Journal of Mathematical Inequalities
Volume 15, Number 3 (2021), 981–1001 doi:10.7153/jmi-2021-15-67

INTEGRALS OF RATIOS OF FOX–WRIGHT AND INCOMPLETE FOX–WRIGHT FUNCTIONS WITH APPLICATIONS

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(*Communicated by T. Burić*)

Abstract. The main focus of the present paper is to establish definite integral formulae for ratios of the Fox–Wright functions. As consequences of the master formula, some novel integral formulae are derived for ratios of other special functions which are associated to Fox–Wright Ψ function, like generalized hypergeometric function, modified Bessel function of the first kind and Mittag–Leffler type functions of two and three parameters. Moreover, closed integral form expressions are obtained for a family of Mathieu-type series and for the associated alternating versions whose terms contain the incomplete Fox-Wright function. As applications, functional bounding inequalities are established for the aforementioned series.

1. Introduction

The Fox-Wright function, which is a generalization of hypergeometric function, is defined as follows [5], [33, p. 4, Eq. (2.4)]:

$$
{}_p\Psi_q\left[\begin{matrix}(a_1,A_1),\cdots,(a_p,A_p)\\(b_1,B_1),\cdots,(b_q,B_q)\end{matrix}\middle|z\right] = {}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle|z\right] = \sum_{k\geqslant 0}\frac{\prod_{l=1}^{p}\Gamma(a_l+kA_l)}{\prod_{l=1}^{q}\Gamma(b_l+kB_l)}\frac{z^k}{k!}, \tag{1.1}
$$

where $A_j \geqslant 0,\ j=1,\cdots,p$, and $B_l \geqslant 0,\ l=1,\cdots,q$. The convergence conditions and convergence radius of the series at the right-hand side of (1.1) immediately follow from the known asymptotic of the Euler Gamma–function. The defining series in (1.1) converges in the whole complex z-plane when

$$
\Delta = 1+\sum_{j=1}^{q}B_j-\sum_{i=1}^{p}A_i > 0.
$$

Mathematics subject classification (2020): 26D15, 33C20, 33C70, 33E12, 40C10.
Keywords and phrases: Complete and incomplete Fox–Wright function, generalized hypergeometric function, Mittag–Leffler functions of two and three parameters, Mathieu–type series, bilateral functional inequalities.
The research of T. K. Pogány has been supported in part by the University of Rijeka, Croatia under the project numbers uniri-pr-prirod-19-16 and uniri-tehnic-18-66.
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If $\Delta = 0$, then the series in (1.1) converges for $|z| < \rho$, and $|z| = \rho$ under the condition $\Re(\mu) > \frac{1}{2}$, where

$$\rho = \left(\prod_{i=1}^{p} A_i^{-A_i}\right)\left(\prod_{j=1}^{q} B_j^{B_j}\right), \qquad \mu = \sum_{j=1}^{q} b_j - \sum_{k=1}^{p} a_k + \frac{p-q}{2}.$$

The Fox–Wright function extends the generalized hypergeometric function ${}_pF_q[z]$ which power series form reads

$${}_pF_q\left[\begin{matrix}\mathbf{a}_p\\ \mathbf{b}_q\end{matrix}\,\middle|\, z\right] = \sum_{k\geqslant 0} \frac{\prod\limits_{l=1}^{p}(a_l)_k}{\prod\limits_{l=1}^{q}(b_l)_k}\frac{z^k}{k!},$$

where, as usual, we make use of the Pochhammer symbol (or raising factorial)

$$(\tau)_0 = 1; \quad (\tau)_k = \tau(\tau+1)\cdots(\tau+k-1) = \frac{\Gamma(\tau+k)}{\Gamma(\tau)}, \qquad k \in \mathbb{N}.$$

In the special case $A_r = B_s = 1$ the Fox–Wright function ${}_p\Psi_q[z]$ reduces (up to the multiplicative constant) to the generalized hypergeometric function

$${}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{1})\\ (\mathbf{b}_q,\mathbf{1})\end{matrix}\,\middle|\, z\right] = \frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_q)}\,{}_pF_q\left[\begin{matrix}\mathbf{a}_p\\ \mathbf{b}_q\end{matrix}\,\middle|\, z\right].$$

The importance of the Fox–Wright function can be found in [9, 12, 13, 14, 15, 16, 19, 20], for instance.

For the exposition of the results in final sections we need the so–called incomplete Fox–Wright function ${}_p\Psi_q^{(\gamma)}[z]$, introduced by Srivastava *et al.* in [28, p. 131, Eq. (6.1)]

$${}_p\Psi_q^{(\gamma)}\left[\begin{matrix}(\mu,M,x),(\mathbf{a}_{p-1},\mathbf{A}_{p-1})\\ (\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\middle|\, z\right] = \sum_{k\geqslant 0}\frac{\gamma(\mu+kM,x)\prod\limits_{j=1}^{p-1}\Gamma(a_j+kA_j)}{\prod\limits_{j=1}^{q}\Gamma(b_j+kB_j)}\frac{z^k}{k!},$$

where $\gamma(a,x)$ denotes the lower incomplete gamma function, which integral expression reads

$$\gamma(a,x) = \int_0^x \mathrm{e}^{-t}t^{a-1}\mathrm{d}t, \qquad x > 0, \Re(a) > 0.$$

The positivity constraint of parameters $M, A_j, B_j > 0$ is linked now to

$$\Delta^{(\gamma)} = 1 + \sum_{j=1}^{q} B_j - M - \sum_{i=1}^{p-1} A_i \geqslant 0;$$

the convergence conditions and characteristics coincide with the ones regarding the 'complete' Fox–Wright ${}_p\Psi_q[z]$.

2. Integral formulae for ratios built by Fox–Wright functions

The aim of this section is to establish certain integral formulae which integrand contains different kind ratios of products of Fox–Wright functions which parameters are contiguous in a specific way. Firstly, we introduce a shorthand for the ratio of two Fox–Wright functions which *one* lower parameter is contiguous as

$$ {}_p\mathscr{R}_q^{\Psi}[(b,B);x] := \frac{{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b+1,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|x\right]}{{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|x\right]}. \tag{2.1}$$

THEOREM 2.1. *For all $x>0$ we have the following integral identity*

$$\int_0^x \frac{{}_p\mathscr{R}_q^{\Psi}[(b,B);t]}{{}_p\mathscr{R}_q^{\Psi}[(b-1,B);t]}\left\{1+(1-B)\,{}_p\mathscr{R}_q^{\Psi}[(b-1,B);t]\right\}\mathrm{d}t = x\left\{1-B\,{}_p\mathscr{R}_q^{\Psi}[(b,B);x]\right\}. \tag{2.2}$$

Moreover, there holds

$$\int_0^x \frac{{}_p\mathscr{R}_q^{\Psi}[(b,1);t]}{{}_p\mathscr{R}_q^{\Psi}[(b-1,1);t]}\,\mathrm{d}t = x\left\{1-{}_p\mathscr{R}_q^{\Psi}[(b,1);x]\right\}. \tag{2.3}$$

Proof. By using the differentiation formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b+1,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|x\right] &= \frac{1}{Bt}\left\{{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|t\right]\right.\\ &\qquad\left.-\,b\,{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b+1,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|t\right]\right\},\end{aligned} \tag{2.4}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_p\mathscr{R}_q^{\Psi}[(b,B);t] = \frac{1}{Bt}\left\{1-{}_p\mathscr{R}_q^{\Psi}[(b,B);t]-\frac{{}_p\mathscr{R}_q^{\Psi}[(b,B);t]}{{}_p\mathscr{R}_q^{\Psi}[(b-1,B);t]}\right\}.$$

In turn, the previous relation yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(t\,{}_p\mathscr{R}_q^{\Psi}[(b,B);t]\right) = \frac{1}{B}\left\{1-(1-B)\,{}_p\mathscr{R}_q^{\Psi}[(b,B);t]-\frac{{}_p\mathscr{R}_q^{\Psi}[(b,B);t]}{{}_p\mathscr{R}_q^{\Psi}[(b-1,B);t]}\right\}.$$

Integrating both sides over $(0,x)$ and rearranging the result we arrive at (2.2) which obviously reduces to (2.3) for $B=1$. □

REMARK 1. The ratio

$$\mathscr{T}^{\Psi}(b;x) = \frac{{}_p\mathscr{R}_q^{\Psi}[(b,B);x]}{{}_p\mathscr{R}_q^{\Psi}[(b-1,B);x]} = \frac{{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b+1,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|x\right]{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b-1,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|x\right]}{{}_p\Psi_q^2\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (b,B),(\mathbf{b}_{q-1},\mathbf{B}_{q-1})\end{matrix}\middle|x\right]},$$

is nothing else then the familiar Turánian expression consisting of Fox–Wright function building blocks, with respect to the lower parameter b. Turán type inequalities have been established for a similar Turánian by Mehrez and Sitnik [17], but in terms of an upper parameter.

Specifying $A_j = B_k = 1$ in (2.1) we get the ratio for two generalized hypergeometric functions which retained the contiguous character *via* the lower parameter b:

$${}_p\mathscr{R}_q^{F}[b;z] := {}_p\mathscr{R}_q^{\Psi}[(b,1);z]\Big|_{\mathbf{A}_p=\mathbf{B}_{q-1}=\mathbf{1}} = \frac{{}_pF_q\left[\begin{matrix}\mathbf{a}_p\\ b+1,\mathbf{b}_{q-1}\end{matrix}\middle|z\right]}{b\,{}_pF_q\left[\begin{matrix}\mathbf{a}_p\\ b,\mathbf{b}_{q-1}\end{matrix}\middle|z\right]}.$$

COROLLARY 2.1. *For all* $\min(a_r, b_s, b-1) > 0$ *and* $x > 0$ *we have*

$$\int_0^x \frac{{}_p\mathscr{R}_q^{F}[b;t]}{{}_p\mathscr{R}_q^{F}[b-1;t]}\,\mathrm{d}t = \frac{bx}{b-1}\left(1 - \frac{{}_p\mathscr{R}_q^{F}[b;x]}{b\,{}_p\mathscr{R}_q^{F}[b-1;x]}\right).$$

The asymptotic expansion of ${}_p\Psi_q[z]$ for large $|z|$ belongs to Wright.

