Journal of Mathematical Inequalities
Volume 15, Number 3 (2021), 1019–1029 doi:10.7153/jmi-2021-15-69

A NEW GENERALIZED REFINEMENTS OF YOUNG'S INEQUALITY AND APPLICATIONS

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(*Communicated by J. Pečarić*)

Abstract. In this work, by the weighted arithmetic-geometric mean inequality, we show if $a,b > 0$ and $0 \leqslant \nu \leqslant 1$. Then for all positive integer m, we have

$$\begin{aligned}\left(a^{\nu}b^{1-\nu}\right)^m &+ r_0^m\left((a+b)^m - 2^m(ab)^{\frac{m}{2}}\right)\\ &+r_m\left[\left((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\nu) + \left((ab)^{\frac{m}{4}} - a^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\nu)\right]\\ &\leqslant \left(\nu a + (1-\nu)b\right)^m,\end{aligned}$$

where $r_0 = \min\{\nu, 1-\nu\}$, $r_m = \min\{2^m r_0^m, (1-r_0)^m - r_0^m\}$ and $\chi_I(\nu)$ the characteristic function. This inequality provides a generalization of an important refinement of the Young inequality obtained by J. Zhao and J. Wu. As applications we give some new generalized refinements of Young type inequalities for the determinants, p-norms and traces, of positive τ-measurable operators.

1. Introduction

The weighted arithmetic-geometric mean (AM-GM) inequality states as follows:

THEOREM 1.1. *Let n be a positive integer. For $k = 1,2,\dots,n$, let $x_k > 0$, and let $\nu_k \geqslant 0$ satisfy $\sum_{k=1}^n \nu_k = 1$. Then, we have*

$$\prod_{k=1}^n x_k^{\nu_k} \leqslant \sum_{k=1}^n \nu_k x_k. \tag{1.1}$$

The special case of the weighted AM-GM inequality ($n = 2$) is the well-known Young's inequality, for positive real numbers a,b and $0 \leqslant \nu \leqslant 1$, we have

$$a^{\nu}b^{1-\nu} \leqslant \nu a + (1-\nu)b. \tag{1.2}$$

The first refinements of Young inequality is the squared version proved in [8] as follows

$$(a^{\nu}b^{1-\nu})^2 + r_0^2(a-b)^2 \leqslant (\nu a + (1-\nu)b)^2, \tag{1.3}$$

Mathematics subject classification (2020): 26D07, 26D15, 46L52, 47A63.
Keywords and phrases: AM-GM inequality, Young's inequality, von Neumann algebras, determinants, p-norms, trace.

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where $r_0 = \min\{\nu, 1-\nu\}$.

Later, Kittaneh and Manasrah [10], obtained the other interesting refinement of Young's inequality

$$a^\nu b^{1-\nu} + r_0(\sqrt{a}-\sqrt{b})^2 \leqslant \nu a + (1-\nu)b, \tag{1.4}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

J. Zhao and J. Wu [15], obtained the following refinement of inequality (1.2) as follows:
if $0 < \nu \leqslant \frac{1}{2}$, then

$$a^\nu b^{1-\nu} + \nu(\sqrt{a}-\sqrt{b})^2 + r_1(\sqrt[4]{ab}-\sqrt{b})^2 \leqslant \nu a + (1-\nu)b, \tag{1.5}$$

if $\frac{1}{2} < \nu \leqslant 1$, then

$$a^\nu b^{1-\nu} + (1-\nu)(\sqrt{a}-\sqrt{b})^2 + r_1(\sqrt[4]{ab}-\sqrt{a})^2 \leqslant \nu a + (1-\nu)b, \tag{1.6}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $r_1 = \min\{2r_0, 1-2r_0\}$.

Manasrah and Kittaneh [1] gave a generalization refinements of (1.2), as follows

$$\left(a^\nu b^{1-\nu}\right)^m + r_0^m\left(a^{\frac{m}{2}} - b^{\frac{m}{2}}\right)^2 \leqslant \left(\nu a + (1-\nu)b\right)^m, \tag{1.7}$$

where $m = 1, 2, 3, \ldots$, and $r_0 = \min\{\nu, 1-\nu\}$.

