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HIGHER DIMENSIONS OPIAL DIAMOND–ALPHA INEQUALITIES ON TIME SCALES

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Abstract. In this paper, we generalize Opial inequality to higher dimensions on time scales. We prove the Opial Delta-nabla inequality of n variables, and then give two diamond-alpha dynamic inequalities of Opial type of n variables. As well, we introduce some special cases.

1. Introduction

To unify the representation of discrete and continuous inequalities, Hilger established the theory of time scales in 1998 [1]. Since then, a number of interesting results on time scales have been presented such as Jensen's equalities, Hardy's equalities, Hölder's inequality, et al [2, 3]. Among them, Bohner and Kaymakçalan gave inequality which cover continuous and discrete opial inequalities [4, 5].

THEOREM 1.1. (see [4]) *Suppose* $0,h\in\mathbb{T}$ *and* $f:[0,h]_{\mathbb{T}}\to\mathbb{R}$ *with* $f(0)=0$ *is delta-differentiable, then*

$$\int_0^h |(f(t)+f^{\sigma}(t))f^{\Delta}(t)|\Delta t\leqslant h\int_0^h |f^{\Delta}(t)|^2\Delta t, \tag{1.1}$$

with equality when $f(x)=ct$.

THEOREM 1.2. (see [5]) *Suppose* $0,h\in\mathbb{T}$ *and* $f:[0,h]_{\mathbb{T}}\to\mathbb{R}$ *with* $f(0)=0$ *is nabla-differentiable, then*

$$\int_0^h |(f(t)+f^{\rho}(t))f^{\nabla}(t)|\nabla t\leqslant h\int_0^h |f^{\nabla}(t)|^2\nabla t, \tag{1.2}$$

with equality when $f(x)=ct$.

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THEOREM 1.3. (see [4]) *Suppose f is Δ-differentiable defined on $[0,h]_{\mathbb{T}}$, then*

$$\int_0^h |(f(t)+f^{\sigma}(t))f^{\Delta}(t)|\Delta t \leqslant \alpha \int_0^h |f^{\Delta}(t)|^2\Delta t + 2\beta \int_0^h |f^{\Delta}(t)|\Delta t, \tag{1.3}$$

where

$$\alpha = \min_{t\in[0,h]_{\mathbb{T}}} \max\{t, h-t\},$$
$$\beta = \max\{|f(0)|,|f(h)|\}.$$

THEOREM 1.4. (see [4]) *Suppose f is ∇-differentiable defined on $[0,h]_{\mathbb{T}}$, then*

$$\int_0^h |(f(t)+f^{\rho}(t))f^{\nabla}(t)|\nabla t \leqslant \alpha \int_0^h |f^{\nabla}(t)|^2\nabla t + 2\beta \int_0^h |f^{\nabla}(t)|\nabla t, \tag{1.4}$$

where

$$\alpha = \min_{t\in[0,h]_{\mathbb{T}}} \max\{t, h-t\},$$

and

$$\beta = \max\{|f(0)|,|f(h)|\}.$$

And many generalized Opial inequalities involving more functions or higher order derivatives on time scales were provided in [4, 14, 15, 16, 17, 18, 20].

THEOREM 1.5. (see [4]) *Suppose $p,q \in C([0,h])$ are positive functions with $\int_0^h \frac{1}{p(t)}\Delta t < \infty$, q nonincreasing. If f is Δ-differentiable defined on $[0,h]_{\mathbb{T}}$ with $f(0) = 0$, then*

$$\int_0^h \Big(q^{\sigma}(t)|(f(t)+f^{\rho}(t))f^{\nabla}(t)|\Delta t \leqslant \Big(\int_0^h \frac{1}{p(t)}\Delta t\Big)\Big(\int_0^h p(t)q(t)|f^{\Delta}(t)|^2\Delta t\Big).$$

