continuous mapping, let $x,y \in B_{r_0}$ and $\varepsilon > 0$ such that:

$$\|x-y\| \leqslant \varepsilon.$$

For all $t \in [0,T]$, we have:

$$\int_0^{|Hx(t)-Hy(t)|} \varphi(\tau)d\tau \leqslant \int_0^{|f(t,x(t))-f(t,y(t))|} \varphi(\tau)d\tau + \int_0^{\frac{1}{\Gamma(\alpha)}\int_0^t \frac{|u(t,x(s))-u(t,y(s))|}{(t-s)^{1-\alpha}}ds} \varphi(\tau)d\tau$$

$$\leqslant \phi(\int_0^{|x(t)-y(t)|} \varphi(\tau)d\tau) + \int_0^{\frac{\omega_\varepsilon}{\Gamma(\alpha+1)}T^\alpha} \varphi(\tau)d\tau,$$

where

$$\omega_\varepsilon = \sup\{|u(t,x(t))-u(t,y(t))|, -r_0 \leqslant x,y \leqslant r_0, |x-y| \leqslant \varepsilon, t \in [0,T]\},$$

the uniform continuity of $u$ on $[0,T]\times[-r_0,r_0]$, implies $\lim_{\varepsilon\to 0}\omega_\varepsilon = 0$, then

$$\int_0^{\|Hx(t)-Hy(t)\|} \varphi(\tau)d\tau \leqslant \phi(\int_0^\varepsilon \varphi(\tau)d\tau).$$

Hence $H$ is continuous on $B_{r_0}$.

Let $W$ be a non empty subset of $B_{r_0}$, for a fixed number $\delta > 0$, $x \in W$ and $t_1, t_2 \in [0;T]$, suppose $t_1 \geqslant t_2$, we have:

$$\int_0^{|Hx(t_2)-Hx(t_1)|} \varphi(\tau)d\tau \leqslant \int_0^{|f(t_2,x(t_2))-f(t_1,x(t_1))|+\frac{1}{\Gamma(\alpha)}(\int_0^{t_2} \frac{|u(t_2,x(s))|}{(t_2-s)^{1-\alpha}}ds - \int_0^{t_1} \frac{|u(t_1,x(s))|}{(t_1-s)^{1-\alpha}}ds)} \varphi(\tau)d\tau$$

$$= \int_0^{|f(t_2,x(t_2))-f(t_1,x(t_1))|+\frac{1}{\Gamma(\alpha)}(\int_0^{t_2} \frac{|u(t_2,x(s))|}{(t_2-s)^{1-\alpha}}ds - \int_0^{t_1} \frac{|u(t_1,x(s))|}{(t_1-s)^{1-\alpha}}ds)} \varphi(\tau)d\tau$$

$$\leqslant \phi(\int_0^{|x(t_2)-x(t_1|} \varphi(\tau)d\tau)$$

$$+ \int_0^{|f(t_1,x(t_2))-f(t_2,x(t_1))|+\frac{\theta(|x|)}{\Gamma(\alpha)}(\int_0^{t_2}(t_2-s)^{\alpha-1}ds - \int_0^{t_1}(t_1-s)^{\alpha-1}ds)} \varphi(\tau)d\tau,$$

$$\leqslant \phi(\int_0^{\omega_x} \varphi(\tau)d\tau) + \int_0^{\omega_f} \varphi(\tau)d\tau + \int_0^{\frac{\theta(r_0)}{\Gamma(\alpha+1)}|t_2^\alpha - t_1^\alpha|} \varphi(\tau)d\tau),$$

where

$$\omega_f = \sup\{|f(t_1,x)-f(t_2,x)|, t_1,t_2 \in [0,T], |t_1-t_2| \leqslant \varepsilon, |x| \leqslant r_0\}$$

and

$$\omega_x = \sup\{|x(t_1) - x(t_2)|, t_1, t_2 \in [0,T], |t_1 - t_2| \leqslant \varepsilon, \}.$$

Letting $\varepsilon \to 0$, we get $\lim_{\varepsilon\to 0} \omega_f = 0$ and

$$\lim_{\varepsilon\to 0} \frac{\theta(r_0)}{\Gamma(\alpha+1)} |t_2^\alpha - t_1^\alpha| = 0,$$

then

$$\omega_0^T(H(X)) \leqslant \phi(\omega_0^T(X)),$$

taking $T \to 0$, we get:

$$\omega_0(H(X)) \leqslant \phi(\omega_0(X)).$$

Consequently, from Corollary 3.4 (with $\psi(t) = t$) the equation (2) has a solution in $E$. □

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