Recently, Choi [3] gave a further generalized refinement of inequalities (1.3) and (1.4) as follows:

$$\left(a^\nu b^{1-\nu}\right)^m + r_0^m\left((a+b)^m - 2^m(ab)^{\frac{m}{2}}\right) \leqslant \left(\nu a + (1-\nu)b\right)^m. \tag{1.8}$$

For more informations on Young's inequality, one can see for instantce the references: [2], [6] and [11]

One of the aims of this paper is to extend the inequalities (1.5) and (1.6) to the following one:

$$\begin{aligned}\left(a^\nu b^{1-\nu}\right)^m &+ r_0^m\left((a+b)^m - 2^m(ab)^{\frac{m}{2}}\right)\\ &+ r_m\left[\left((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\right)^2\chi_{(0,\frac{1}{2}]}(\nu) + \left((ab)^{\frac{m}{4}} - a^{\frac{m}{2}}\right)^2\chi_{(\frac{1}{2},1]}(\nu)\right]\\ &\leqslant \left(\nu a + (1-\nu)b\right)^m,\end{aligned}$$

where $r_0 = \min\{\nu, 1-\nu\}$, and $r_m = \min\{2^m r_0^m, (1-r_0)^m - r_0^m\}$. As application of this inequality, we give a new refinements to certain Young's type inequalities for the determinants, p-norms and traces of positive τ-measurable operators.

2. A new generalized refinements of Young's inequality

In this section, we are concerned by the investigation of generalized refinements of Young's inequality.

Before stating and proving our results, we need the following lemma.

LEMMA 2.1. *Let m be a positive integer and let ν a positive number, such that $0 \leqslant \nu \leqslant 1$. Then we have*

$$\sum_{k=1}^{m} \binom{m}{k} k\nu^k (1-\nu)^{m-k} = m\nu, \tag{2.1}$$

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k)\nu^k (1-\nu)^{m-k} = m(1-\nu) \tag{2.2}$$

and

$$\sum_{k=1}^{m-1} \binom{m}{k} k = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) = m2^{m-1}, \tag{2.3}$$

where $\binom{m}{k}$ is the binomial coefficient.

Proof. for any nonnegative real numbers x_1 and x_2, we have

$$(x_1+x_2)^m = \sum_{k=0}^{m} \binom{m}{k} x_1^k x_2^{m-k}, \tag{2.4}$$

by derivation of (2.4) with respect x_1 and x_2 respectively we find that

$$m(x_1+x_2)^{m-1} = \sum_{k=1}^{m} \binom{m}{k} k x_1^{k-1} x_2^{m-k}, \tag{2.5}$$

and

$$m(x_1+x_2)^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) x_1^k x_2^{m-k-1}. \tag{2.6}$$

By multiplying (2.5) and (2.6) by x_1 and x_2 respectively

$$mx_1(x_1+x_2)^{m-1} = \sum_{k=1}^{m} \binom{m}{k} k x_1^k x_2^{m-k}, \tag{2.7}$$

and

$$mx_2(x_1+x_2)^{m-1} = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) x_1^k x_2^{m-k}. \tag{2.8}$$

By setting $x_1 = \nu$ and $x_2 = 1-\nu$ in (2.7) and (2.8) respectively we deduces the result.

The equalities (2.3) follows by setting $x_1 = 1$ and $x_2 = 1$ in (2.7) and (2.8) respectively. This completes the proof. □

The main result of this section reads as follows.