THEOREM 2.2. ([33, 34]) *If* $z \in \mathbb{C}$ *and* $|\arg(z)| \leqslant \pi - \varepsilon$ $(0 < \varepsilon < \pi)$, *then the asymptotic behaviour of the Fox–Wright function at infinity equals*

$${}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\ (\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle|z\right] = I(Z), \tag{2.5}$$

where for any $M \in \mathbb{N}$

$$I(Z) = Z^{-\mu}\mathrm{e}^{Z}\left\{\sum_{m=0}^{M-1} A_m Z^{-m} + \mathscr{O}(Z^{-M})\right\}$$

$$Z = \Delta\left(\frac{|z|}{\rho}\right)^{\frac{1}{\Delta}} \exp\left\{\frac{\mathrm{i}\arg(z)}{\Delta}\right\},$$

and

$$A_0 = (2\pi)^{\frac{1}{2}(p-q)}\,\Delta^{-\frac{1}{2}+\mu}\prod_{r=1}^{p} A_r^{a_r-\frac{1}{2}}\prod_{r=1}^{q} B_r^{\frac{1}{2}-b_r}.$$

COROLLARY 2.2. *Let $b>1$ and $B>0$. Then we have*

$$\lim_{x\to\infty} x^{\frac{1}{\Delta}}\left\{1-\frac{1}{x}\int_0^x \frac{{}_p\mathscr{R}_q^{\Psi}[(b,B);t]}{{}_p\mathscr{R}_q^{\Psi}[(b-1,B);t]}\left\{1+(1-B)\,{}_p\mathscr{R}_q^{\Psi}[(b-1,B);t]\right\}\mathrm{d}t\right\}=\rho^{\frac{1}{\Delta}}.$$

Proof. Bearing in mind the asymptotic expansion (2.5) of Theorem 2.3, we conclude

$$\lim_{z\to\infty}\left(\frac{z}{\rho}\right)^{\frac{1}{\Delta}}{}_p\mathscr{R}_q^{\Psi}[(b+1,B);z]=\frac{1}{B}.$$

Taking this limit and (2.2) straightforward calculations complete the proof. □

Now, putting in Corollary 2.2 $B=1;\mathbf{A}_p=\mathbf{1}=\mathbf{B}_{q-1}$ we obtain

COROLLARY 2.3. *Assume $b>1$. Then se have the asymptotic*

$$\lim_{x\to\infty} x^{\frac{1}{1+q-p}}\left\{1-(1-b^{-1})\frac{1}{x}\int_0^x \frac{{}_p\mathscr{R}_q^{F}[b;t]}{{}_p\mathscr{R}_q^{F}[b-1;t]}\,\mathrm{d}t\right\}=1.$$

The four parameter Wright function is defined by the power series [10, Eq. (21)]

$$\phi\big((a,\mu),(b,\nu);z\big)=\sum_{k\geqslant 0}\frac{z^k}{\Gamma(a+k\mu)\Gamma(b+k\nu)};\qquad \mu,\nu\in\mathbb{R};\ a,b\in\mathbb{C}.$$

The series is absolutely convergent for all $z\in\mathbb{C}$ for $\mu+\nu>0$. When $\mu+\nu=0$, the series is absolutely convergent inside the open disc $|z|<|\mu|^{\mu}|\nu|^{\nu}$ and on the circle $|z|=|\mu|^{\mu}|\nu|^{\nu}$ under the constraint $\Re(a+b)>2$. Some of the basic properties of the four parameters Wright function were proved in [10, Lemma 3.1 and 3.2].

REMARK 2. It is worth to mention that

$$\phi\big((a,\mu),(b,\nu);z\big)={}_1\Psi_2\left[\begin{matrix}(1,1)\\(a,\mu),(b,\nu)\end{matrix}\middle|z\right].$$

On the other hand this function we meet under the name of generalized four–parameter Mittag–Leffler function $E_{(\alpha,\beta)_2}(z)$ [1], see [17].

For $b=1=\nu$, the function $\phi(z)$ reduces to the two–parameter Wright function

$$W_{\alpha,\beta}(z)={}_0\Psi_2\left[\begin{matrix}-\\(\beta,\alpha),(1,1)\end{matrix}\middle|z\right]=\sum_{k\geqslant 0}\frac{z^k}{\Gamma(\alpha k+\beta)\,k!},\qquad z>0$$

which was intensively studied by the inventor E. M. Wright [33, 35] and B. Stanković [30], among others. Here the parameters' range is $(\alpha,\beta)\in\mathbb{R}_+^2$.

[1] We point out that the 'ordinary' four–parameter Mittag–Leffler function

$$E_{\alpha,\beta}^{\gamma,\kappa}(z)=\sum_{n\geqslant 0}\frac{(\gamma)_{\kappa n}}{\Gamma(\alpha+\beta n)}\frac{z^n}{n!};\qquad z,\beta,\gamma\in\mathbb{C};\ \Re(\alpha)>\big(\Re(\kappa)-1\big)_+;\ \Re(\kappa)>0,$$

should be distinguished from $E_{(\alpha,\beta)_2}(z)$, consult [29].

Next, by means of the integral formula (2.2) we obtain the associated expression for $\phi(z)$.

COROLLARY 2.4. *Assume $\Re(a), \Re(b) > 1$ and the four-parameter Wright function absolutely converges. Then we have*

$$\int_0^x \frac{\phi\big((a+1,\mu),(b,\nu);t\big)}{\phi\left((a,\mu),(b,\nu);t\right)} \left\{ \frac{\phi\big((a-1,\mu),(b,\nu);t\big)}{\phi\left((a,\mu),(b,\nu);t\right)} + 1 - \mu \right\} \mathrm{d}t$$
$$= x\left(1 - \mu \frac{\phi\big((a+1,\mu),(b,\nu);x\big)}{\phi\big((a,\mu),(b,\nu);x\big)}\right),$$

for all $x > 0$. Moreover, for all $\alpha, \beta - 1 > 0$ and $x > 0$ there holds

$$\int_0^x \frac{W_{\alpha,\beta+1}(t)}{W_{\alpha,\beta}(t)} \left\{ \frac{W_{\alpha,\beta+1}(t)}{W_{\alpha,\beta}(t)} + 1 - \alpha \right\} \mathrm{d}t = x\left(1 - \alpha \frac{W_{\alpha,\beta+1}(x)}{W_{\alpha,\beta}(x)}\right). \tag{2.6}$$

Recall the series definition of the modified Bessel function of the first kind I_p of the order p:

$$I_p(x) = \sum_{k \geqslant 0} \frac{\left(\frac{x}{2}\right)^{2k+p}}{k!\Gamma(k+p+1)}; \qquad x \in \mathbb{R}.$$

It is worth mentioning that in particular we have

$$W_{1,p+1}(x) = x^{-\frac{p}{2}} I_p(2\sqrt{x}), \qquad x \in \mathbb{R},$$

which relates the modified function to the display (2.6) of Corollary 2.4.

COROLLARY 2.5. *For any $p > -1$ and any positive $x > 0$ we have*

$$\int_0^x \frac{tI_{p+1}(t)I_{p-1}(t)}{I_p^2(t)} \mathrm{d}t = \frac{x^2}{2}\left(1 - \frac{2I_{p+1}(x)}{xI_p(x)}\right). \tag{2.7}$$

Moreover, for $x > 0$ it holds

$$\int_0^x \left(\frac{t\cosh^2 t - \cosh t \sinh t}{\sinh^2 t}\right) \mathrm{d}t = \frac{x^2}{2}\left(1 + \frac{2}{x^2} - \frac{2\cosh x}{x \sinh x}\right). \tag{2.8}$$

Proof. Specifying $a = 1$ and $\beta = p+1$ in (2.6) we immediately obtain the stated formula (2.7). However, taking $p = -1/2$ in (2.7), having in mind the familiar formulae

$$I_{\frac{1}{2}}(x) = \frac{\sqrt{2}\sinh x}{\sqrt{\pi x}}; \quad I_{-\frac{1}{2}}(x) = \frac{\sqrt{2}\cosh x}{\sqrt{\pi x}}; \quad I_{-\frac{3}{2}}(x) = \frac{\sqrt{2}\cosh x}{\sqrt{\pi x}} - \frac{\sqrt{2}\sinh x}{x\sqrt{\pi x}},$$

we readily establish (2.8) as well. $\square$

REMARK 3. It is important to mention the another proof of the integral (2.7). Namely, Joshi and Bissu [8, Eq. (3.6)] showed that

$$\frac{tI_{p+1}(t)I_{p-1}(t)}{I_p^2(t)} = t - \left(\frac{t\,I_p'(t)}{I_p(t)}\right)'.$$

Integrating over $(0,x)$ and rearranging the resulting expression we get

$$\int_0^x \frac{tI_{p+1}(t)I_{p-1}(t)}{I_p^2(t)}\mathrm{d}t = \frac{x^2}{2} - \frac{xI_p'(x)}{I_p(x)} + \lim_{x\to 0}\frac{xI_p'(x)}{I_p(x)}.$$

Thanking to the recurrence relation [31, p. 79]

$$\frac{I_p'(x)}{I_p(x)} = \frac{I_{p+1}(x)}{I_p(x)} + \frac{p}{x},$$

which implies

$$\frac{xI_p'(x)}{I_p(x)} = \frac{xI_{p+1}(x)}{I_p(x)} + p,$$

that is a fortiori

$$\lim_{x\to 0}\frac{xI_p'(x)}{I_p(x)} = p.$$

Collecting these relations we deduce the stated formula (2.7).

