Journal of Mathematical Inequalities
Volume 15, Number 3 (2021), 1003–1018 doi:10.7153/jmi-2021-15-68

ON (n,k)–QUASI CLASS Q^* OPERATORS

Ilmi Hoxha and Naim L. Braha *

(*Communicated by J. Mićić Hot*)

Abstract. Let T be a bounded linear operator on a complex Hilbert space H. In this paper we introduce a new class of operators: (n,k)-quasi class Q^* operators, superclass of (n,k)-quasi-$*$-paranormal operators.

An operator T is said to be (n,k)-quasi class Q^* if it satisfies

$$\|T^*(T^kx)\|^2 \leqslant \frac{1}{n+1}\left(\|T^{n+1}(T^kx)\|^2 + n\|T^kx\|^2\right),$$

for all $x \in H$ and for some nonnegative integers n and k. We will prove structural and spectral properties of this class of operators, and also prove the spectrum continuity of this class of operators.

1. Introduction

Throughout this paper, let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $L(H)$ denote the C^* algebra of all bounded operators on H. For $T \in L(H)$, we denote by $\ker(T)$ the null space and by $T(H)$ the range of T. The null operator and the identity on H will be denoted by O and I, respectively. If T is an operator, then T^* is its adjoint, and $\|T\| = \|T^*\|$.

We shall denote the set of all complex numbers by $\mathbb{C}$, the set of all positive integers by $\mathbb{N}$, the set of all nonnegative integers by $\mathbb{N}_0$ and the complex conjugate of a complex number λ by $\overline{\lambda}$. The closure of a set M will be denoted by $\overline{M}$ and we shall henceforth shorten $T-\lambda I$ to $T-\lambda$. An operator $T \in L(H)$ is a positive operator, $T \geqslant O$, if $\langle Tx,x\rangle \geqslant 0$ for all $x \in H$.

We write $\sigma(T)$, $\sigma_p(T)$ and $\sigma_a(T)$ for the spectrum, point spectrum and approximate point spectrum, respectively. Sets of isolated points and accumulation points of $\sigma(T)$ are denoted by $\mathrm{iso}\sigma(T)$ and $\mathrm{acc}\sigma(T)$, respectively. We write $r(T)$ for the spectral radius. It is well known that $r(T) \leqslant \|T\|$. The operator T is called normaloid if $r(T) = \|T\|$.

We write $\alpha(T) = \dim\ker T$, $\beta(T) = \dim\left(H \setminus \overline{T(H)}\right)$. An operator $T \in L(H)$ is called an upper semi-Fredholm, if it has a closed range and $\alpha(T) < \infty$, while T is called a lower semi-Fredholm if $\beta(T) < \infty$. However, T is called a semi-Fredholm operator if

Mathematics subject classification (2020): Primary 47B20; Secondary 47A80, 47B37.
Keywords and phrases: (n,k)-quasi class Q^*, (n,k)-quasi-$*$-paranormal operators, spectrum continuity.

* Corresponding author.

T is either an upper or a lower semi-Fredholm, and T is said to be a Fredholm operator if it is both an upper and a lower semi-Fredholm. If $T \in L(H)$ is semi-Fredholm, then the index is defined by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

An operator $T \in L(H)$, is said to be paranormal [3], if

$$\|Tx\|^2 \leqslant \|T^2x\|$$

for any unit vector x in H. An operator $T \in L(H)$, is said to be $*$-paranormal [1], if

$$\|T^*x\|^2 \leqslant \|T^2x\|$$

for any unit vector x in H.

In papers [8, 9], the author has proved that a k-quasi-$*$-class A operator is a k-quasi-$*$-paranormal operator.

Hoxha and Braha, [6] introduced a new class of operators called k-quasi-$*$-paranormal operators. An operator T is called k-quasi-$*$-paranormal if

$$\|T^*T^kx\|^2 \leqslant \|T^{k+2}x\|\|T^kx\|,$$

for all $x \in H$, where k is a nonnegative integer number.

Q. Zeng and H. Zhong [12] introduced a new class of operators called (n,k)-quasi-$*$-paranormal operators: An operator $T \in L(H)$ is said to be (n,k)-quasi-$*$-paranormal operators if

$$\|T^*(T^kx)\| \leqslant \|T^{n+1}(T^kx)\|^{\frac{1}{n+1}}\|T^kx\|^{\frac{n}{n+1}},$$

for all $x \in H$ and for some nonnegative integers n and k.

2. Structural properties

Now we introduce the class of (n,k)-quasi class Q^* operators defined as follows:

DEFINITION 2.1. An operator $T \in L(H)$ is said to be (n,k)-quasi class Q^* if

$$\|T^*(T^kx)\|^2 \leqslant \frac{1}{n+1}\left(\|T^{n+1}(T^kx)\|^2 + n\|T^kx\|^2\right),$$

for all $x \in H$ and for some nonnegative integer numbers n and k.

A $(1,k)$-quasi class Q^* operator is a k-quasi class Q^* operator:

$$\|T^*(T^kx)\|^2 \leqslant \frac{1}{2}\left(\|T^{k+2}x\|^2 + \|T^kx\|^2\right);$$

a $(1,1)$-quasi class Q^* operator is a quasi class Q^* operator:

$$\|T^*(Tx)\|^2 \leqslant \frac{1}{2}\left(\|T^3x\|^2 + \|Tx\|^2\right);$$

a $(1,0)$-quasi class Q^* operator is a class Q^* operator:

$$\|T^*x\|^2 \leqslant \frac{1}{2}\left(\|T^2x\|^2 + \|x\|^2\right);$$

an $(n,0)$-quasi class Q^* operator is an n-class Q^* operator

$$\|T^*x\|^2 \leqslant \frac{1}{n+1}\left(\|T^{n+1}x\|^2 + n\|x\|^2\right).$$

Q. Zeng and H. Zhong [12, Lemma 2.2] prove that an operator $T \in L(H)$ is of the (n,k)-quasi-$*$-paranormal if and only if

$$T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)\lambda^n TT^* + n\lambda^{n+1}I\right)T^k \geqslant O, \text{ for all } \lambda > 0.$$

THEOREM 2.2. *An operator $T \in L(H)$ is of the (n,k)-quasi class Q^*, if and only if*

$$T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^k \geqslant O,$$

where k and n are nonnegative integer numbers.