THEOREM 1.6. (see [4]) *Suppose $m,n \in \mathbb{N}$, if Δ^n-differentiable function $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$ with $f(0) = f^{\Delta}(0) = \cdots = f^{\Delta^{n-1}}(0) = 0$, then*

$$\int_0^h \Big|\Big(\sum_{k=0}^{m} f^k(t)(f^{\sigma})^{m-k}(t)\Big)f^{\Delta^n}(t)\Big|\Delta t \leqslant h^{nm}\int_0^h \Big|f^{\Delta^n}(t)\Big|^{m+1}\Delta t.$$

In [14], it was presented that

THEOREM 1.7. *Suppose $m,n \in \mathbb{N}$ and $g,p : [0,h]_{\mathbb{T}} \to \mathbb{R}$ are positive Δ-differentiable with g nonincreasing, and $\int_0^h \frac{1}{p(t)}\Delta t < \infty$. If $f : [0,h]_{\mathbb{T}} \to \mathbb{R}$ is Δ^n-differentiable with $f(0) = f^{\Delta}(0) = \cdots = f^{\Delta^{n-1}}(0)$, then*

$$\begin{aligned}&\int_0^h g^{\sigma}(t)\Big|\Big(\sum_{k=1}^{m} f^k(t)(f^{\sigma})^{m-k}(t)\Big)f^{\Delta^n}(t)\Big|\Delta t\\ &\leqslant h^{(n-1)m}\Big(\int_0^h \frac{1}{p(t)}\Delta t\Big)^m\Big(\int_0^h p^m(t)g^{\sigma}(t)\Big|f^{\Delta^n}(t)\Big|^{m+1}\Delta t\Big).\end{aligned} \tag{1.5}$$

Recently, researchers have provided $\diamond_\alpha$ as a weighting between Δ and ∇ dynamic derivatives. Many diamond-alpha inequalities as the generalization of delta inequalities and nabla inequalities were provided, such as [6, 7, 8, 9, 10].

According to Theorems 1.1 and 1.2, Opial diamond-alpha inequality we need to prove is

$$\int_0^h \left|\frac{\partial f^2(v)}{\diamond_\alpha v}\right| \diamond_\alpha v \leqslant h \int_0^h \left|\frac{\partial f(v)}{\diamond_\alpha v}\right|^2 \diamond_\alpha v. \tag{1.6}$$

However, it doesn't hold when $h=1$, $\alpha=\frac{1}{2}$, $\mathbb{T}=\{-1,0,1,2\}$, $f(0)=f(-1)=0$, $f(1)=f(2)=1$.

In [11], the following inequality was discovered in 2010. For convenience, we call it Right-Opial Diamond-Alpha inequality of one variable.

THEOREM 1.8. (Right-Opial Diamond-Alpha inequality of one variable) *Suppose* $0,h\in\mathbb{T}$ *and* $f:[0,h]_{\mathbb{T}}\to\mathbb{R}$ *with* $f(0)=0$ *is* $\diamond_\alpha$*-differentiable, if* $f^\Delta f^\nabla$ *is nonnegative, then*

$$\alpha^3\int_0^h |(f(t)+f^\sigma(t))f^\Delta(t)|\Delta t+(1-\alpha)^3\int_0^h |(f(t)+f^\rho(t))f^\nabla(t)|\nabla t\leqslant h\int_0^h (f^{\diamond_\alpha}(t))^2\diamond_\alpha t \tag{1.7}$$

In [15], a generalization about Theorem 1.8 was given that

THEOREM 1.9. *Suppose* $p>1$, $q=\frac{p}{p-1}$, $h>0$, $h\in\mathbb{T}$, $w:[0,h]_{\mathbb{T}}\to(0,\infty)$ *is continuous function,* $f\in C_{\diamond_\alpha}([0,h]_{\mathbb{T}},\mathbb{R})$ *with* $f(0)=0$*. If both* f^Δ *and* f^∇ *are nonnegative, then*

$$\begin{aligned}&\alpha^4\int_0^h |(f^2)^\Delta(t)|\Delta t+(1-\alpha^4)\int_0^h |(f^2)^\nabla(t)|\nabla t\\ &\leqslant \Big(\int_0^h w^{1-q}(t)\diamond_\alpha t\Big)^{\frac{2}{q}}\Big(\int_0^h w(t)|f^{\diamond_\alpha}(t)|^p\diamond_\alpha t\Big)^{\frac{2}{p}}.\end{aligned} \tag{1.8}$$

In this paper, we will firstly give the following inequality called Left-Opial Diamond-Alpha inequality of one variable.