COROLLARY 2.6. *Let $p>-1$, then we have the following functional upper bound*

$$\frac{I_{p+1}(x)}{I_p(x)} \leqslant \frac{x}{2(p+1)}, \qquad x>0. \tag{2.9}$$

Proof. We recall the bilateral Turán type inequality reported by Baricz [1, Theorem 2.1]:

$$\frac{p}{p+1} \leqslant \frac{I_{p+1}(x)I_{p-1}(x)}{I_p^2(x)} \leqslant 1.$$

Routine algebra and the use of the integral formula (2.7) result in (2.9). $\square$

REMARK 4. The inequality (2.9) was firstly proved by Ifantis and Siafarikas in [7, Eq. (2.21)].

COROLLARY 2.7. *For all $p>-1$ there holds*

$$\frac{1}{2x^2}\int_0^x \frac{tI_{p+1}(t)I_{p-1}(t)}{I_p^2(t)}\,\mathrm{d}t = \sum_{k\geqslant 1}\frac{(p+1)(j_{p,k}^2+x^2)-j_{p,k}^2}{j_{p,k}^2(j_{p,k}^2+x^2)}, \tag{2.10}$$

where $0<j_{p,1}<j_{p,2}<\ldots<j_{p,n}<\ldots$ are the positive zeros of the Bessel function of the first kind $J_p(x)$. In particular, we have

$$\lim_{x\to 0}\frac{1}{x^2}\int_0^x \frac{tI_{p+1}(t)I_{p-1}(t)}{I_p^2(t)}\,\mathrm{d}t = \frac{p}{2(p+1)}. \tag{2.11}$$

Proof. By using the formula (2.7) we thus get

$$\frac{1}{x^2}\int_0^x \frac{tI_{p+1}(t)I_{p-1}(t)}{I_p^2(t)}\,\mathrm{d}t = \frac{1}{2} - \frac{I_{p+1}(x)}{xI_p(x)}. \tag{2.12}$$

On the other hand the Mittag-Leffler expansion reads [4, Eq. 7.9.3]

$$\frac{I_{p+1}(x)}{I_p(x)} = \sum_{k\geqslant 1}\frac{2x}{j_{p,k}^2+x^2},$$

while the celebrated first Rayleigh sum

$$\sigma_p^{(2)} = \sum_{k\geqslant 1}\frac{1}{j_{p,k}^2} = \frac{1}{4(p+1)},$$

combined with the relation (2.12) gives get the stated relation (2.10). Finally, letting $x \longrightarrow 0$ in the transformed (2.12) we obtain (2.11). $\square$

Similarly to the Corollary 2.2 we now deduce certain seemingly novel integral formulae for the four parameters Wright function $\phi(x)$ and the classical Wright function $W_{\alpha,\beta}(x)$.

COROLLARY 2.8. *For all $\Re(a),\Re(b) > -1$ we have*

$$\lim_{x\to\infty} x^{\frac{1}{\mu+\nu}}\left\{1-\frac{1}{x}\int_0^x \frac{\phi\big((a+1,\mu),(b,\nu);t\big)}{\phi\left((a,\mu),(b,\nu);t\right)}\left[\frac{\phi\big((a-1,\mu),(b,\nu);t\big)}{\phi\left((a,\mu),(b,\nu);t\right)}+1-\mu\right]\mathrm{d}t\right\} = (\mu^\mu\nu^\nu)^{\frac{1}{\mu+\nu}}.$$

Moreover,

$$\lim_{x\to\infty} x^{\frac{1}{\alpha+1}}\left\{1-\frac{1}{x}\int_0^x \frac{W_{\alpha,\beta+1}(t)}{W_{\alpha,\beta}(t)}\left[\frac{W_{\alpha,\beta-1}(t)}{W_{\alpha,\beta}(t)}+1-\alpha\right]\mathrm{d}t\right\} = \alpha^{\frac{\alpha}{\alpha+1}}.$$

The three-parameter Mittag-Leffler function (else Prabhakar's function [27]) $E_{\alpha,\beta}^\gamma(z)$ is defined by [27, 25]

$$E_{\alpha,\beta}^\gamma(z) = \sum_{k\geqslant 0}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!};\qquad \Re(\alpha),\Re(\beta),\Re(\gamma)>0; z\in\mathbb{C}.$$

However, we have that

$$E_{\alpha,\beta}^\gamma(z) = \frac{1}{\Gamma(\gamma)}\,{}_1\Psi_1\left[\begin{matrix}(\gamma,1)\\(\beta,\alpha)\end{matrix}\middle| z\right]. \tag{2.13}$$

For $\gamma = 1$ we recover the two–parameter Mittag-Leffler function (also known as the Wiman function [32])

$$E_{\alpha,\beta}(z) = E_{\alpha,\beta}^1(z) = \sum_{k\geqslant 0}\frac{z^k}{\Gamma(\alpha k+\beta)}.$$

Bearing in mind the tools of Theorem 2.1, Corollary 2.2 and the last formula (2.13) we derive the following statement regarding the Prabhakar's and the Wiman functions, involving the Turánians with respect to the second parameter β.

COROLLARY 2.9. *For all $\beta > 1$ and all positive $x > 0$ we have*

$$\int_0^x \frac{E^{\gamma}_{\alpha,\beta+1}(t)}{E^{\gamma}_{\alpha,\beta}(t)} \left[\frac{E^{\gamma}_{\alpha,\beta-1}(t)}{E^{\gamma}_{\alpha,\beta}(t)} + 1 - \alpha\right] \mathrm{d}t = x\left(1 - \alpha\frac{E^{\gamma}_{\alpha,\beta+1}(x)}{E^{\gamma}_{\alpha,\beta}(x)}\right),$$

and

$$\lim_{x\to\infty} x^{\frac{1}{\alpha}} \left\{1 - \frac{1}{x}\int_0^x \frac{E^{\gamma}_{\alpha,\beta+1}(t)}{E^{\gamma}_{\alpha,\beta}(t)} \left[\frac{E^{\gamma}_{\alpha,\beta-1}(t)}{E^{\gamma}_{\alpha,\beta}(t)} + 1 - \alpha\right] \mathrm{d}t\right\} = \alpha.$$

REMARK 5. The results of the Corollary 2.9 are invariant with respect to the parameter γ. Therefore, by setting $\gamma = 1$ in Corollary 2.9 we obtain the same integral and asymptotic formulae for the two–parametric Mittag-Leffler or Wiman function $E_{\alpha,\beta}(x)$ when $\beta > 1$ and $x > 0$.

3. Mathieu-type series of incomplete Fox-Wright function terms

In this section, we investigate the Mathieu-type series $\mathscr{K}$ and its alternating variant $\widetilde{\mathscr{K}}$, which building blocks are incomplete Fox-Wright functions. These series are defined as

$$\mathscr{K}_x^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right) = \sum_{j\geqslant 1} \frac{\gamma(\mu,(r+c_j)x)}{c_j^{\lambda}(\alpha_j+r)^{\mu}}\, {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda,1,c_jx),(\mathbf{a}_p,\mathbf{A}_p)\\ (\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\Big|\,\frac{r}{c_j}\right], \tag{3.1}$$

and

$$\widetilde{\mathscr{K}}_x^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right) = \sum_{j\geqslant 1} \frac{(-1)^{j-1}\gamma(\mu,(r+c_j)x)}{c_j^{\lambda}(\alpha_j+r)^{\mu}}\, {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda,1,c_jx),(\mathbf{a}_p,\mathbf{A}_p)\\ (\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\Big|\,\frac{r}{c_j}\right], \tag{3.2}$$

respectively. Here it is tacitly assumed that all of the required constraints for the parameters involved are satisfied for the convergence of both series (3.1) and (3.2).

THEOREM 3.1. *Let $x > 0,\ \mu > 0,\ \lambda > 0,\ r > 0$ and let the real sequence $\mathbf{c} = (c_n)_{n\geqslant 1}$ be monotone increasing to infinity. Then the Mathieu–type series $\mathscr{K}$ (3.1) possesses the integral representation*

$$\mathscr{K}_x^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right) = \mathscr{I}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda+1,\mu) + \mathscr{I}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu+1), \tag{3.3}$$

where

$$\mathscr{I}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu) = \int_{c_1}^{\infty} \frac{\gamma(\mu,(t+r)x)\lfloor c^{-1}(t)\rfloor}{t^{\lambda}(t+r)^{\mu}}\, {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda,1,tx),(\mathbf{a}_p,\mathbf{A}_p)\\ (\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\Big|\,\frac{r}{t}\right] \mathrm{d}t, \tag{3.4}$$

and $c:\mathbb{R}^+\to\mathbb{R}^+$ *is positive monotone increasing function which reduction* $c|_{\mathbb{N}}=\mathbf{c}$*, while* $c^{-1}(x)$ *denotes the inverse of* $c(x)$ *and* $\lfloor c^{-1}(x)\rfloor$ *signifies the integer part of* $c^{-1}(x)$.