Proof. Since T is of the (n,k)-quasi class Q^*, then

$$(n+1)\|T^*(T^kx)\|^2 \leqslant \left(\|T^{n+1}(T^kx)\|^2 + n\|T^kx\|^2\right),$$

for all $x \in H$, where $k,n \in \mathbb{N}_0$. Then,

$$\left\langle T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^kx, x\right\rangle \geqslant 0$$

for all $x \in H$, where k and n are nonnegative integer numbers. The last relation is equivalent to

$$T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^k \geqslant O. \quad \square$$

LEMMA 2.3. *For positive real numbers $a > 0$ and $b > 0$,*

$$\lambda a + \mu b \geqslant a^\lambda b^\mu$$

holds for $\lambda > 0$ and $\mu > 0$ such that $\lambda + \mu = 1$.

LEMMA 2.4. *If T is an (n,k)-quasi-$*$-paranormal operator, then T is an (n,k)-quasi class Q^* operator.*

Proof. Let T be an (n,k)-quasi-$*$-paranormal operator. Then, we have

$$\begin{aligned}\|T^*(T^kx)\|^2 &\leqslant \|T^{n+1}(T^kx)\|^{\frac{2}{1+n}}\|T^kx\|^{\frac{2n}{n+1}} \\ &\leqslant \frac{1}{n+1}\|T^{n+1}(T^kx)\|^2 + \frac{n}{n+1}\|T^kx\|^2\end{aligned}$$

so, T is an (n,k)-quasi class Q^* operator. $\square$

An operator $T \in \mathscr{L}(\mathscr{H})$, is said to belong to k-quasi class $\mathscr{A}_n^*$ operator ([7]) if

$$T^{*k}\left(|T^{n+1}|^{\frac{2}{n+1}} - |T^*|^2\right)T^k \geqslant O$$

for $n,k \in \mathbb{N}_0$.

From [7, Theorem 2.5] if T is a k-quasi class $\mathscr{A}_n^*$ operator, then T is an (n,k)-quasi-$*$-paranormal operator, from the above theorem T is an (n,k)-quasi class Q^* operator.

If T is an (n,k)-quasi class Q^* operator, then T is an $(n,k+1)$-quasi class Q^* operator. The inverse is not true, as it can be seen below.

Consider the unilateral weighted shift operators as an infinite dimensional Hilbert space operator. Recall that given a bounded sequence of a positive numbers $\alpha : \alpha_1, \alpha_2, \alpha_3, \alpha_4, \ldots$ (called weights) the unilateral weighted shift W_α associated with weight α is the operator on $H = l_2$ defined by $W_\alpha e_m = \alpha_m e_{m+1}$ for all $m \geqslant 1$, where $\{e_m\}_{m=1}^{\infty}$ is the canonical orthonormal basis on l_2.

$$\mathbf{W}_\alpha = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \ldots \\ \alpha_1 & 0 & 0 & 0 & 0 \ldots \\ 0 & \alpha_2 & 0 & 0 & 0 \ldots \\ 0 & 0 & \alpha_3 & 0 & 0 \ldots \\ 0 & 0 & 0 & \alpha_4 & 0 \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots \ddots \end{pmatrix}$$

Let $\operatorname{diag}(\{\alpha_m\}_{m=1}^{\infty}) = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3, \ldots)$ denote an infinite diagonal matrix on l_2. Then,

$$\begin{aligned}
&W_\alpha^{*k}\left(W_\alpha^{*(n+1)}W_\alpha^{(n+1)} - (n+1)W_\alpha W_\alpha^* + n\right)W_\alpha^k \\
&= \operatorname{diag}(\{\alpha_m^2\alpha_{m+1}^2 \cdot \ldots \cdot \alpha_{m+k-2}^2\alpha_{m+k-1}^2\alpha_{m+k}^2\alpha_{m+k+1}^2 \cdot \ldots \cdot \alpha_{m+k+n-1}^2\alpha_{m+k+n}^2\}_{m=1}^{\infty}) \\
&\quad -(n+1)\operatorname{diag}(\{\alpha_m^2\alpha_{m+1}^2 \cdot \ldots \cdot \alpha_{m+k-2}^2\alpha_{m+k-1}^2\alpha_{m+k-1}^2\}_{m=1}^{\infty}) \\
&\quad +n\operatorname{diag}(\{\alpha_m^2\alpha_{m+1}^2 \cdot \ldots \cdot \alpha_{m+k-1}^2\}_{m=1}^{\infty})
\end{aligned}$$

Then,

$$\alpha_m^2\alpha_{m+1}^2 \cdot \ldots \cdot \alpha_{m+k-2}^2\alpha_{m+k-1}^2(\alpha_{m+k}^2\alpha_{m+k+1}^2 \cdot \ldots \cdot \alpha_{m+k+n-1}^2\alpha_{m+k+n}^2 \\ -(n+1)\alpha_{m+k-1}^2 + n) \geqslant 0.$$

Thus, W_α is an (n,k)-quasi class Q^* operator, if and only if,

$$\alpha_{m+k}^2\alpha_{m+k+1}^2 \cdot \ldots \cdot \alpha_{m+k+n-1}^2\alpha_{m+k+n}^2 - (n+1)\alpha_{m+k-1}^2 + n \geqslant 0,$$

for $m \geqslant 1$.

If $\alpha_1 = 2$ and $\alpha_m = 1$ for $m \geqslant 2$, then W_α is a $(2,2)$-quasi class Q^* operator but is not a $(2,1)$-quasi class Q^* operator.

Since (n,k)-quasi-$*$-paranormal is not a normaloid operator [12, Example 2.3(4)], then (n,k)-quasi class Q^* is not a normaloid operator for $k \geqslant 2$.