THEOREM 2.1. *Let a and b be two positive numbers and $0 \leqslant \nu \leqslant 1$. Then for all positive integer m, we have*

$$\begin{aligned}&\Big(a^{\nu}b^{1-\nu}\Big)^m + r_0^m\Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big)\\ &\quad + r_m\Big[\Big((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu) + \Big((ab)^{\frac{m}{4}} - a^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big]\\ &\leqslant \Big(\nu a + (1-\nu)b\Big)^m,\end{aligned}$$

where $r_0 = \min\{\nu, 1-\nu\}$, $r_m = \min\{2^m r_0^m, (1-r_0)^m - r_0^m\}$ and $\chi_I(\nu)$ the characteristic function defined by

$$\chi_I(x) = \begin{cases} 1 & if\ x \in I \\ 0 & if\ x \notin I. \end{cases}$$

Proof. Suppose that $0 < \nu \leqslant \frac{1}{2}$. We claim that

$$\Big(\nu a + (1-\nu)b\Big)^m - \nu^m\Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big) - r_m\Big((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\Big)^2 \geqslant \Big(a^{\nu}b^{1-\nu}\Big)^m. \tag{2.9}$$

We have, the following identities

$$\begin{aligned}&\Big(\nu a + (1-\nu)b\Big)^m - \nu^m\Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big) - r_m\Big((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\Big)^2\\ &= \sum_{k=0}^{m}\binom{m}{k}\nu^k(1-\nu)^{m-k}a^k b^{m-k} - \nu^m\Big(\sum_{k=0}^{m}\binom{m}{k}a^k b^{m-k} - 2^m(ab)^{\frac{m}{2}}\Big)\\ &\quad - r_m\Big((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\Big)^2\\ &= \sum_{k=0}^{m}\binom{m}{k}\Big(\nu^k(1-\nu)^{m-k} - \nu^m\Big)a^k b^{m-k} + 2^m\nu^m(ab)^{\frac{m}{2}}\\ &\quad - r_m\Big((ab)^{\frac{m}{2}} + b^m - 2(ab)^{\frac{m}{4}}b^{\frac{m}{2}}\Big)\\ &= \sum_{k=1}^{m}\binom{m}{k}\Big(\nu^k(1-\nu)^{m-k} - \nu^m\Big)a^k b^{m-k}\\ &= \Big((1-\nu)^m - \nu^m - r_m\Big)b^m + \Big(2^m\nu^m - r_m\Big)(ab)^{\frac{m}{2}} + 2r_m(ab)^{\frac{m}{4}}b^{\frac{m}{2}}\\ &= \sum_{k=0}^{m+2}\nu_k x_k,\end{aligned}$$

where x_k is given by:

$$x_0 := b^m, \quad \text{with} \quad \nu_0 := \Big((1-\nu)^m - \nu^m - r_m\Big),$$

and for $1 \leqslant k \leqslant m$,

$$x_k := a^k b^{m-k}, \quad \text{with} \quad \nu_k := \binom{m}{k}\Big(\nu^k(1-\nu)^{m-k} - \nu^m\Big),$$

and

$$x_{m+1} := (ab)^{\frac{m}{2}}, \quad \text{with} \quad \nu_{m+1} := \Big(2^m \nu^m - r_m\Big),$$

and

$$x_{m+2} := (ab)^{\frac{m}{4}} b^{\frac{m}{2}}, \quad \text{with} \quad \nu_{m+2} := 2r_m.$$

We have

1. $x_k > 0$ for all $k \in \{0,1,\ldots,m+1,m+2\}$,
2. $\nu_k \geqslant 0$ for all $k \in \{0,1,\ldots,m+1,m+2\}$, with $\sum_{k=0}^{m+2} \nu_k = 1$.

Hence by Theorem 1.1, we get

$$\begin{aligned}\Big(\nu a + (1-\nu)b\Big)^m &- \nu^m\Big((a+b)^m - 2^m(ab)^{\frac{m}{2}}\Big) - r_m\Big((ab)^{\frac{m}{4}} - b^{\frac{m}{2}}\Big)^2 \\ &\geqslant \prod_{k=0}^{m+2} x_k^{\nu_k} = a^{\alpha(m)} b^{\beta(m)},\end{aligned}$$