Proof. Consider the Laplace transform of $z\mapsto z^{b-1}{}_p\Psi_q[z]\,\chi_{[0,x]}(z)$ [2]

$$\int_0^x \mathrm{e}^{-zt}z^{s-1}{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\middle|\,rz\right]\mathrm{d}z = t^{-s}{}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(s,1,tx),(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\middle|\,\frac{r}{t}\right].$$

Thus, in view of (3.1) we get

$$\mathscr{K}_x^{(\lambda,\mu)}\big({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\big)=\sum_{j\geqslant 1}\frac{\gamma(\mu,(r+c_j)x)}{(\alpha_j+r)^\mu}\int_0^x \mathrm{e}^{-c_jz}z^{\lambda-1}{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\middle|\,rz\right]\mathrm{d}z. \tag{3.5}$$

Moreover, by the incomplete gamma function properties we have

$$\frac{\gamma(\mu,(r+c_j)x)}{(c_j+r)^\mu}=\int_0^x \xi^{\mu-1}e^{-(c_j+r)\xi}\,\mathrm{d}\xi.$$

Bearing in mind the above formula and (3.5) we find that

$$\mathscr{K}_x^{(\lambda,\mu)}\big({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\big)=\int_0^x\int_0^x\Big(\sum_{j\geqslant 1}\mathrm{e}^{-c_j(z+\xi)}\Big)\xi^{\mu-1}\mathrm{e}^{-r\xi}z^{\lambda-1}{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\middle|\,rz\right]\mathrm{d}z\mathrm{d}\xi.$$

Using the Cahen formula [3, 23], [2, p. 9, Eq. (1.21)] for expressing in the integral form the resulting Dirichlet series

$$\mathscr{D}_{\mathbf{c}}(z+\xi)=\sum_{j\geqslant 1}\mathrm{e}^{-c_j(z+\xi)}=(z+\xi)\int_{c_1}^\infty \mathrm{e}^{-(z+\xi)x}\lfloor c^{-1}(x)\rfloor\,\mathrm{d}x,$$

we obtain

$$\begin{aligned}\mathscr{K}_x^{(\lambda,\mu)}\big({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\big)&=\int_{c_1}^\infty\int_0^x\int_0^x(z+\xi)\xi^{\mu-1}z^{\lambda-1}\mathrm{e}^{-(z+\xi)x-r\xi}\\&\qquad\times{}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\,\middle|\,rz\right]\lfloor c^{-1}(x)\rfloor\,\mathrm{d}z\mathrm{d}\xi\,\mathrm{d}x=:\mathscr{I}_z+\mathscr{I}_\xi,\end{aligned}\tag{3.6}$$

[2]The indicator function of the set $S\subseteq\mathbb{R}$ we write $\chi_S[z]$. So, the 'finite' Laplace transform

$$\mathscr{L}_x\{f;t\}=\int_0^\infty \mathrm{e}^{-tz}f(z)\,\chi_{[0,x]}(z)\,\mathrm{d}z=\int_0^x \mathrm{e}^{-tz}f(z)\,\mathrm{d}z,$$

that is, $\mathscr{L}_x\{f;t\}=\mathscr{L}\{f\cdot\chi_{[0,x]};t\}$.

where

$$
\begin{aligned}
\mathscr{I}_z &= \int_{c_1}^{\infty}\int_0^x\int_0^x \xi^{\mu-1} z^{\lambda} \mathrm{e}^{-(z+\xi)t-r\xi}\, {}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| rz\right] \lfloor c^{-1}(t)\rfloor\, \mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t \\
&= \int_{c_1}^{\infty}\int_0^x \xi^{\mu-1}\mathrm{e}^{-(t+r)\xi}\left\{\int_0^x \mathrm{e}^{-zt} z^{\lambda}\, {}_p\Psi_q\left[\begin{matrix}(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| rz\right]\right\} \lfloor c^{-1}(t)\rfloor\, \mathrm{d}z\,\mathrm{d}\xi\,\mathrm{d}t \\
&= \int_{c_1}^{\infty}\int_0^x \frac{\xi^{\mu-1}\mathrm{e}^{-(t+r)\xi}}{t^{\lambda+1}}\, {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda+1,1,tx),(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| \frac{r}{t}\right] \lfloor c^{-1}(t)\rfloor\, \mathrm{d}\xi\,\mathrm{d}t \\
&= \int_{c_1}^{\infty}\left\{\int_0^x \xi^{\mu-1}\mathrm{e}^{-(t+r)\xi}\,\mathrm{d}\xi\right\} {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda+1,1,tx),(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| \frac{r}{t}\right] \frac{\lfloor c^{-1}(t)\rfloor}{t^{\lambda+1}}\,\mathrm{d}t \\
&= \int_{c_1}^{\infty}\frac{\gamma(\mu,(t+r)x)\lfloor c^{-1}(t)\rfloor}{t^{\lambda+1}(t+r)^{\mu}}\, {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda+1,1,tx),(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| \frac{r}{t}\right]\mathrm{d}t.
\end{aligned}
\tag{3.7}
$$

In a similar way we earn

$$
\mathscr{I}_\xi = \int_{c_1}^{\infty}\frac{\gamma(\mu+1,(t+r)x)\lfloor c^{-1}(t)\rfloor}{t^{\lambda}(t+r)^{\mu+1}}\, {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda,1,tx),(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| \frac{r}{t}\right]\mathrm{d}t. \tag{3.8}
$$

Now, collecting (3.6), (3.7) and (3.8) we confirm the stated representation (3.3). □

Closing the integral expression derivation part of the section, by mimicking the above used proving procedure employing the formula (see for details for instance [2, 22, 24])

$$
\widetilde{\mathscr{D}}_{\mathbf{c}}(z+\xi) = \sum_{j\geqslant 1}(-1)^{j-1}\mathrm{e}^{-c_j(z+\xi)} = (z+\xi)\int_{c_1}^{\infty}\mathrm{e}^{-(z+\xi)x}\sin^2\left(\frac{\pi}{2}\lfloor c^{-1}(x)\rfloor\right)\mathrm{d}x,
$$

we obtain the following result for the alternating Mathieu-type series. So, this result we list without proof.

THEOREM 3.2. *Let $x>0,\mu>0,\lambda>0,r>0$ and the real sequence $\mathbf{c}=(c_n)_{n\geqslant 1}$ monotone increases and tends to ∞. Then the Mathieu–type series $\widetilde{\mathscr{K}}$ (3.2) has the integral representation, reads as follows*

$$
\widetilde{\mathscr{K}}_x^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right) = \widetilde{\mathscr{I}}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda+1,\mu) + \widetilde{\mathscr{I}}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu+1), \tag{3.9}
$$

where

$$
\widetilde{\mathscr{I}}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu) = \int_{c_1}^{\infty}\frac{\gamma(\mu,(t+r)x)\sin^2\left(\frac{\pi}{2}\lfloor c^{-1}(t)\rfloor\right)}{t^{\lambda}(t+r)^{\mu}}\, {}_{p+1}\Psi_q^{(\gamma)}\left[\begin{matrix}(\lambda,1,tx),(\mathbf{a}_p,\mathbf{A}_p)\\(\mathbf{b}_q,\mathbf{B}_q)\end{matrix}\middle| \frac{r}{t}\right]\mathrm{d}t. \tag{3.10}
$$

In the next proposition we present a bilateral Luke–type exponential inequality for the incomplete Fox–Wright function, which is a result interesting by itself being independent from the here studied topic. We point out that its proof is an immediate consequence of [26, Theorem 3] applied to the incomplete Fox–Wright function, there it is omitted.

PROPOSITION 3.1. *Denote*

$$\Phi_m^{(\gamma)} := \frac{\gamma(a_1+mA_1,x)\prod_{j=2}^p \Gamma(a_j+mA_j)}{\prod_{j=1}^q \Gamma(b_j+mB_j)}; \qquad m=0,1,2.$$

If $\Phi_2^{(\gamma)} < \Phi_1^{(\gamma)}$ *and* $(\Phi_1^{(\gamma)})^2 < \Phi_0^{(\gamma)}\Phi_2^{(\gamma)}$, *then for all* $z\in\mathbb{R}$ *we have the following bilateral functional inequality*

$$\Phi_0^{(\gamma)} \mathrm{e}^{\Phi_1^{(\gamma)}(\Phi_0^{(\gamma)})^{-1}|z|} \leqslant {}_p\Psi_q^{(\gamma)}\left[\begin{array}{c}(a_1,A_1,x),(\mathbf{a}_{p-1},\mathbf{A}_{p-1})\\ (\mathbf{b}_q,\mathbf{B}_q)\end{array}\Big| z\right] \leqslant \Phi_0^{(\gamma)} - \Phi_1^{(\gamma)}(1-\mathrm{e}^{|z|}).$$

In continuation two sets of bilateral exponential bounding inequalities are established for the $\mathscr{I}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu)$ and $\widetilde{\mathscr{I}}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu)$ applying Proposition 3.1 in conjunction with (3.4) and (3.10).

PROPOSITION 3.2. *Let the parameter space the same as in* Theorem 3.2 *and* Proposition 3.1. *Then we have*

$$L_{x,\mathbf{c},r}(\lambda,\mu) \leqslant \mathscr{I}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu) \leqslant R_{x,\mathbf{c},r}(\lambda,\mu), \tag{3.11}$$

and

$$\widetilde{L}_{x,\mathbf{c},r}(\lambda,\mu) \leqslant \widetilde{\mathscr{I}}_{x,\mathbf{c},r}^{\Psi^{(\gamma)}}(\lambda,\mu) \leqslant \widetilde{R}_{x,\mathbf{c},r}(\lambda,\mu), \tag{3.12}$$

where

$$L_{x,\mathbf{c},r}(\lambda,\mu) = \Phi_0^{(\gamma)} \int_{c_1}^\infty \frac{\gamma(\mu,(t+r)x)\lfloor c^{-1}(t)\rfloor}{t^\lambda (t+r)^\mu} \mathrm{e}^{(\Phi_1^{(\gamma)}r)/(\Phi_0^{(\gamma)}t)}\,\mathrm{d}t, \tag{3.13}$$

$$\begin{aligned} R_{x,\mathbf{c},r}(\lambda,\mu) &= (\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)}) \int_{c_1}^\infty \frac{\gamma(\mu,(t+r)x)\lfloor c^{-1}(t)\rfloor}{t^\lambda (t+r)^\mu} dt \\ &\quad + \frac{\Phi_1^{(\gamma)}}{r^{\lambda+\mu-1}} \int_0^{\frac{r}{c_1}} \frac{\gamma(\mu,(t^{-1}+1)rx)\lfloor c^{-1}(r/t)\rfloor}{(t+1)^\mu} t^{\lambda+\mu-2}\,\mathrm{e}^t\,\mathrm{d}t, \end{aligned} \tag{3.14}$$