THEOREM 2.5. *If T is an (n,k)-quasi class Q^* operator, which commutes with a unitary S, then TS is an (n,k)-quasi class Q^* operator.*

Proof. Let $A = TS$, $TS = ST$, $S^*T^* = T^*S^*$ and $SS^* = S^*S = I$.

$$\begin{aligned}&A^{*(n+k+1)}A^{(n+k+1)} - (n+1)A^{*(k)}AA^*A^k + nA^{*k}A^k \\ =&(TS)^{*(n+k+1)}(TS)^{(n+k+1)} - (n+1)(TS)^{*k}(TS)(TS)^*(TS)^k + n(TS)^{*k}(TS)^k \\ =&S^{*k}T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^kS^k \geqslant O,\end{aligned}$$

so that TS is an (n,k)-quasi class Q^* operator. $\square$

THEOREM 2.6. *Let T be an (n,k)-quasi class Q^* operator. If T is unitarily equivalent to an operator S, then S is an (n,k)-quasi class Q^* operator.*

Proof. Since T is unitarily equivalent to an operator S, then $S = U^*TU$. We have

$$\begin{aligned}&S^{*k}\left(S^{*(n+1)}S^{(n+1)} - (n+1)SS^* + nI\right)S^k \\ =&(U^*TU)^{*k}\left((U^*TU)^{*(n+1)}(U^*TU)^{(n+1)} - (n+1)(U^*TU)(U^*TU)^* + nI\right)(U^*TU)^k \\ =&U^*T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^kU \geqslant O\end{aligned}$$

so that S is an (n,k)-quasi class Q^* operator. $\square$

THEOREM 2.7. *If T does not have a dense range, then the following statements are equivalent:*

(1) T is an (n,k)-quasi class Q^* operator

(2)

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad \text{on } H = \overline{T^k(H)} \oplus \ker(T^{*k}),$$

where $A^{*(n+1)}A^{(n+1)} - (n+1)(AA^* + BB^*) + nI \geqslant O$, and $C^k = O$. Furthermore, $\sigma(T) = \sigma(A) \cup \{0\}$.

Proof. $(1) \Rightarrow (2)$ Consider the matrix representation of T with respect to the decomposition $H = \overline{T^k(H)} \oplus \ker(T^{*k})$:

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix}.$$

Let P be the projection onto $\overline{T^k(H)}$. Since T is an (n,k)-quasi class Q^* operator, we have

$$P\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)P \geqslant O.$$

Therefore

$$A^{*(n+1)}A^{(n+1)} - (n+1)(AA^* + BB^*) + nI \geqslant O.$$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H = \overline{T^k(H)} \oplus \ker(T^{*k})$. Then,

$$\langle C^k x_2, x_2 \rangle = \left\langle T^k(I-P)x, (I-P)x \right\rangle = \left\langle (I-P)x, T^{*k}(I-P)x \right\rangle = 0,$$

thus $C^k = O$.

By [4, Corollary 7], $\sigma(A) \cup \sigma(C) = \sigma(T) \cup \vartheta$, where ϑ is the union of the holes in $\sigma(T)$, which happen to be a subset of $\sigma(A) \cap \sigma(C)$, and $\sigma(A) \cap \sigma(C)$ has no interior points. Therefore $\sigma(T) = \sigma(A) \cup \sigma(C)$. Since C is nilpotent, we have $\sigma(T) = \sigma(A) \cup \{0\}$.

$(2) \Rightarrow (1)$ Suppose $T = \begin{pmatrix} A & B \\ O & C \end{pmatrix}$ on $H = \overline{T^k(H)} \oplus \ker(T^{*k})$, where

$$A^{*(n+1)}A^{(n+1)} - (n+1)(AA^* + BB^*) + nI \geqslant O$$

and $C^k = O$. Since

$$T^k = \begin{pmatrix} A^k & \sum_{j=0}^{k-1} A^j B C^{k-1-j} \\ O & O \end{pmatrix}$$

we have

$$\begin{aligned}
&T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^k \\
&= \begin{pmatrix} A & B \\ O & C \end{pmatrix}^{*k} \left(\begin{pmatrix} A & B \\ O & C \end{pmatrix}^{*(n+1)} \begin{pmatrix} A & B \\ O & C \end{pmatrix}^{(n+1)} - (n+1) \begin{pmatrix} A & B \\ O & C \end{pmatrix} \begin{pmatrix} A & B \\ O & C \end{pmatrix}^{*} + nI \right) \begin{pmatrix} A & B \\ O & C \end{pmatrix}^{k} \\
&= \begin{pmatrix} A^{*k} & O \\ (\sum_{j=0}^{k-1} A^j B C^{k-1-j})^* & O \end{pmatrix} \begin{pmatrix} D & E \\ E^* & F \end{pmatrix} \begin{pmatrix} A^k & \sum_{j=0}^{k-1} A^j B C^{k-1-j} \\ O & O \end{pmatrix} \\
&= \begin{pmatrix} A^{*k} D A^k & A^{*k} D \sum_{j=0}^{k-1} A^j B C^{k-1-j} \\ (\sum_{j=0}^{k-1} A^j B C^{k-1-j})^* D A^k & (\sum_{j=0}^{k-1} A^j B C^{k-1-j})^* D \sum_{j=0}^{k-1} A^j B C^{k-1-j} \end{pmatrix}
\end{aligned}$$

where

$$D = A^{*(n+1)}A^{(n+1)} - (n+1)(AA^* + BB^*) + n$$

$$E = A^{*(n+1)} \sum_{j=0}^{n} A^j B C^{n-j} - (n+1)BC^*$$

$$F = (\sum_{j=0}^{n} A^j B C^{n-j})^* (\sum_{j=0}^{n} A^j B C^{n-j}) + C^{*(n+1)}C^{(n+1)} - (n+1)CC^* + n$$

Let $v = x \oplus y$ be a vector in $H = \overline{T^k(H)} \oplus \ker(T^{*k})$, where $x \in \overline{T^k(H)}$ and $y \in \ker(T^{*k})$. Then

$$\begin{aligned}
&\left\langle T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^k v, v\right\rangle \\
&= \left\langle A^{*k}DA^k x, x\right\rangle + \left\langle A^{*k}D\sum_{j=0}^{k-1}A^jBC^{k-1-j}y, x\right\rangle + \left\langle (\sum_{j=0}^{k-1}A^jBC^{k-1-j})^*DA^k x, y\right\rangle \\
&\quad + \left\langle (\sum_{j=0}^{k-1}A^jBC^{k-1-j})^*D\sum_{j=0}^{k-1}A^jBC^{k-1-j}y, y\right\rangle \\
&= \left\langle D(A^k x + \sum_{j=0}^{k-1}A^jBC^{k-1-j}y), (A^k x + \sum_{j=0}^{k-1}A^jBC^{k-1-j}y)\right\rangle
\end{aligned}$$

Since

$$D = A^{*(n+1)}A^{(n+1)} - (n+1)(AA^* + BB^*) + n \geqslant O$$

we have

$$\left\langle T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^k v, v\right\rangle \geqslant 0,$$

hence

$$T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^k \geqslant O.$$

Thus, T is an (n,k)-quasi class Q^* operator. □

COROLLARY 2.8. *If T is an (n,k)-quasi class Q^* operator and $T^k(H)$ is not dense range, then*

$$T = \begin{pmatrix} A & B \\ O & C \end{pmatrix} \quad on\ H = \overline{T^k(H)} \oplus \ker(T^{*k}),$$

where A is an n-class Q^* operator on $\overline{T^k(H)}$, and $C^k = O$.