THEOREM 1.10. (Left-Opial Diamond-Alpha inequality of one variable) *Suppose* $0,h\in\mathbb{T}$, $f:\mathbb{T}\to\mathbb{R}$ *with* $f(\rho(0))=f(\sigma(0))=f(0)=0$*, and* f^Δ,f^∇ *are continuous functions, then*

$$\begin{aligned}\int_0^h \left|\frac{\partial f^2(t)}{\diamond_\alpha t}\right|\diamond_\alpha t &\leqslant \alpha^2 h\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2\Delta t+\alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2\Delta t\\ &\quad+\alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2\nabla t+(1-\alpha)^2h\int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2\nabla t,\end{aligned} \tag{1.9}$$

where $\sigma(t)=\inf\{s\in\mathbb{T}:s>t\}$ *and* $\rho(t)=\sup\{s\in\mathbb{T}:s<t\}$.

Secondly, the following inequality called Opial Delta-Nabla Inequality of n variables will be given.

THEOREM 1.11. (Opial Delta-Nabla Inequality of n variables) *Suppose* $0, h_i \in \mathbb{T}_i$ $(i = 1, 2, \cdots, n)$, *diamond-*α_i $(i = 1, 2, \cdots, n)$ *integrable and differential function* $f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \times \cdots \times [0, h_n]_{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ *satisfies that* $f(v_1, v_2, \cdots, v_n) = 0$ *when one of* v_i $(i = 1, 2, \cdots, n) = 0$, *then*

$$\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n(f^2(v_1,v_2,\cdots,v_n))}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n}\right|\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n$$
$$\leqslant \prod_{i=1}^n h_i \int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n}\right|^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n, \quad (1.10)$$

where $\omega_i(i = 1, 2, \cdots, n) = 0$ *or* 1.

Theorem 1.11 may be helpful when generalizing Theorems 1.3, 1.5, 1.6 and 1.7 to higher dimension.

And then we will prove the following Theorem via Theorem 1.11.

THEOREM 1.12. (Right-Opial Diamond-Alpha inequality of n variables) *Suppose* $0, h_i \in \mathbb{T}_i$ $(i = 1, \cdots, n)$, *diamond-*α_i $(i = 1, \cdots, n)$ *integrable and differential function* $f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \times \cdots \times [0, h_n]_{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ *satisfies*
(i) $f(v_1, v_2, \cdots, v_n) = 0$ *when one of* v_i $(i = 1, \cdots, n) = 0$;
(ii) $f_{1,2,\cdots,n}^{\diamond_{\omega_1},\diamond_{\omega_2},\cdots\diamond_{\omega_n}} \geqslant 0$ *for all* ω_i $(i = 1, \cdots, n) = 0$ *or* 1;
then

$$\sum_{\omega_1,\cdots,\omega_n=0 \text{ or } 1}\psi_1^3(\omega_1)\cdots\psi_n^3(\omega_n)\int_0^{h_n}\cdots\int_0^{h_1}\left|\frac{\partial^n(f^2(v_1,\cdots,v_n))}{\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n}\right|\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n$$
$$\leqslant \prod_{i=1}^n h_i \int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n}\right|^2\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n, \quad (1.11)$$

where $\omega_i(i = 1, \cdots, n) = 0$ *or* 1, *functions* $\psi_i : \{0, 1\} \to [0, 1](i = 1, \cdots, n)$ *are defined as follows,*

$$\psi_i(x) = \begin{cases} \alpha_i & \text{if } x = 1, \\ 1 - \alpha_i & \text{if } x = 0. \end{cases}$$