$$\begin{aligned} \widetilde{L}_{x,\mathbf{c},r}(\lambda,\mu) &= \Phi_0^{(\gamma)} \int_{c_1}^\infty \frac{\gamma(\mu,(t+r)x)\sin^2\left(\frac{\pi}{2}\lfloor c^{-1}(t)\rfloor\right)}{t^\lambda (t+r)^\mu} \mathrm{e}^{(\Phi_1^{(\gamma)}r)/(\Phi_0^{(\gamma)}t)}\,\mathrm{d}t, \\ \widetilde{R}_{x,\mathbf{c},r}(\lambda,\mu) &= (\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)}) \int_{c_1}^\infty \frac{\gamma(\mu,(t+r)x)\sin^2\left(\frac{\pi}{2}\lfloor c^{-1}(t)\rfloor\right)}{t^\lambda (t+r)^\mu}\,\mathrm{d}t \\ &\quad + \frac{\Phi_1^{(\gamma)}}{r^{\lambda+\mu-1}} \int_0^{\frac{r}{c_1}} \frac{\gamma(\mu,(t^{-1}+1)rx)\sin^2\left(\frac{\pi}{2}\lfloor c^{-1}(r/t)\rfloor\right)}{(t+1)^\mu} t^{\lambda+\mu-2}\mathrm{e}^t\,\mathrm{d}t. \end{aligned} \tag{3.15}$$

At this point we are ready to formulate our functional inequality result precising the growth rate of the function $c(x)$ in the previous proposition controlling the behavior of both the inverse function $c^{-1}(x)$ and the sequence $\mathbf{c}$.

PROPOSITION 3.3. *Let the parameters' range* $\lambda,\mu,\varepsilon>0$ *such that* $\mu>\varepsilon$; $\lambda+\mu>1+\varepsilon$ *together with another constraints in* Theorem 3.1 *and* Proposition 3.2. *In addition assume that the growth rate of the positive monotone increasing (to infinity) function* $c(x)$ *is at least polynomial:*

$$c(x)\geqslant M_c\, x^{\frac{1}{\lambda+\mu-1-\varepsilon}}, \qquad x>0;\ M_c>0. \tag{3.16}$$

Then

$$L^1_{x,\mathbf{c},r}(\lambda,\mu)\leqslant \mathscr{J}^{\Psi^{(\gamma)}}_{x,\mathbf{c},r}(\lambda,\mu)\leqslant R^1_{x,\mathbf{c},r}(\lambda,\mu), \tag{3.17}$$

where

$$\begin{aligned}
L^1_{x,\mathbf{c},r}(\lambda,\mu) &= \frac{\Phi_0^{(\gamma)}((r+c_1)x)^{\mu}\mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{\mu(\lambda+\mu-1)c_1^{\lambda+\mu-1}}\left(1+\frac{\lambda+\mu-1}{\lambda+\mu}\,\frac{\Phi_1^{(\gamma)}r}{\Phi_0^{(\gamma)}c_1}\right)^{-\mu} \\
&\quad+\frac{r\Phi_1^{(\gamma)}((r+c_1)x)^{\mu}\mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{\mu(\lambda+\mu)c_1^{\lambda+\mu}}\left(1+\frac{\lambda+\mu}{\lambda+\mu+1}\,\frac{\Phi_1^{(\gamma)}r}{\Phi_0^{(\gamma)}c_1}\right)^{-\mu} \\
&\quad-\frac{(r+c_1)^{\mu}x^{\mu+1}\Phi_0^{(\gamma)}(\Phi_1^{(\gamma)})^{\mu}\mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{(\mu+1)(\lambda+\mu-2)c_1^{\lambda+\mu-2}\left(1+\frac{(\lambda+\mu-2)\Phi_1^{(\gamma)}r}{(\lambda+\mu-1)\Phi_0^{(\gamma)}c_1}\right)^{\mu}} \\
&\quad-\frac{r(r+c_1)^{\mu}x^{\mu+1}(\Phi_1^{(\gamma)})^{\mu+1}\mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{(\mu+1)(\lambda+\mu-1)c_1^{\lambda+\mu-1}\left(1+\frac{(\lambda+\mu-1)\Phi_1^{(\gamma)}r}{(\lambda+\mu)\Phi_0^{(\gamma)}c_1}\right)^{\mu}}, \\
R^1_{x,\mathbf{c},r}(\lambda,\mu) &= \frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{c_1^{\varepsilon}(\varepsilon+1)M_c^{\lambda+\mu-1-\varepsilon}}\left[\frac{1}{\varepsilon}+\frac{c_1^{\mu}}{(r+c_1)^{\mu}}+\frac{\mathrm{e}^{\frac{r}{c_1}}-1}{\varepsilon+2}\left(1+\frac{(\varepsilon+1)c_1^{\mu}}{(r+c_1)^{\mu}}\right)\right] \\
&\quad+\frac{\Gamma(\varepsilon)\Gamma(\mu-\varepsilon)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})}{r^{\varepsilon}M_c^{\lambda+\mu-1-\varepsilon}}.
\end{aligned}$$

Proof. As the sequence $\mathbf{c}$ monotone increases, for all $x\geqslant c_1$ it is $\lfloor c^{-1}(x)\rfloor\geqslant 1$. From (3.13) the estimate $e^t\geqslant 1+t, t\in\mathbb{R}$ implies that

$$\begin{aligned}
L_{x,\mathbf{c},r}(\lambda,\mu) &\geqslant \Phi_0^{(\gamma)}\int_{c_1}^{\infty}\frac{\gamma(\mu,(t+r)x)}{t^{\lambda}(t+r)^{\mu}}(1+(\Phi_1^{(\gamma)}r)/(\Phi_0^{(\gamma)}t))dt, \\
&= \frac{\left(\Phi_0^{(\gamma)}\right)^{\lambda+\mu}}{r^{\lambda+\mu-1}(\Phi_1^{(\gamma)})^{\lambda-1}}\int_0^{\frac{\Phi_1^{(\gamma)}r}{\Phi_0^{(\gamma)}c_1}} u^{\lambda+\mu-2}(1+u)\frac{\gamma\big(\mu,(1+(\Phi_1^{(\gamma)}/\Phi_0^{(\gamma)})rxu\big)}{\left(\Phi_1^{(\gamma)}+\Phi_0^{(\gamma)}u\right)^{\mu}}\,\mathrm{d}u.
\end{aligned}$$

Now, applying the lower bound in the inequality [21, Theorem 4.1]

$$\frac{x^{\mu}}{\mu}\mathrm{e}^{-\frac{\mu x}{\mu+1}}\leqslant\gamma(\mu,x)\leqslant\frac{1+\mu\mathrm{e}^{-x}}{\mu+1},$$

we get

$$L_{x,\mathbf{c},r}(\lambda,\mu) \geqslant \frac{((r+c_1)x)^\mu \left(\Phi_0^{(\gamma)}\right)^{\lambda+\mu} \mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{\mu r^{\lambda+\mu-1}(\Phi_1^{(\gamma)})^{\lambda-1}}$$
$$\times \int_0^{\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}} \frac{u^{\lambda+\mu-2}(1+u)}{\left(\Phi_1^{(\gamma)}+\Phi_0^{(\gamma)} u\right)^\mu} \left(1-\frac{\mu r x \Phi_1^{(\gamma)}}{(\mu+1)\Phi_0^{(\gamma)} u}\right) \mathrm{d}u$$
$$= \frac{((r+c_1)x)^\mu \left(\Phi_0^{(\gamma)}\right)^{\lambda+\mu} e^{-\frac{\mu r x}{\mu+1}}}{\mu r^{\lambda+\mu-1}(\Phi_1^{(\gamma)})^{\lambda-1}} \int_0^{\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}} \frac{u^{\lambda+\mu-2}(1+u)}{\left(\Phi_1^{(\gamma)}+\Phi_0^{(\gamma)} u\right)^\mu} \mathrm{d}u$$
$$- \frac{x((r+c_1)x)^\mu \left(\Phi_0^{(\gamma)}\right)^{\lambda+\mu-1} e^{-\frac{\mu r x}{\mu+1}}}{(\mu+1) r^{\lambda+\mu-2}(\Phi_1^{(\gamma)})^{\lambda-2}} \int_0^{\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}} \frac{u^{\lambda+\mu-3}(1+u)}{\left(\Phi_1^{(\gamma)}+\Phi_0^{(\gamma)} u\right)^\mu} \mathrm{d}u.$$