THEOREM 2.9. *If T is an (n,k)-quasi class Q^* operator and M is an invariant subspace for T, then the restriction $T|_M$ is also an (n,k)-quasi class Q^* operator.*

Proof. Let P be the projection onto M. Then $TP = PTP$, so that $(T|_M)^* = PT^*P$. Hence, for $x \in M$ we have

$$\begin{aligned}
\|(T|_M)^*((T|_M)^k x)\|^2 &= \|(PT^*P)(PTP)^k x\|^2 = \|P(T^*T^k x)\|^2 \leqslant \|T^*(T^k x)\|^2 \\
&\leqslant \frac{1}{n+1}\left(\|T^{n+1}(T^k x)\|^2 + n\|T^k x\|^2\right) \\
&= \frac{1}{n+1}\left(\|(T|_M)^{n+1}((T|_M)^k x)\|^2 + n\|(T|_M)^k x\|^2\right). \quad \square
\end{aligned}$$

THEOREM 2.10. *Let $T \in L(H)$. If $\lambda^{-\frac{1}{2}}T$ is an operator of the (n,k)-quasi class Q^* then T is (n,k)-quasi-$*$-paranormal for all $\lambda > 0$.*

Proof. Let $\lambda^{-\frac{1}{2}}T$ be an operator of (n,k)-quasi class Q^*, then

$$(\lambda^{-\frac{1}{2}}T)^{*k}\left((\lambda^{-\frac{1}{2}}T)^{*(n+1)}(\lambda^{-\frac{1}{2}}T)^{(n+1)}-(n+1)(\lambda^{-\frac{1}{2}}T)(\lambda^{-\frac{1}{2}}T)^{*}+nI\right)(\lambda^{-\frac{1}{2}}T)^{k}\geqslant O$$

$$\lambda^{-\frac{k}{2}}T^{*k}\left(\lambda^{-(n+1)}T^{*(n+1)}T^{(n+1)}-(n+1)\lambda^{-1}TT^{*}+nI\right)\lambda^{-\frac{k}{2}}T^{k}\geqslant O,$$

$$\frac{1}{\lambda^{n+k+1}}T^{*k}\left(T^{*(n+1)}T^{(n+1)}-(n+1)\lambda^{n}TT^{*}+n\lambda^{(n+1)}\right)T^{k}\geqslant O,$$

$$T^{*k}\left(T^{*(n+1)}T^{(n+1)}-(n+1)\lambda^{n}TT^{*}+n\lambda^{(n+1)}\right)T^{k}\geqslant O$$

for all $\lambda>0$. By this it is proved that the operator T is an (n,k)-quasi-$*$-paranormal operator. □

THEOREM 2.11. *Let us suppose that T is (n,k)-quasi class Q^* and $T\geqslant 42I$, then it follows that $2\cdot T$, is $\left(\frac{n-3}{4},k\right)$-quasi class Q^*, for $n=4r+3$ and r integer greater then* 10.

Proof. Let us suppose that T is (n,k)-quasi class Q^* and $T\geqslant 42I$. Then we have to prove that

$$\|(2T)^{*}((2T)^{k}x)\|^{2}\leqslant\frac{1}{\frac{n-3}{4}+1}\left(\|(2T)^{1+\frac{n-3}{4}}((2T)^{k}x)\|^{2}+\frac{n-3}{4}\|(2T)^{k}x\|^{2}\right),$$

for every $n\geqslant 44$. Let us suppose that last relation is valid, then we get

$$\|T^{*}(T^{k}x)\|^{2}\leqslant\frac{1}{n+1}\left(2^{\frac{n-3}{2}}\|T^{\frac{n+1}{4}}(T^{k}x)\|^{2}+\frac{n-3}{4^{2}}\|T^{k}x\|^{2}\right).$$

From fact that T is (n,k)-quasi class Q^* and $T\geqslant 42I$, we obtain that

$$\frac{1}{n+1}\left(\|T^{n+1}(T^{k}x)\|^{2}+n\|T^{k}x\|^{2}\right)\leqslant\frac{1}{n+1}\left(2^{\frac{n-3}{2}}\|T^{\frac{n+1}{4}}(T^{k}x)\|^{2}+\frac{n-3}{4^{2}}\|T^{k}x\|^{2}\right).$$

Hence, $2\cdot T$, is $\left(\frac{n-3}{4},k\right)$-quasi class Q^*, for $n\geqslant 44$. □

3. Spectral properties

For $T\in L(H)$, the smallest nonnegative integer p such that $\ker(T^{p})=\ker(T^{p+1})$ is called the ascent of T and is denoted by $p(T)$. If no such integer exists, we set $p(T)=\infty$. We say that $T\in L(H)$ is of finite ascent if $p(T-\lambda)<\infty$, for all $\lambda\in\mathbb{C}$.

PROPOSITION 3.1. *If T is (n,k)-quasi class Q^*, and $T\geqslant 42I$, then it is $\left(\frac{n-3}{4},k\right)$-quasi-$*$-paranormal, for $n=4r+3$ and r integer greater then* 10. *T has finite ascent under above conditions.*

Proof. From theorem 2.11 we obtain that $2\cdot T$, is $\left(\frac{n-3}{4},k\right)$-quasi class Q^*. On the other side from theorem 2.10, we have that $\sqrt{4}\cdot T$ is $\left(\frac{n-3}{4},k\right)$-quasi class Q^*, respectively T is $\left(\frac{n-3}{4},k\right)$-quasi-$*$-paranormal for every $\lambda > 0$. Now from corollary 4.5 ([12]), it follows that T has finite ascent. □

THEOREM 3.2. *If T is an (n,k)-quasi class Q^* operator, $0 \neq \lambda \in \sigma_p(T)$ and T is the form $T = \begin{pmatrix} \lambda & A \\ O & B \end{pmatrix}$ on $H = \mathrm{ker(T-\lambda)} \oplus (\mathrm{ker(T-\lambda)})^{\perp}$, then $A = O$.*