where

$$\begin{aligned}\alpha(m) &= \sum_{k=1}^{m-1}\binom{m}{k} k(\nu^k(1-\nu)^{m-k} - \nu^m) + \frac{m}{2}\Big(2^m\nu^m - r_m\Big) + \frac{m}{2} r_m \\ &= \sum_{k=1}^{m}\binom{m}{k} k\nu^k(1-\nu)^{m-k} - \nu^m \sum_{k=1}^{m}\binom{m}{k} k + 2^{m-1} m\nu^m \\ &= m\nu, \ \text{(by Lemma 2.1)}\end{aligned}$$

and

$$\begin{aligned}\beta(m) &= \sum_{k=1}^{m-1}\binom{m}{k}(m-k)(\nu^k(1-\nu)^{m-k} - \nu^m) + m\Big((1-\nu)^m - \nu^m - r_m\Big) \\ &\quad + \frac{m}{2}\Big(2^m\nu^m - r_m\Big) + \frac{3m}{2} r_m \\ &= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)(\nu^k(1-\nu)^{m-k} - \nu^m) + 2^{m-1} m\nu^m \\ &= \sum_{k=0}^{m-1}\binom{m}{k}(m-k)\nu^k(1-\nu)^{m-k} - \nu^m \sum_{k=0}^{m-1}\binom{m}{k}(m-k) + 2^{m-1} m\nu^m \\ &= m(1-\nu) \ \text{(by Lemma 2.1)}.\end{aligned}$$

If $\nu \in [\frac{1}{2}, 1]$, then $1-\nu \in [0, \frac{1}{2}]$. So by changing two elements a, b and two weights $\nu, 1-\nu$ in inequality (2.9), the desired inequality is obtained. □

REMARK 2.1. If we set $m = 1$ in Theorem 2.1, then we recapture the inequalities (1.5) and (1.6).

3. Applications to refined Young type inequalities for the traces, determinants, and p-norms

In this section, we focus on the Young inequality for the traces, determinants, and p-norms of positive τ-measurable operators.

Before stating and proving our results, we need to recall certain useful definitions.

3.1. Recalls and preliminaries

Throughout this section, $\mathscr{M} \subset B(\mathscr{H})$ will denote a von Neumann algebra on a separable Hilbert space $\mathscr{H}$, namely $\mathscr{M}$ is a $*$-subalgebra of $B(\mathscr{H})$ containing the identity 1. A trace τ on the von Neumann algebra $\mathscr{M}$ is a map $\tau : \mathscr{M}^+ \mapsto [0, +\infty)$ which is additive, positively homogeneous and unitarily invariant, that is, $\tau(x) = \tau(u^*xu)$ for all $x \in \mathscr{M}^+$ and unitary $u \in \mathscr{M}$, where $\mathscr{M}^+ = \{x \in \mathscr{M}, x \geqslant 0\}$. A trace τ is called

1. faithful if for all $x \in \mathscr{M}^+$, $\tau(x) = 0$ implies that $x = 0$,
2. semi-finite if for every $x \in \mathscr{M}^+$, with $\tau(x) > 0$, there exists $0 \leqslant y \leqslant x$, such that $0 < \tau(y) < \infty$,
3. normal if $x_i \uparrow_i x \in \mathscr{M}^+$, implies that $\tau(x_i) \uparrow_i \tau(x)$.

Note that a trace is called finite if $\tau(1) < \infty$.

$L_p(\mathscr{M}, \tau)$, $0 < p < +\infty$ denoted the set of all τ-measurable operators x affiliated with $\mathscr{M}$ where

$$||x||_p = \tau(|x|^p)^{\frac{1}{p}} < +\infty.$$

Namely $L_p(\mathscr{M}, \tau)$ is a Banach space under $||.||_p$ for $1 \leqslant p < +\infty$, For more informations on the $L_p(\mathscr{M}, \tau)$ spaces, one can see for instantce the references: [9] and [13].

DEFINITION 3.1. [7] Let $\mathscr{M}$ be a finite von Neumann algebra acting on a separable Hilbert space $\mathscr{H}$, with a normal faithful finite trace τ. For $x \in \mathscr{M}$, we define the determinant of x by $\Delta_\tau(x) = \exp \tau(\log |x|)$ if $|x|$ is invertible, and otherwise we define $\Delta_\tau(x) = \inf \Delta_\tau(|x| + \varepsilon 1)$, the infimum takes over all scalars $\varepsilon > 0$.