With the aid of the formula [6, Eq. 3.194(1), p. 313]:

$$\int_0^a \frac{t^{b-1}}{(1+t)^c} \mathrm{d}t = \frac{a^b}{b} \, {}_2F_1\left[\begin{matrix} b, c \\ b+1 \end{matrix} \,\middle|\, -a\right], \tag{3.18}$$

where $\Re(b) > 0$, $|\arg(1+a)| < \pi$, and following Luke's inequality [11, Theorem 13, Eq. (4.20)]

$$\frac{1}{(1+\theta z)^\sigma} \leqslant {}_{p+1}F_p\left[\begin{matrix} \sigma, \, \mathbf{a}_p \\ \mathbf{b}_p \end{matrix} \,\middle|\, -z\right] \leqslant 1-\theta+\frac{\theta}{(1+z)^\sigma}; \qquad z, \sigma > 0, \tag{3.19}$$

where $\theta = \prod_{j=1}^p (a_j/b_j); b_j \geqslant a_j, j = 1, \cdots, p$, we obtain

$$L_{x,\mathbf{c},r}(\lambda,\mu) \geqslant \frac{((r+c_1)x)^\mu \left(\Phi_0^{(\gamma)}\right)^{\lambda+\mu} \mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{\mu r^{\lambda+\mu-1}(\Phi_1^{(\gamma)})^{\lambda+\mu-1}} \int_0^{\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}} \frac{u^{\lambda+\mu-2}(1+u)\,\mathrm{d}u}{\left(1+(\Phi_0^{(\gamma)})/\Phi_1^{(\gamma)})u\right)^\mu}$$
$$= \frac{\Phi_0^{(\gamma)}((r+c_1)x)^\mu e^{-\frac{\mu r x}{\mu+1}}}{\mu(\lambda+\mu-1)c_1^{\lambda+\mu-1}} \, {}_2F_1\left[\begin{matrix} \mu, \, \lambda+\mu-1 \\ \lambda+\mu \end{matrix} \,\middle|\, -\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}\right]$$
$$+ \frac{r\Phi_1^{(\gamma)}((r+c_1)x)^\mu \mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{\mu(\lambda+\mu)c_1^{\lambda+\mu}} \, {}_2F_1\left[\begin{matrix} \mu, \, \lambda+\mu \\ \lambda+\mu+1 \end{matrix} \,\middle|\, -\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}\right]$$
$$- \frac{(r+c_1)^\mu x^{\mu+1}\Phi_0^{(\gamma)}(\Phi_1^{(\gamma)})^\mu \mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{(\mu+1)(\lambda+\mu-2)c_1^{\lambda+\mu-2}} \, {}_2F_1\left[\begin{matrix} \mu, \, \lambda+\mu-2 \\ \lambda+\mu-1 \end{matrix} \,\middle|\, -\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}\right]$$
$$- \frac{r(r+c_1)^\mu x^{\mu+1}(\Phi_1^{(\gamma)})^{\mu+1} \mathrm{e}^{-\frac{\mu r x}{\mu+1}}}{(\mu+1)(\lambda+\mu-1)c_1^{\lambda+\mu-1}} \, {}_2F_1\left[\begin{matrix} \mu, \, \lambda+\mu-1 \\ \lambda+\mu \end{matrix} \,\middle|\, -\frac{\Phi_1^{(\gamma)} r}{\Phi_0^{(\gamma)} c_1}\right]$$

$$
\begin{aligned}
&\geqslant \frac{\Phi_0^{(\gamma)}((r+c_1)x)^\mu}{\mu(\lambda+\mu-1)c_1^{\lambda+\mu-1}}\left(1+\frac{\lambda+\mu-1}{\lambda+\mu}\frac{\Phi_1^{(\gamma)}r}{\Phi_0^{(\gamma)}c_1}\right)^{-\mu}\\
&\quad+\frac{r\Phi_1^{(\gamma)}((r+c_1)x)^\mu}{\mu(\lambda+\mu)c_1^{\lambda+\mu}}\left(1+\frac{\lambda+\mu}{\lambda+\mu+1}\frac{\Phi_1^{(\gamma)}r}{\Phi_0^{(\gamma)}c_1}\right)^{-\mu}\\
&\quad-\frac{(r+c_1)^\mu x^{\mu+1}\Phi_0^{(\gamma)}(\Phi_1^{(\gamma)})^\mu \mathrm{e}^{-\frac{\mu rx}{\mu+1}}}{(\mu+1)(\lambda+\mu-2)c_1^{\lambda+\mu-2}\left(1+\frac{(\lambda+\mu-2)\Phi_1^{(\gamma)}r}{(\lambda+\mu-1)\Phi_0^{(\gamma)}c_1}\right)^{\mu}}\\
&\quad-\frac{r(r+c_1)^\mu x^{\mu+1}(\Phi_1^{(\gamma)})^{\mu+1} \mathrm{e}^{-\frac{\mu rx}{\mu+1}}}{(\mu+1)(\lambda+\mu-1)c_1^{\lambda+\mu-1}\left(1+\frac{(\lambda+\mu-1)\Phi_1^{(\gamma)}r}{(\lambda+\mu)\Phi_0^{(\gamma)}c_1}\right)^{\mu}}\\
&=: L^1_{x,\mathbf{c},r}(\lambda,\mu).
\end{aligned}
$$

This inequality proves the lower bound in (3.17).

Now, denote $R^{11}_{x,\mathbf{c},r}(\lambda,\mu)$ and $R^{12}_{x,\mathbf{c},r}(\lambda,\mu)$, say, the first and the second integrals on the right-hand-side in the display (3.14), respectively. Then according to the growth rate declaration (3.16) of the function $c(x)$ we have for its inverse that

$$c^{-1}(x)\leqslant M_c^{1+\varepsilon-\lambda-\mu}\,x^{\lambda+\mu+1-\varepsilon};\qquad x>0\,.$$

In conjunction with the obvious estimate $\gamma(\mu,(t+r)x)\leqslant\Gamma(\mu)$, it readily follows

$$
\begin{aligned}
R^{11}_{x,\mathbf{c},r}(\lambda,\mu)&\leqslant\Gamma(\mu)\int_0^\infty\frac{\lfloor c^{-1}(t)\rfloor}{t^\lambda(t+r)^\mu}\,\mathrm{d}t\leqslant\frac{\Gamma(\mu)}{M_c^*}\int_0^\infty\frac{t^{\mu-1-\varepsilon}}{(t+r)^\mu}\,\mathrm{d}t\\
&=\frac{\Gamma(\mu)\mathrm{B}(\varepsilon,\mu-\varepsilon)}{r^\varepsilon M_c^*}=\frac{\Gamma(\varepsilon)\Gamma(\mu-\varepsilon)}{r^\varepsilon M_c^*},
\end{aligned}\tag{3.20}
$$

where the shorthand $M_c^*=M_c^{\lambda+\mu-1-\varepsilon}$ is used and $\mathrm{B}(x,y)$ stands for the beta function.

Similar calculation procedure yields

$$
\begin{aligned}
R^{12}_{x,\mathbf{c},r}(\lambda,\mu)&\leqslant\Gamma(\mu)\int_0^{\frac{r}{c_1}}\frac{\left[c^{-1}(r/t)\right]}{(t+1)^\mu}t^{\lambda+\mu-2}\mathrm{e}^t\,\mathrm{d}t\\
&\leqslant\Gamma(\mu)\left(\frac{r}{M_c}\right)^{\lambda+\nu-1-\varepsilon}\int_0^{\frac{r}{c_1}}\frac{t^{\varepsilon-1}\mathrm{e}^t}{(t+1)^\mu}\,\mathrm{d}t.
\end{aligned}\tag{3.21}
$$

The convexity of the exponential function $t\mapsto\mathrm{e}^t$ implies that

$$\mathrm{e}^t\leqslant 1+\frac{c_1}{r}(\mathrm{e}^{\frac{r}{c_1}}-1)t;\qquad t\in\left[0,\tfrac{r}{c_1}\right].\tag{3.22}$$

Hence, collecting (3.21), (3.22) and (3.18) we have

$$
\begin{aligned}
R^{12}_{x,\mathbf{c},r}(\lambda,\mu)&\leqslant\Gamma(\mu)\left(\frac{r}{M_c}\right)^{\lambda+\nu-1-\varepsilon}\left\{\int_0^{\frac{r}{c_1}}\frac{t^{\varepsilon-1}}{(t+1)^\mu}\mathrm{d}t+\frac{c_1}{r}(\mathrm{e}^{\frac{r}{c_1}}-1)\int_0^{\frac{r}{c_1}}\frac{t^\varepsilon}{(t+1)^\mu}dt\right\}\\
&=\frac{\Gamma(\mu)r^{\lambda+\nu-1}}{c_1^\varepsilon M_c^*}\left\{\frac{1}{\varepsilon}\,{}_2F_1\left[\begin{matrix}\mu,\varepsilon\\ \varepsilon+1\end{matrix}\,\middle|-\frac{r}{c_1}\right]+\frac{\mathrm{e}^{\frac{r}{c_1}}-1}{\varepsilon+1}\,{}_2F_1\left[\begin{matrix}\mu,\varepsilon+1\\ \varepsilon+2\end{matrix}\,\middle|-\frac{r}{c_1}\right]\right\}.
\end{aligned}
$$

Combining the above inequality with (3.19) we obtain the following bound

$$R^{12}_{x,\mathbf{c},r}(\lambda,\mu) \leqslant \frac{\Gamma(\mu)r^{\lambda+\nu-1}}{c_1^{\varepsilon}(\varepsilon+1)M_c^*}\left[\frac{1}{\varepsilon}+\frac{c_1^{\mu}}{(r+c_1)^{\mu}}+\frac{\mathrm{e}^{\frac{r}{c_1}}-1}{\varepsilon+2}\left(1+\frac{(\varepsilon+1)c_1^{\mu}}{(r+c_1)^{\mu}}\right)\right]. \tag{3.23}$$

Finally, inserting the estimates (3.20) and (3.23) into (3.14) we deduce the right-hand-side upper bound of inequality (3.17). The rest is obvious. □

We present now a bilateral functional inequality for $\mathscr{K}$ and $\widetilde{\mathscr{K}}$. These results are in view of Theorems 3.1 and 3.2 the Propositions 3.1, 3.2 and 3.3.