Proof. Let P be the orthogonal projection onto $\mathrm{ker(T-\lambda)}$ and $x \in \mathrm{ker(T-\lambda)}$. Since T is an (n,k)-quasi class Q^* operator, and $x = \frac{1}{\lambda^k}T^k x \in T^k(H)$, we have

$$P\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + n\right)P \geqslant O.$$

then

$$\lambda^{2(n+1)} - (n+1)(\lambda^2 + AA^*) + n \geqslant 0$$

which yields that

$$\lambda^{2(n+1)} - (n+1)\lambda^2 + n \geqslant (n+1)AA^*$$

hence $A = O$. □

COROLLARY 3.3. *If T is an (n,k)-quasi class Q^* operator, $0 \neq \lambda$ then $Tx = \lambda x$ implies $T^*x = \overline{\lambda}x$.*

Proof. We may assume that $x \neq 0$. Let M be a span of $\{x\}$. Then M is an invariant subspace of T. We have

$$T = \begin{pmatrix} \lambda & A \\ O & B \end{pmatrix} \quad \text{on } H = M \oplus M^{\perp},$$

and P the orthogonal projection of H onto M. Since T is an (n,k)-quasi class Q^* operator, then we have

$$P\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + n\right)P \geqslant O.$$

From above theorem we have $A = O$. Thus

$$(T-\lambda)^*x = \begin{pmatrix} O & O \\ O & B-\lambda \end{pmatrix}^* \begin{pmatrix} x \\ 0 \end{pmatrix} = 0. \quad \square$$

A complex number λ is said to be in the point spectrum $\sigma_p(T)$ of T if there is a nonzero $x \in H$ such that $(T-\lambda)x = 0$. If in addition, $(T-\lambda)^*x = 0$, then λ is said to be in the joint point spectrum $\sigma_{jp}(T)$ of T. Clearly $\sigma_{jp}(T) \subseteq \sigma_p(T)$. In general $\sigma_{jp}(T) \neq \sigma_p(T)$.

COROLLARY 3.4. *If T is an (n,k)-quasi class Q^* operator, then $\sigma_{jp}(T)\setminus\{0\} = \sigma_p(T)\setminus\{0\}$.*

COROLLARY 3.5. *Let T be an (n,k)-quasi class Q^* operator, then $\ker(T-\lambda) = \ker(T-\lambda)^2$, for $\lambda \neq 0 \in \mathbb{C}$.*

Proof. We have to tell that $\ker(T-\lambda) = \ker(T-\lambda)^2$. To do that, it is sufficient enough to show that $\ker(T-\lambda)^2 \subseteq \ker(T-\lambda)$, since $\ker(T-\lambda) \subseteq \ker(T-\lambda)^2$ is clear.

Let $x \in \ker(T-\lambda)^2$, then $(T-\lambda)^2 x = 0$. From Corollary 3.3 we have $(T-\lambda)^*(T-\lambda)x = 0$. Hence,

$$\|(T-\lambda)x\|^2 = \langle (T-\lambda)^*(T-\lambda)x, x\rangle = 0,$$

so we have $(T-\lambda)x = 0$, which implies $\ker(T-\lambda)^2 \subseteq \ker(T-\lambda)$. □

REMARK 3.6. If T is an (n,k)-quasi class Q^* operator, and $\lim_{n\to\infty} \frac{\|T^{n+1}(T^k x)\|^2}{n+1} < \infty$, for $k \in \mathbb{N}$. Then there exists an (∞,k)-quasi class Q^* operator.

THEOREM 3.7. *Let T be an (∞,k)-quasi class Q^* operator, and $(T-\lambda)x_m \to 0$, as $m\to\infty$ for $\lambda \neq 0$. Then (x_m) is in the resolvent set of $(T^* - \overline{\lambda})$.*

Proof. Let $(T-\lambda)x_m \to 0$, as $m\to\infty$, $||x_m|| = 1$ and $r \in \mathbb{N}$. Then we have

$$(T^r - \lambda^r)x_m \to 0,$$

because

$$T^r = (T-\lambda+\lambda)^r = \sum_{j=1}^{r} \binom{r}{j} \lambda^{r-j}(T-\lambda)^j + \lambda^r.$$

From the last relation we get:

$$\begin{aligned} |||\lambda^r x_m|| - ||(T^r-\lambda^r)x_m||| &\leqslant ||T^r x_m|| = ||\lambda^r x_m + (T^r-\lambda^r)x_m|| \\ &\leqslant ||\lambda^r x_m|| + ||(T^r-\lambda^r)x_m||, \end{aligned}$$

and $||T^r x_m|| \to |\lambda|^r$. Knowing that

$$|||T^*(\lambda^r x_m)|| - ||T^*(T^r-\lambda^r)x_m|||^2 \leqslant ||T^*(T^r x_m)||^2$$

and that T is an (n,k)-quasi class Q^* operator, we obtain

$$\lim_{m\to\infty} \sup ||T^* x_m||^2 \leqslant \frac{1}{n+1}\left(|\lambda|^{2n+2} + n\right).$$

Now we will distinguish two cases:

I) If $0<|\lambda|\leqslant 1$, then we have

$$||T^*x_m-\overline{\lambda}x_m||^{2n+2} = \left(||T^*x_m-\overline{\lambda}x_m||^2\right)^{n+1}$$
$$= \left(||T^*x_m||^2-\overline{\lambda}\langle (T-\lambda)x_m,x_m\rangle-\lambda\langle x_m,(T-\lambda)x_m\rangle-|\lambda|^2\right)^{n+1},$$

and

$$\limsup_{m\to\infty}||T^*x_m-\overline{\lambda}x_m||^{2n+2}\leqslant\left(\frac{|\lambda|^{2n+2}}{n+1}+\frac{n}{n+1}-|\lambda|^2\right)^{n+1}\to 0, \text{as} \quad n\to\infty.$$

From the last relation we get that

$$||T^*x_m-\overline{\lambda}x_m||<1,$$

respectively $(T^*-\overline{\lambda})$ is invertible, and $(x_m)\in\rho(T^*-\overline{\lambda})$.

II) Let now consider that $|\lambda|>1$. Then we have

$$\lim_{n\to\infty}\lim_{m\to\infty}\sup||T^*x_m-\overline{\lambda}x_m||^{2n+2}\leqslant\lim_{n\to\infty}\left(\frac{|\lambda|^{2n+2}}{n+1}+\frac{n}{n+1}-|\lambda|^2\right)^{n+1}=$$
$$e^{\lim_{n\to\infty}(\lambda^{2n+2}-\lambda^2(n+1)-1)}\to 0, \text{as} \quad n\to\infty.$$

Hence, like the above, we have that $(T^*-\overline{\lambda})$ is invertible, and $(x_m)\in\rho(T^*-\overline{\lambda})$. □

COROLLARY 3.8. *If T is an (n,k)-quasi class Q^* operator, and $\alpha,\beta\in\sigma_p(T)\setminus\{0\}$ with $\alpha\neq\beta$, then* $\ker(T-\alpha)\perp\ker(T-\beta)$.