Now we shall stat some nown properties of determinants of τ-measurable operators (see [4], [5]) which we shall need later

1. $\Delta_\tau(\lambda x) = |\lambda| \Delta_\tau(x), \quad \forall \lambda \in \mathbb{R}$,
2. $\Delta_\tau(xy) = \Delta_\tau(x)\Delta_\tau(y)$.

The version Young's inequalities for the determinants, p-norm and traces states as follows: for any $x,y,z \in \mathscr{M}^+$ and for all positive integer m, we have

$$\left(\Delta_\tau(x^\nu y^{1-\nu})\right)^m \leqslant \left(\Delta_\tau(\nu x+(1-\nu)y)\right)^m, \tag{3.1}$$

$$||x^\nu zy^{1-\nu}||_p^m \leqslant \left(\nu||xz||_p+(1-\nu)||zy||_p\right)^m. \tag{3.2}$$

$$\left((\tau(x^\nu y^{1-\nu})\right)^m \leqslant \left(\tau(\nu x+(1-\nu)y)\right)^m, \tag{3.3}$$

J. Shao [12] showed the above three inequalities as:

$$\left(\Delta_\tau(x^\nu y^{1-\nu})\right)^m + r_0^m\left((\Delta_\tau(x)+\Delta_\tau(y))^m - 2^m(\Delta_\tau(xy))^{\frac{m}{2}}\right) \leqslant \left(\Delta_\tau(\nu x+(1-\nu)y)\right)^m,$$

$$||x^\nu zy^{1-\nu}||_p^m + r_0^m\left((||xz||_p+||zy||_p)^m - 2^m(||xz||_p||zy||_p)^{\frac{m}{2}}\right) \leqslant \left(\nu||xz||_p+(1-\nu)||zy||_p\right)^m$$

and

$$\left((\tau(x^\nu y^{1-\nu})\right)^m + r_0^m\left((\tau(x)+\tau(y))^m - 2^m(\tau(x)\tau(y))^{\frac{m}{2}}\right) \leqslant \left(\tau(\nu x+(1-\nu)y)\right)^m.$$

Next, as a consequence of Theorem 2.1, we present a refinement of Young's type inequality for the determinants of positive τ-measurable operators.

3.2. Refinement of Young's type inequality for determinants

Before giving our result, we need the following lemma.

LEMMA 3.1. [7] *Let $x,y \in \mathscr{M}^+$. Then we have*

$$\Delta_\tau(x)+\Delta_\tau(y) \leqslant \Delta_\tau(x+y)$$

THEOREM 3.1. *Let $x,y \in \mathscr{M}^+$, and $0 \leqslant \nu \leqslant 1$. Then for all positive integer m, we have*

$$\begin{aligned}&\left(\Delta_\tau(x^\nu y^{1-\nu})\right)^m + r_0^m\left((\Delta_\tau(x)+\Delta_\tau(y))^m - 2^m(\Delta_\tau(xy))^{\frac{m}{2}}\right) + r_m\\ &\quad\times\left[\left([\Delta_\tau(xy)]^{\frac{m}{4}} - (\Delta_\tau(y))^{\frac{m}{2}}\right)^2 \chi_{(0,\frac{1}{2}]}(\nu) + \left([\Delta_\tau(xy)]^{\frac{m}{4}} - (\Delta_\tau(x))^{\frac{m}{2}}\right)^2 \chi_{(\frac{1}{2},1]}(\nu)\right]\\ &\leqslant \Delta_\tau\left(\nu x+(1-\nu)y\right)^m,\end{aligned}$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $r_m = \min\{2^m r_0^m, (1-r_0)^m - r_0^m\}$.