THEOREM 3.3. *Assume that the parameters $\lambda,\mu,\varepsilon>0$ so, that $\mu>\varepsilon$; $\lambda+\mu>1+\varepsilon$ and there hold another constraints of* Theorem 3.1 *and* Proposition 3.2*. When the growth rate of the positive monotone increasing to infinity function $c(x)$ is constrained by* (3.16)*, we have*

$$L^2_{x,\mathbf{c},r}(\lambda,\mu) \leqslant \mathscr{K}_x^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right) \leqslant R^2_{x,\mathbf{c},r}(\lambda,\mu), \tag{3.24}$$

where

$$\begin{aligned}
L^2_{x,\mathbf{c},r}(\lambda,\mu) &= \frac{\Phi_0^{(\gamma)}((r+c_1)x)^{\mu}e^{-\frac{\mu rx}{\mu+1}}}{\mu(\lambda+\mu)c_1^{\lambda+\mu}}\left(1+\frac{\lambda+\mu}{\lambda+\mu+1}\frac{\Phi_1^{(\gamma)}r}{\Phi_0^{(\gamma)}c_1}\right)^{-\mu} \\
&\quad\times\left\{1+\frac{(r+c_1)\mu x}{\mu+1}\frac{c_1(\lambda+\mu+1)\Phi_0^{(\gamma)}\mathrm{e}^{-\frac{rx}{(\mu+1)(\mu+2)}}}{(\lambda+\mu+1)\Phi_0^{(\gamma)}c_1+(\lambda+\mu)\Phi_1^{(\gamma)}r}\right\} \\
&\quad+\frac{r\Phi_1^{(\gamma)}((r+c_1)x)^{\mu}e^{-\frac{\mu rx}{\mu+1}}}{\mu(\lambda+\mu+1)c_1^{\lambda+\mu+1}}\left(1+\frac{\lambda+\mu+1}{\lambda+\mu+2}\frac{\Phi_1^{(\gamma)}r}{\Phi_0^{(\gamma)}c_1}\right)^{-\mu} \\
&\quad\times\left\{1+\frac{(r+c_1)\mu x}{\mu+1}\frac{c_1(\lambda+\mu+2)\Phi_0^{(\gamma)}\mathrm{e}^{-\frac{rx}{(\mu+1)(\mu+2)}}}{(\lambda+\mu+2)\Phi_0^{(\gamma)}c_1+(\lambda+\mu+1)\Phi_1^{(\gamma)}r}\right\} \\
&\quad-\frac{x^{\mu+1}(r+c_1)^{\mu}\Phi_0^{(\gamma)}(\Phi_1^{(\gamma)})^{\mu}\,\mathrm{e}^{-\frac{\mu rx}{\mu+1}}}{(\mu+1)(\lambda+\mu-1)c_1^{\lambda+\mu-1}\left(1+\frac{r(\lambda+\mu-1)\Phi_1^{(\gamma)}}{c_1(\lambda+\mu)\Phi_0^{(\gamma)}}\right)} \\
&\quad\times\left[1+\frac{xc_1(\mu+1)(r+c_1)(\lambda+\mu)\Phi_0^{(\gamma)}\Phi_1^{(\gamma)}\,\mathrm{e}^{-\frac{rx}{(\mu+1)(\mu+2)}}}{(\mu+2)\left(c_1(\lambda+\mu)\Phi_0^{(\gamma)}+r(\lambda+\mu-1)\Phi_1^{(\gamma)}\right)}\right] \\
&\quad-\frac{rx^{\mu+1}(r+c_1)^{\mu}\Phi_0^{(\gamma)}(\Phi_1^{(\gamma)})^{\mu}\,\mathrm{e}^{-\frac{\mu rx}{\mu+1}}}{(\mu+1)(\lambda+\mu)c_1^{\lambda+\mu}\left(1+\frac{r(\lambda+\mu)\Phi_1^{(\gamma)}}{c_1(\lambda+\mu+1)\Phi_0^{(\gamma)}}\right)} \\
&\quad\times\left[1+\frac{xc_1(\mu+1)(r+c_1)(\lambda+\mu+1)\Phi_0^{(\gamma)}\Phi_1^{(\gamma)}\,\mathrm{e}^{-\frac{rx}{(\mu+1)(\mu+2)}}}{(\mu+2)\left(c_1(\lambda+\mu+1)\Phi_0^{(\gamma)}+r(\lambda+\mu)\Phi_1^{(\gamma)}\right)}\right]
\end{aligned}$$

$$R^2_{x,\mathbf{c},r}(\lambda,\mu) = \frac{(1+\mu-\varepsilon)\Gamma(\varepsilon)\Gamma(\mu-\varepsilon)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})}{r^{\varepsilon}M_c^{\lambda+\mu-\varepsilon}} + \frac{\Gamma(\mu)\Phi_1^{(\gamma)})}{c_1^{\varepsilon}(\varepsilon+1)M_c^{\lambda+\mu-\varepsilon}}$$
$$\times\left\{\frac{\mathrm{e}^{\frac{r}{c_1}}-1}{\varepsilon+2}\left(1+\mu+\frac{(\varepsilon+1)(r+(\mu+1)c_1)c_1^{\mu}}{(r+c_1)^{\mu+1}}\right)+\frac{1+\mu}{\varepsilon}+\frac{c_1^{\mu}(r+(\mu+1)c_1)}{(r+c_1)^{\mu+1}}\right\}.$$

PROPOSITION 3.4. *Let $x,r>0$ and the positive sequence $\mathbf{c}=(c_n)_{n\geqslant 1}$ monotone increases to ∞. Assume that the parameters $\lambda\in(0,1),\mu>0$ so, that $\lambda+\mu>1$. Then we have*

$$0\leqslant \widetilde{\mathscr{J}}^{\Psi(\gamma)}_{x,\mathbf{c},r}(\lambda,\mu)\leqslant \widetilde{R}^1_{x,\mathbf{c},r}(\lambda,\mu), \tag{3.25}$$

where

$$\begin{aligned}\widetilde{R}^1_{x,\mathbf{c},r}(\lambda,\mu) &= \frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{c_1^{\lambda+\mu-1}(\lambda+\mu)}\left(\frac{1}{\lambda+\mu-1}+\frac{c_1^{\mu}}{(c_1+r)^{\mu}}\right)\\ &\quad+\frac{\Gamma(\mu)\Phi_1^{(\gamma)}(\mathrm{e}^{\frac{r}{c_1}}-1)}{c_1^{\lambda+\mu-1}(\lambda+\mu+1)}\left(\frac{1}{\lambda+\mu}+\frac{c_1^{\mu}}{(c_1+r)^{\mu}}\right)\\ &\quad+\frac{\Gamma(\lambda+\mu-1)\Gamma(1-\lambda)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})}{r^{\lambda+\mu-1}}.\end{aligned}$$

Proof. Since $\sin^2(x)\leqslant 1, x\in\mathbb{R}$, keeping in mind that [6, p. 313, Eq. 3.194.3]

$$\int_0^{\infty}\frac{\mathrm{d}t}{t^{\lambda}(t+r)^{\mu}} = \frac{1}{r^{\lambda+\mu-1}}\mathrm{B}(\lambda+\mu-1,1-\lambda) = \frac{\Gamma(\lambda+\mu-1)\Gamma(1-\lambda)}{\Gamma(\mu)\,r^{\lambda+\mu-1}}, \tag{3.26}$$

and the convex sum of two integrals of the form [6, p. 313, Eq. 3.194.1] which reads

$$\begin{aligned}\int_0^{\frac{r}{c_1}}\frac{t^{\lambda+\mu-1}}{(t+1)^{\mu}}\Big(1+\frac{c_1}{r}(\mathrm{e}^{\frac{r}{c_1}}-1)t\Big)\mathrm{d}t &= \left(\frac{r}{c_1}\right)^{\lambda+\mu}\frac{1}{\lambda+\mu}\,{}_2F_1\left[\begin{matrix}\mu,\lambda+\mu\\ \lambda+\mu+1\end{matrix}\,\Big|\,-\frac{r}{c_1}\right]\\ &\quad+\left(\frac{r}{c_1}\right)^{\lambda+\mu}\frac{\mathrm{e}^{\frac{r}{c_1}}-1}{\lambda+\mu+1}\,{}_2F_1\left[\begin{matrix}\mu,\lambda+\mu+1\\ \lambda+\mu+2\end{matrix}\,\Big|\,-\frac{r}{c_1}\right],\end{aligned} \tag{3.27}$$

the estimate of (3.15) becomes

$$\begin{aligned}\widetilde{R}_{x,\mathbf{c},r}(\lambda,\mu) &\leqslant \Gamma(\mu)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\int_0^{\infty}\frac{\mathrm{d}t}{t^{\lambda}(t+r)^{\mu}}+\frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{r^{\lambda+\mu-1}}\int_0^{\frac{r}{c_1}}\frac{t^{\lambda+\mu-2}\mathrm{e}^{t}}{(t+1)^{\mu}}\mathrm{d}t\\ &\leqslant \frac{\Gamma(\mu)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})}{r^{\lambda+\mu-1}}\mathrm{B}(\lambda+\mu-1,1-\lambda)+\frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{r^{\lambda+\mu-1}}\int_0^{\frac{r}{c_1}}\frac{t^{\lambda+\mu-2}}{(t+1)^{\mu}}\mathrm{d}t\\ &\quad+\frac{c_1\Gamma(\mu)\Phi_1^{(\gamma)}(\mathrm{e}^{\frac{r}{c_1}}-1)}{r^{\lambda+\mu}}\int_0^{\frac{r}{c_1}}\frac{t^{\lambda+\mu-1}}{(t+1)^{\mu}}\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(\lambda+\mu-1)\Gamma(1-\lambda)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})}{r^{\lambda+\mu-1}} \\
&\quad + \frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{c_1^{\lambda+\mu-1}(\lambda+\mu-1)}\, {}_2F_1\left[\begin{matrix}\mu,\lambda+\mu-1\\ \lambda+\mu\end{matrix}\,\Big|\, -\frac{r}{c_1}\right] \\
&\quad + \frac{\Gamma(\mu)\Phi_1^{(\gamma)}(\mathrm{e}^{\frac{r}{c_1}}-1)}{c_1^{\lambda+\mu-1}(\lambda+\mu)}\, {}_2F_1\left[\begin{matrix}\mu,\lambda+\mu\\ \lambda+\mu+1\end{matrix}\,\Big|\, -\frac{r}{c_1}\right] =: H_{x,\mathbf{c}}(r).
\end{aligned}
$$