Proof. Let $x\in\ker(T-\alpha)$ and $y\in\ker(T-\beta)$. Then $Tx=\alpha x$ and $Ty=\beta y$. Therefore

$$\alpha\langle x,y\rangle=\langle Tx,y\rangle=\langle x,T^*y\rangle=\langle x,\overline{\beta}y\rangle=\beta\langle x,y\rangle,$$

then $\langle x,y\rangle=0$. Therefore, $\ker(T-\alpha)\perp\ker(T-\beta)$. □

Let $Hol(\sigma(T))$ be the space of all analytic functions in an open neighborhood of $\sigma(T)$. We say that $T\in L(H)$ has the single valued extension property (SVEP) at $\lambda\in\mathbb{C}$, if for every open neighborhood U of λ the only analytic function $f:U\to H$ which satisfies the equation $(T-\lambda)f(\lambda)=0$, is the constant function $f\equiv 0$ on U. The operator T is said to have SVEP if T has SVEP at every $\lambda\in\mathbb{C}$.

COROLLARY 3.9. *If T is an (n,k)-quasi class Q^* operator, then T has SVEP.*

Proof. Let f be an analytic function such that $(T-\lambda)f(\lambda)=0$ on an open set U. By assumption, $f(\lambda)\in\ker(T-\lambda)$ for each $\lambda\in U$. Thus $f(\lambda)\perp f(\mu)$ for any two different nonzero numbers λ and μ in U by Corollary 3.8. Therefore, for any sequence $\{\mu_n\}$ of non-zero complex numbers such that $\mu_n\longrightarrow\lambda$, thus

$$\|f(\lambda)\|^2=\lim_{m\longrightarrow\infty}\langle f(\lambda),f(\mu_m)\rangle=0.$$

That is, T has SVEP. □

LEMMA 3.10. [2] *Let H be a complex Hilbert space. Then there exists a Hilbert space Y such that $H \subset Y$ and a map $\varphi : L(H) \to L(Y)$ such that:*

(1). φ is a faithful $*$-representation of the algebra $L(H)$ on Y, so:

$$\varphi(I_H) = I_Y,\ \varphi(T^*) = (\varphi(T))^*,\ \varphi(TS) = \varphi(T)\varphi(S)$$

$$\varphi(\alpha T + \beta S) = \alpha\varphi(T) + \beta\varphi(S) \text{ for any } T,S \in L(H) \text{ and } \alpha,\beta \in \mathbb{C},$$

(2). $\varphi(T) \geqslant 0$ for any $T \geqslant 0$ in $L(H)$,

(3). $\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$ for any $T \in L(H)$,

COROLLARY 3.11. *If T is an (n,k)-quasi class Q^* operator, then $\varphi(T)$ is an (n,k)-quasi class Q^* operator.*

Proof. Let $\varphi : L(H) \to L(K)$ be Berberian's faithful $*$-representation. First we show that $\varphi(T)$ belongs to the (n,k)-quasi class Q^*. Since T is a (n,k)-quasi class Q^* we have

$$\begin{aligned}&(\varphi(T))^{*k}\left[(\varphi(T))^{*(n+1)}(\varphi(T))^{(n+1)} - (n+1)\varphi(T)(\varphi(T))^* + n\right](\varphi(T))^k\\ =&\varphi\left(T^{*k}\left(T^{*(n+1)}T^{(n+1)} - (n+1)TT^* + nI\right)T^k\right) \geqslant O\end{aligned}$$

thus $\varphi(T)$ is an (n,k)-quasi class Q^* operator. □

COROLLARY 3.12. *If T is an (n,k)-quasi class Q^* operator, for every non zero $\lambda \in \sigma_p(T)$, the matrix representation of T with respect to the decomposition $H = \ker(\mathrm{T}-\lambda) \oplus \ker(\mathrm{T}-\lambda)^\perp$ is $T = \begin{pmatrix}\lambda & O\\ O & B\end{pmatrix}$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$.*

Proof. By Corollary 3.3, if $\lambda \neq 0$ and $\lambda \in \sigma_p(T)$, we have $\ker(\mathrm{T}-\lambda)$ reduces T. So we have that $T = \begin{pmatrix}\lambda & O\\ O & B\end{pmatrix}$ on $H = \ker(\mathrm{T}-\lambda) \oplus (\ker(\mathrm{T}-\lambda))^\perp$ for some operator B satisfying $\lambda \notin \sigma_p(B)$ and $\sigma(T) = \{\lambda\} \cup \sigma(B)$. □

4. Spectrum continuity on the set of (n,k)-quasi class Q^* operator

Let $\{E_m\}_{m\in\mathbb{N}}$ be a sequence of compact subsets of $\mathbb{C}$. Let's define the inferior and superior limits of $\{E_m\}_{m\in\mathbb{N}}$, denoted respectively by $\liminf_{m\to\infty}\{E_m\}$ and $\limsup_{m\to\infty}\{E_m\}$ as it follows:

1). $\liminf_{m\to\infty}\{E_m\} = \{\lambda \in \mathbb{C} : \text{for every } \varepsilon > 0, \text{there exists } N \in \mathbb{N} \text{ such that } B(\lambda,\varepsilon) \cap E_m \neq \emptyset \text{ for all } m > N\}$,

2). $\limsup_{m\to\infty}\{E_m\} = \{\lambda \in \mathbb{C} : \text{for every } \varepsilon > 0, \text{there exists } J \subseteq \mathbb{N} \text{ infinite such that } B(\lambda,\varepsilon) \cap E_m \neq \emptyset \text{ for all } m \in J\}$.

If $\liminf_{m\to\infty}\{E_m\} = \limsup_{m\to\infty}\{E_m\}$, then $\lim_{m\to\infty}\{E_m\}$ is defined by this common limit.

A mapping p, defined on $L(H)$, whose values are compact subsets on $\mathbb{C}$ is said to be upper semi-continuous at T, if $T_m \to T$ then $\limsup_{m\to\infty} p(T_m) \subset p(T)$, and lower semi-continuous at T, if $T_m \to T$ then $p(T) \subset \liminf_{m\to\infty} p(T_m)$. If p is both upper and lower semi-continuous at T, then it is said to be continuous at T and in this case $\lim_{m\to\infty} p(T_m) = p(T)$.