Proof. By using Theorem 2.1 and Lemma 3.1, we have

$$\begin{aligned}
\Delta_\tau\Big(\nu x+(1-\nu)y\Big)^m &\geqslant \Big[\nu\Delta_\tau(x)+(1-\nu)\Delta_\tau(y)\Big]^m \text{ (by Lemma 3.1)}\\
&\geqslant \Big[\Big(\Delta_\tau(x)\Big)^\nu\Big(\Delta_\tau(y)\Big)^{1-\nu}\Big]^m\\
&\quad +r_0^m\Big((\Delta_\tau(x)+\Delta_\tau(y))^m-2^m(\Delta_\tau(x)\Delta_\tau(y))^{\frac{m}{2}}\Big)^2\\
&\quad +r_m\Big[\Big([\Delta_\tau(x)\Delta_\tau(y)]^{\frac{m}{4}}-(\Delta_\tau(y))^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&\quad +\Big([\Delta_\tau(x)\Delta_\tau(y)]^{\frac{m}{4}}-(\Delta_\tau(x))^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big] \qquad \text{(by Theorem 2.1)}\\
&= \Big(\Delta_\tau(x^\nu y^{1-\nu})\Big)^m+r_0^m\Big((\Delta_\tau(x)+\Delta_\tau(y))^m-2^m(\Delta_\tau(xy))^{\frac{m}{2}}\Big)^2\\
&\quad +r_m\Big[\Big([\Delta_\tau(xy)]^{\frac{m}{4}}-(\Delta_\tau(y))^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&\quad +\Big([\Delta_\tau(xy)]^{\frac{m}{4}}-(\Delta_\tau(x))^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big]. \qquad \square
\end{aligned}$$

3.3. Refinement of Young's type inequality for p-norms

In this subsection, we are concerned by establishing a new refinement of Young's inequality for p-norms of positive τ-measurable operators.

Before giving our result, we need the following lemma.

LEMMA 3.2. [14] *Let $x,y\in L_p(\mathscr{M},\tau)$ be a positive operators, where $1\leqslant p<+\infty$, $z\in\mathscr{M}$, and $0\leqslant\nu\leqslant 1$. Then we have*

$$||x^\nu zy^{1-\nu}||_p\leqslant ||xz||_p^\nu||zy||_p^{1-\nu}.$$

In particular,

$$\tau(x^\nu y^{1-\nu})\leqslant\tau(x)^\nu\tau(y)^{1-\nu}.$$

THEOREM 3.2. *Let $x,y,z\in\mathscr{M}^+$, and $0\leqslant\nu\leqslant 1$. Then for all positive integer m, we have*

$$\begin{aligned}
||x^\nu zy^{1-\nu}||_p^m &+ r_0^m\Big((||xz||_p+||zy||_p)^m-2^m(||xz||_p||zy||_p)^{\frac{m}{2}}\Big)\\
&+r_m\Big[\Big((||xz||_p||zy||_p)^{\frac{m}{4}}-(||zy||_p)^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&+\Big((||xz||_p||zy||_p)^{\frac{m}{4}}-(||xz||_p)^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big]\\
&\leqslant \Big[\nu||xz||_p+(1-\nu)||zy||_p\Big]^m,
\end{aligned}$$

where $r_0=\min\{\nu,1-\nu\}$ and $r_m=\min\{2r_0^m,(1-r_0)^m-r_0^m\}$.

Proof. By using Theorem 2.1 and Lemma 3.2, we have

$$
\begin{aligned}
&||x^{\nu}zy^{1-\nu}||_p^m + r_0^m\Big((||xz||_p+||zy||_p)^m - 2^m(||xz||_p||zy||_p)^{\frac{m}{2}}\Big)\\
&+r_m\Big[\Big((||xz||_p||zy||_p)^{\frac{m}{4}} - (||zy||_p)^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&+\Big((||xz||_p||zy||_p)^{\frac{m}{4}} - (||xz||_p)^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big]\\
\leqslant\ &\Big[||xz||_p^{\nu}||zy||_p^{1-\nu}\Big]^m + r_0^m\Big((||xz||_p+||zy||_p)^m - 2^m(||xz||_p||zy||_p)^{\frac{m}{2}}\Big)\\
&+r_m\Big[\Big((||xz||_p||zy||_p)^{\frac{m}{4}} - (||zy||_p)^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&+\Big((||xz||_p||zy||_p)^{\frac{m}{4}} - (||xz||_p)^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big] \text{ (by Lemma 3.2)}\\
\leqslant\ &\Big[\nu||xz||_p+(1-\nu)||zy||_p\Big]^m \text{ (by Theorem 2.1).} \quad \square
\end{aligned}
$$