Ergo, combining the resulting expression $H_{x,\mathbf{c}}(r)$ with Luke's hypergeometric inequality (3.19) we arrive at the asserted upper bound $\widetilde{R}^1_{x,\mathbf{c},r}(\lambda,\mu)$. □

THEOREM 3.4. *Let $x,r>0$ and the positive sequence $\mathbf{c}=(c_n)_{n\geqslant 1}$ monotone increases to ∞. Assume that the parameters $\lambda\in(0,1), \mu>0$ so, that $\lambda+\mu>1$. Then we have*

$$0 \leqslant \widetilde{\mathscr{K}}_x^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right) \leqslant \widetilde{R}^2_{x,\mathbf{c},r}(\lambda,\mu), \tag{3.28}$$

where

$$
\begin{aligned}
\widetilde{R}^2_{x,\mathbf{c},r}(\lambda,\mu) &= \frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{c_1^{\lambda+\mu}(\lambda+\mu+1)}\left\{\frac{\mu+1}{\lambda+\mu}+\frac{c_1^{\mu}}{(c_1+r)^{\mu}}\left(1+\frac{\mu c_1}{c_1+r}\right)\right\} \\
&\quad + \frac{\Gamma(\mu)\Phi_1^{(\gamma)}\,(\mathrm{e}^{\frac{r}{c_1}}-1)}{c_1^{\lambda+\mu}(\lambda+\mu+2)}\left\{\frac{\mu+1}{\lambda+\mu+1}+\frac{c_1^{\mu}}{(c_1+r)^{\mu}}\left(1+\frac{\mu c_1}{c_1+r}\right)\right\} \\
&\quad + \frac{\Gamma(\lambda+\mu)\Gamma(2-\lambda)}{\lambda\, r^{\lambda+\mu}}\left(\Phi_1^{(\gamma)}-\Phi_0^{(\gamma)}\right).
\end{aligned}
$$

Proof. Having in mind the relations (3.9), (3.12) and (3.15) we get

$$
\begin{aligned}
\widetilde{\mathscr{K}}_x^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right) &\leqslant \widetilde{\mathscr{J}}_{x,\mathbf{c},r}(\lambda+1,\mu)+\widetilde{\mathscr{J}}_{x,\mathbf{c},r}(\lambda,\mu+1) \\
&\leqslant \widetilde{R}^1_{x,\mathbf{c},r}(\lambda+1,\mu)+\widetilde{R}^1_{x,\mathbf{c},r}(\lambda,\mu+1) =: \widetilde{R}^2_{x,\mathbf{c},r}(\lambda,\mu),
\end{aligned}
$$

where the second–line estimate we conclude by (3.25). After routine calculation we derive

$$
\begin{aligned}
\widetilde{R}^2_{x,\mathbf{c},r}(\lambda,\mu) &= \frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{c_1^{\lambda+\mu}(\lambda+\mu+1)}\left\{\frac{\mu+1}{\lambda+\mu}+\frac{c_1^{\mu}}{(c_1+r)^{\mu}}\left(1+\frac{\mu c_1}{c_1+r}\right)\right\} \\
&\quad + \frac{\Gamma(\mu)\Phi_1^{(\gamma)}\,(\mathrm{e}^{\frac{r}{c_1}}-1)}{c_1^{\lambda+\mu}(\lambda+\mu+2)}\left\{\frac{\mu+1}{\lambda+\mu+1}+\frac{c_1^{\mu}}{(c_1+r)^{\mu}}\left(1+\frac{\mu c_1}{c_1+r}\right)\right\} \\
&\quad + \frac{\Gamma(\lambda+\mu)}{r^{\lambda+\mu}}\left(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)}\right)\left(\Gamma(1-\lambda)+\Gamma(-\lambda)\right).
\end{aligned}
$$

Observing that $\Gamma(1-\lambda)+\Gamma(-\lambda)=-\lambda^{-1}\Gamma(2-\lambda)$, we finish the derivation of the stated bound by routine steps. □

Finally, we point out that there is a way to avoid Proposition 3.4 in extracting the upper bound for $\widetilde{\mathscr{K}_x}^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right)$, we can also apply (3.9), (3.12) but now starting with (3.15) in derivation. Thus,

$$\begin{aligned}
&\widetilde{\mathscr{K}_x}^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right)\\
&\leqslant \widetilde{R}_{x,\mathbf{c},r}(\lambda+1,\mu)+\widetilde{R}_{x,\mathbf{c},r}(\lambda,\mu+1)\\
&\leqslant (\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\int_{c_1}^{\infty}\frac{\Gamma(\mu)\mathrm{d}t}{t^{\lambda+1}(t+r)^{\mu}}+\frac{\Phi_1^{(\gamma)}}{r^{\lambda+\mu}}\int_0^{\frac{r}{c_1}}\frac{\Gamma(\mu)t^{\lambda+\mu-1}\mathrm{e}^t}{(t+1)^{\mu}}\mathrm{d}t\\
&\quad+(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\int_{c_1}^{\infty}\frac{\Gamma(\mu+1)\mathrm{d}t}{t^{\lambda}(t+r)^{\mu+1}}+\frac{\Phi_1^{(\gamma)}}{r^{\lambda+\mu}}\int_0^{\frac{r}{c_1}}\frac{\Gamma(\mu+1)t^{\lambda+\mu-1}\mathrm{e}^t}{(t+1)^{\mu+1}}\mathrm{d}t\\
&=(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\int_{c_1}^{\infty}\frac{\Gamma(\mu)}{t^{\lambda}(t+r)^{\mu}}\left(\frac{1}{t}+\frac{\mu}{t+r}\right)\mathrm{d}t\\
&\quad+\frac{\Gamma(\mu)\Phi_1^{(\gamma)}}{r^{\lambda+\mu}}\int_0^{\frac{r}{c_1}}\frac{t^{\lambda+\mu-1}\mathrm{e}^t}{(t+1)^{\mu}}\left(1+\frac{\mu}{1+t}\right)\mathrm{d}t\\
&\leqslant (\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\left(\frac{1}{c_1}+\frac{\mu}{c_1+r}\right)\int_{c_1}^{\infty}\frac{\Gamma(\mu)}{t^{\lambda}(t+r)^{\mu}}\mathrm{d}t\\
&\quad+\frac{(\mu+1)\Gamma(\mu)\Phi_1^{(\gamma)}}{r^{\lambda+\mu}}\int_0^{\frac{r}{c_1}}\frac{t^{\lambda+\mu-1}\mathrm{e}^t}{(t+1)^{\mu}}\mathrm{d}t\\
&\leqslant (\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\left(\frac{1}{c_1}+\frac{\mu}{c_1+r}\right)\int_{0}^{\infty}\frac{\Gamma(\mu)}{t^{\lambda}(t+r)^{\mu}}\mathrm{d}t\\
&\quad+\frac{(\mu+1)\Gamma(\mu)\Phi_1^{(\gamma)}}{r^{\lambda+\mu}}\int_0^{\frac{r}{c_1}}\frac{t^{\lambda+\mu-1}}{(t+1)^{\mu}}\left(1+\frac{c_1}{r}(\mathrm{e}^{\frac{r}{c_1}}-1)t\right)\mathrm{d}t.
\end{aligned}$$

Combining this estimate with the integrals (3.26) and (3.27) we obtain

$$\begin{aligned}
\widetilde{\mathscr{K}_x}^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right)\leqslant{}&\frac{(c_1(1+\mu)+r)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\Gamma(\lambda+\mu-1)\Gamma(1-\lambda)}{c_1(c_1+r)r^{\lambda+\mu-1}}\\
&+\frac{(\mu+1)\Gamma(\mu)\Phi_1^{(\gamma)}}{(\lambda+\mu)c_1^{\lambda+\mu}}\,{}_2F_1\left[\begin{matrix}\mu,\lambda+\mu\\ \lambda+\mu+1\end{matrix}\,\middle|\,-\frac{r}{c_1}\right]\\
&+\frac{(\mu+1)\Gamma(\mu)\Phi_1^{(\gamma)}(\mathrm{e}^{\frac{r}{c_1}}-1)}{(\lambda+\mu+1)c_1^{\lambda+\mu}}\,{}_2F_1\left[\begin{matrix}\mu,\lambda+\mu+1\\ \lambda+\mu+2\end{matrix}\,\middle|\,-\frac{r}{c_1}\right].
\end{aligned}$$

Now, Luke's inequality (3.19) transforms this bound into

$$0\leqslant\widetilde{\mathscr{K}_x}^{(\lambda,\mu)}\left({}_{p+1}\Psi_q^{(\gamma)};\mathbf{c};r\right)\leqslant\widetilde{\mathscr{R}}^{3}_{x,\mathbf{c},r}(\lambda,\mu),$$

where

$$\widetilde{\mathscr{R}}^{3}_{x,\mathbf{c},r}(\lambda,\mu) = \frac{(c_1(1+\mu)+r)(\Phi_0^{(\gamma)}-\Phi_1^{(\gamma)})\Gamma(\lambda+\mu-1)\Gamma(1-\lambda)}{c_1(c_1+r)r^{\lambda+\mu-1}}$$
$$+\frac{(\mu+1)\Gamma(\mu)\Phi_1^{(\gamma)}}{(\lambda+\mu)(\lambda+\mu+1)c_1^{\lambda+\mu}}\left(1+\frac{\lambda+\mu}{(\lambda+\mu+1)(1+r/c_1)^{\mu}}\right)$$
$$+\frac{(\mu+1)\Gamma(\mu)\Phi_1^{(\gamma)}(\mathrm{e}^{\frac{r}{c_1}}-1)}{(\lambda+\mu+1)(\lambda+\mu+2)c_1^{\lambda+\mu}}\left(1+\frac{\lambda+\mu+1}{(\lambda+\mu+2)(1+r/c_1)^{\mu}}\right),$$

which completes the discussion.

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(Received June 4, 2020)

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