The spectrum $\sigma : T \to \sigma(T)$ is upper semi-continuous by [5, Problem 102], but it is not continuous in general, [11, Example 4.6]

LEMMA 4.1. [10] *If* $\{T_m\} \subset L(H)$ *and* $T \in L(H)$ *are such that* T_m *converges, according to the operator norm topology to* T, *then* $\mathrm{iso}\,\sigma(T) \subseteq \liminf_{m\to\infty} \sigma(T_m)$.

THEOREM 4.2. *The spectrum* σ *is continuous on the set of* (n,k)*-quasi class* Q^* *operators.*

Proof. Let $\{T_m\}$ be a sequence of operators from (n,k)-quasi class Q^* operators and $\lim_{m\to\infty} \|T_m - T\| = 0$, where T is an (n,k)-quasi class Q^* operator. Since the function σ is upper semi-continuous, $\limsup_{m\to\infty} \sigma(T_m) \subset \sigma(T)$. Therefore, to prove the theorem, it will be sufficient to prove that $\sigma(T) \subset \liminf_{m\to\infty} \sigma(T_m)$. From Lemma 4.1 it will be sufficient to prove $\mathrm{acc}\,\sigma(T) \subset \liminf_{m\to\infty} \sigma(T_m)$.

Let $\varphi(T)$ be the Berberian extension to T. From Lemma 3.10 we have $\sigma(T) = \sigma(\varphi(T))$, $\sigma(T_m) = \sigma(\varphi(T)_m)$ and $\sigma_a(T) = \sigma_a(\varphi(T)) = \sigma_p(\varphi(T))$.

If T is an (n,k)-quasi class Q^* operator, from Corollary 3.11 $\varphi(T)$ is an (n,k)-quasi class Q^* operator, therefore

$$\mathrm{acc}\,\sigma(T) \subset \liminf_{m\to\infty} \sigma(T_m) \Longleftrightarrow \mathrm{acc}\,\sigma(\varphi(T)) \subset \liminf_{m\to\infty} \sigma(\varphi(T)_m).$$

Now, assume that $\lambda \in \mathrm{acc}\,\sigma(\varphi(T))$. Consider two cases:

Case I: Let

$$\lambda \in \sigma(\varphi(T)) \setminus \sigma_a(\varphi(T)) \tag{4.1}$$

So $\varphi(T) - \lambda$ is upper semi-Fredholm operator and $\alpha(\varphi(T) - \lambda) = 0$.

Suppose the contrary, $\lambda \notin \liminf_{m\to\infty} \sigma(\varphi(T)_m)$. Then, there exists a $\delta > 0$, a neighborhood $\mathscr{D}_\delta(\lambda)$ of λ and a subsequence $\{\varphi(T)_{m_l}\}$ of $\{\varphi(T)_m\}$ such that $\sigma(\varphi(T)_{m_l}) \cap \mathscr{D}_\delta(\lambda) = \emptyset$ for every $l \geqslant 1$. This implies that $\varphi(T)_{m_l} - \mu$ is a Fredholm operator and $\mathrm{ind}(\varphi(T)_{m_l} - \mu) = 0$ for every $\mu \in \mathscr{D}_\delta(\lambda)$ and

$$\lim_{m\to\infty} \|(\varphi(T)_{m_l} - \mu) - (\varphi(T) - \mu)\| = 0.$$

From the continuity of the index follows that $\mathrm{ind}(\varphi(T) - \mu) = 0$ and $\varphi(T) - \mu$ is a Fredholm operator. Since $\alpha(\varphi(T) - \mu) = 0$, $\mu \notin \sigma(\varphi(T))$ for every μ in a ε-neighborhood of λ. This is a contradiction of relation 4.1, therefore we must have $\lambda \in \liminf_{m\to\infty} \sigma(\varphi(T)_m)$.

Case II: Let $\lambda \in \sigma_a(\varphi(T))$. Then $\lambda \in \sigma_p(\varphi(T))$. By Corollary 3.12 $\varphi(T)$ has a representation

$$\varphi(T) = \lambda \oplus B \text{ on } H = \ker(\varphi(T) - \lambda) \oplus (\ker(\varphi(T) - \lambda))^\perp \text{ and } \ker(B - \lambda) = \{0\}.$$

Therefore $B-\lambda$ is upper semi-Fredholm operator and $\alpha(B-\lambda)=0$. There exists a $\varepsilon>0$ such that $B-(\lambda-\mu_0)$ is upper semi-Fredholm operator with $\operatorname{ind}(B-(\lambda-\mu_0))=\operatorname{ind}(B-\lambda)$ and $\alpha(B-(\lambda-\mu_0))=0$ for every μ_0 such that $0<|\mu_0|<\varepsilon$.

Choose $0<\varepsilon<\delta$ and set $\mu=\lambda-\mu_0$, and we have $\varphi(T)-\mu=(\lambda-\mu)\oplus(B-\mu)$ is upper semi-Fredholm operator, $\operatorname{ind}(\varphi(T)-\mu)=\operatorname{ind}(B-\mu)$ and $\alpha(\varphi(T)-\mu)=0$.

Suppose the contrary, $\lambda\notin\liminf_{m\to\infty}\sigma(\varphi(T)_m)$. Then $\varphi(T)_{m_l}-\mu$ is a Fredholm operator and $\operatorname{ind}(\varphi(T)_{m_l}-\mu)=0$ and

$$\lim_{m\to\infty}\|(\varphi(T)_{m_l}-\mu)-(\varphi(T)-\mu)\|=0.$$

It follows from the continuity of the index that $\operatorname{ind}(\varphi(T)-\mu)=0$ and $\varphi(T)-\mu$ is a Fredholm operator. Since $\alpha(\varphi(T)-\mu)=0$, $\mu\notin\sigma(\varphi(T))$ for every μ in a ε-neighborhood of λ. This contradicts the assumption $\lambda\in\sigma_a(\varphi(T))$, therefore we must have $\lambda\in\liminf_{m\to\infty}\sigma(\varphi(T)_m)$. $\square$