3.4. Refinement of Young's type inequality for traces

We end this paper by giving an inequality for traces of positive τ-measurable operators by using Theorem 2.1. Precisely, we show the following result.

THEOREM 3.3. *Let $x,y \in \mathscr{M}^+$, and $0 \leqslant \nu \leqslant 1$. Then for all positive integer m, we have*

$$
\begin{aligned}
&\Big(\tau(x^{\nu}y^{1-\nu})\Big)^m + r_0^m\Big((\tau(x)+\tau(y))^m - 2^m(\tau(x)\tau(y))^{\frac{m}{2}}\Big) + r_m\\
&\quad\times\Big[\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(y))^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu) + \Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(x))^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big]\\
&\leqslant \Big[\tau(\nu x+(1-\nu)y)\Big]^m,
\end{aligned}
$$

where $r_0 = \min\{\nu, 1-\nu\}$ and $r_m = \min\{2^m r_0^m, (1-r_0)^m - r_0^m\}$.

Proof. By using Theorem 2.1 and Lemma 3.2, we have

$$
\begin{aligned}
&\Big(\tau(x^{\nu}y^{1-\nu})\Big)^m + r_0^m\Big((\tau(x)+\tau(y))^m - 2^m(\tau(x)\tau(y))^{\frac{m}{2}}\Big)\\
&\qquad+\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(y))^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu) +\\
&\qquad\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(x))^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big)\\
&\qquad\leqslant \Big[(\tau(x))^{\nu}(\tau(y))^{1-\nu}\Big]^m + r_0^m\Big((\tau(x)+\tau(y))^m - 2^m(\tau(x)\tau(y))^{\frac{m}{2}}\Big)\\
&\qquad+r_m\Big[\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(y))^{\frac{m}{2}}\Big)^2\chi_{(0,\frac{1}{2}]}(\nu)\\
&\qquad+\Big([\tau(x)\tau(y)]^{\frac{m}{4}} - (\tau(x))^{\frac{m}{2}}\Big)^2\chi_{(\frac{1}{2},1]}(\nu)\Big] \text{ (by Lemma 3.2)}\\
&\qquad\leqslant \Big[\tau(\nu x+(1-\nu)y)\Big]^m \text{ (by Theorem 2.1).} \quad \square
\end{aligned}
$$

4. Concluding remarks

The paper starts with and an introduction in which we make some recalls concern (scalar) Young's inequality and its refinements obtained by several authors.

The purpose of this work is devoted to generalize some refinement of Young's inequality and provide several applications.

In section 2, we establish in Theorem 2.1 a new generalized refinement of Young inequality. This theorem will generalize the result (see inequalities (1.5) and (1.6)), obtained by J. Zhao and J. Wu.

In section 3, we make some recalls concern the determinants, p-norms and traces of τ-measurable operators.

As a consequence of Theorem 2.1, we deduce (see Theorem 3.1) a new refinement of Young's type inequality for the determinants of positive τ-measurable operators.

A second application of Theorem 2.1 is to give (see Theorem 3.2) a new refinement of Young's type inequality for the p-norm of positive τ-measurable operators.

In the last application of Theorem 2.1, we provide a new refinement of Young's type inequality (see Theorem 3.3), for the traces.

We hope that our work will provide more other applications.

Acknowledgements. The authors would like to express their deep thanks to the anonymous referees for their helpful comments and suggestions on the initial version of the manuscript which lead to the improvement of this paper.

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(Received June 16, 2020)

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