5. Tensor product

Let T be any (n,k)-quasi class Q^* operator. We claim that $T\otimes I$ and $I\otimes T$ are both (n,k)-quasi class Q^* operator. This can be seen by using the fact that the tensor product of two positive operators is positive and the following computations:

$$\begin{aligned}&(T\otimes I)^{*k}\left((T\otimes I)^{*(n+1)}(T\otimes I)^{(n+1)}-(n+1)(T\otimes I)(T\otimes I)^*+n(I\otimes I)\right)(T\otimes I)^k\\ =&(T^{*k}\otimes I)\left((T^{*(n+1)}T^{(n+1)})\otimes I-(n+1)(TT^*)\otimes I+n(I\otimes I)\right)(T^k\otimes I)\\ =&\left[T^{*k}\left(T^{*(n+1)}T^{(n+1)}-(n+1)TT^*+nI\right)T^k\right]\otimes I\geqslant O\end{aligned}$$

By the above relation, we proved that if T is an (n,k)-quasi class Q^* operator, then the tensor product $T\otimes I$ is an (n,k)-quasi class Q^* operator. However, if T is an (n,k)-quasi class Q^* operator, then $T\otimes T$ is not necessarily (n,k)-quasi class Q^* operator. Let's see this through an example:

LEMMA 5.1. *Let $S=\oplus_{n=1}^{\infty}H_n$, where $H_n\cong\mathbb{R}^2$. For given positive operators A,B on $\mathbb{R}^2$ and for any fixed $n\in\mathbb{N}$, the operator $T=T_{A,B}$ on S is defined as follows:*

$$T(x_1,x_2,\ldots)=(0,Ax_1,Bx_2,Bx_3,Bx_4,\ldots),$$

and the adjoint operator of T is

$$T^*(x_1,x_2,\ldots)=(Ax_2,Bx_3,Bx_4,Bx_5,\ldots).$$

The operator $T_{A,B}$ is an $(n,1)$-quasi class Q^* operator, if and only if,

$$A(B^{2n+2}-(n+1)A^2+n)A\geqslant O$$

and

$$B^{2n+2}-(n+1)B^4+nB^2\geqslant O.$$

EXAMPLE 5.2. In this example we will prove that if T is an $(2,1)$-quasi class Q^*, then their tensorial product is not $(2,1)$-quasi class Q^*.

Take A and B as

$$A = \begin{pmatrix} 1 & 0.94 \\ 0.94 & 2 \end{pmatrix}^{\frac{1}{2}} \text{ and } B = \begin{pmatrix} 1 & 2.82 \\ 2.82 & 8 \end{pmatrix}^{\frac{1}{6}}.$$

Then

$$B^6 - 3B^4 + 2B^2 = \begin{pmatrix} 0.2534 & -0.0263 \\ -0.0263 & 0.1882 \end{pmatrix} \geqslant 0,$$

and

$$A(B^6 - 3A^2 + 2I)A = \begin{pmatrix} 0.6912 & 2.2477 \\ 2.2477 & 7.3088 \end{pmatrix} \geqslant 0.$$

So, $T_{A,B}$ is a $(2,1)$-quasi class Q^* operator, and

$$AB^6A \otimes AB^6A - 3A^4 \otimes A^4 + 2A^2 \otimes A^2 = \begin{pmatrix} 10.2094 & 24.2748 & 24.2748 & 55.8379 \\ 24.2748 & 54.3848 & 55.8379 & 120.9860 \\ 24.2748 & 55.8379 & 54.3848 & 120.9860 \\ 55.8379 & 120.9860 & 120.9860 & 258.9975 \end{pmatrix}$$

is not positive. Then, $T \otimes T$ is not a $(2,1)$-quasi class Q^* operator.

Conclusion. In this paper we have defined new class of operators named (n,k)-quasi class Q^* and operator T is from that class if it satisfies

$$\|T^*(T^k x)\|^2 \leqslant \frac{1}{n+1}\left(\|T^{n+1}(T^k x)\|^2 + n\|T^k x\|^2\right),$$

for all $x \in H$ and for some nonnegative integers n and k. We have prove structural and spectral properties of this class of operators, and also it is proven the spectrum continuity of this class of operators.

REFERENCES

[1] S. C. ARORA AND J. K. THUKRAL, *On a class of operators*, Glasnik Math. **21** (41) (1986), 381–386.
[2] S. K. BERBERIAN, *Approximate proper vectors*, Proc. Amer. Math. Soc. 10 (1959), 175–182.
[3] T. FURUTA, *On The Class of Paranormal Operators*, Proc. Jap. Acad. **43** (1967), 594–598.
[4] J. K. HAN, H. Y. LEE, AND W. Y. LEE, *Invertible completions of* 2×2 *upper triangular operator matrices*, Proceedings of the American Mathematical Society, vol. **128** (2000), 119–123.
[5] P. R. HALMOS, *A Hilbert Space Problem Book*, Springer–Verlag, New York, 1982.
[6] I. HOXHA AND N. L. BRAHA, *A note on* k*-quasi-*$*$*-paranormal operators*, Journal of Inequalities and Applications 2013, 2013:350.
[7] I. HOXHA AND N. L. BRAHA, *On* k*-Quasi Class* $\mathscr{A}_n^*$ *Operators*, Bulletin of Mathematical Analysis and Applications, Volume 6 Issue **1** (2014), pages 23–33.
[8] S. MECHERI, *On a new class of operators and Weyl type theorems*, Filomat 27:4 (2013), 629–636.
[9] S. MECHERI, *Isolated points of spectrum of* k*-quasi-*$*$*-class* A *operators*, Studia Math. 208 (2012), no. 1, 87–96.

[10] J. D. NEWBURGH, *The variation of Spectra*, Duke Math. J. **18** (1951), 165–176.
[11] S. SANCHEZ-PERALES AND V. A. CRUZ-BARRIGUETE, *Continuity of approximate point spectrum on the algebra* $B(X)$, Commun. Korean Math. Soc. **28** (2013), no. 3, pp. 487–500.
[12] Q. ZENG AND H. ZHONG, *On* (n,k) *-quasi-*$*$*-paranormal operators*, arXiv 1209.5050v1 [math. FA], 2012.

(Received June 7, 2020)

Ilmi Hoxha
Faculty of Education
University of Gjakova "Fehmi Agani"
Avenue "Ismail Qemali" nn Gjakovë, 50000, Kosova
e-mail: ilmihoxha011@gmail.com

Naim L. Braha
Department of Mathematics and Computer Sciences
University of Prishtina
Avenue "Mother Teresa" 5, Prishtinë, 10000, Kosova
e-mail: nbraha@yahoo.com