In the end, we will obtain

THEOREM 1.13. (Left-Opial Diamond-Alpha inequality of n variables) *Suppose* $0, h_i \in \mathbb{T}_i$ $(i = 1, \cdots, n)$, *diamond-*α_i $(i = 1, \cdots, n)$ *integral and differentiable function* $f : \mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ *satisfies*
(i) $f(v_1, v_2, \cdots, v_n) = 0$ *when one of* v_i $(i = 1, \cdots, n) = 0$;
(ii) $f(v_1, v_2, \cdots, v_n) = 0$ *when one of* v_i $(i = 1, \cdots, n) = \rho(0)$;
(iii) $f(v_1, v_2, \cdots, v_n) = 0$ *when one of* v_i $(i = 1, \cdots, n) = \sigma(0)$;
(iv) $f_i^{\Delta_i}(v_1, v_2, \cdots, v_i, \cdots, v_n) = f_i^{\nabla_i}(v_1, v_2, \cdots, \sigma(v_i), \cdots, v_n)$ *for all* $i = 1, 2, \cdots, n$;
(v) $f_i^{\nabla_i}(v_1, v_2, \cdots, v_i, \cdots, v_n) = f_i^{\Delta_i}(v_1, v_2, \cdots, \rho(v_i), \cdots, v_n)$ *for all* $i = 1, 2, \cdots, n$;

then

$$\begin{aligned}
&\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f^2(v_1,v_2,\cdots,v_n)}{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n}\right|\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n\\
\leqslant&\sum_{\omega_1,\omega_1,\cdots,\omega_n=0\ or\ 1}\psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)\\
&\times\sum_{\omega_1',\omega_2',\cdots,\omega_n'=0\ or\ 1}\psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')\\
&\times\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2\cdots\diamond_{\omega_n'}v_n}\right|^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n,
\end{aligned}\tag{1.12}$$

where $\omega_i(i=1,\cdots,n)=0$ *or* 1, *functions* $\psi_i:\{0,1\}\to[0,1](i=1,\cdots,n)$ *are defined as follows,*

$$\psi_i(x)=\begin{cases}\alpha_i & if\ x=1,\\ 1-\alpha_i & if\ x=0.\end{cases}$$

The structure of this paper is: In section 2, some preliminaries will be mentioned. And we will prove Theorem 1.10 in section 3. In section 4, we prove Theorems from 1.11 to 1.13 via some lemmas, and introduce some special cases.

2. Preliminaries

For more basic knowledge about time scales, readers can consult [13].

See [6], one-order partial Diamond-Alpha dynamic derivative is denoted by

$$\frac{\partial f(v_1,v_2,\cdots,v_n)}{\diamond_{\alpha_i}v_i}\qquad(i=1,2,\cdots,n).$$

Further,

$$\frac{\partial}{\diamond_{\alpha_j}v_j}\frac{\partial f(v_1,v_2,\cdots,v_n)}{\diamond_{\alpha_i}v_i}=\frac{\partial^2 f(v_1,v_2,\cdots,v_n)}{\diamond_{\alpha_i}v_i\diamond_{\alpha_j}v_j}\qquad(1\leqslant i,j\leqslant n),$$

and higher order partial Diamond-Alpha dynamic derivatives can be defined in the same way.

However in [15], the authors give an another denotations: one-order partial Delta dynamic derivative is denoted by

$$f_i^{\Delta_i}(v_1,v_2,\cdots,v_n)\qquad(i=1,2,\cdots,n),$$

similarly, one-order partial Diamond-Alpha dynamic derivatives can be denoted by

$$f_i^{\diamond_{\alpha_i}}(v_1,v_2,\cdots,v_n)\qquad(i=1,2,\cdots,n).$$

Further, for all $1 \leqslant i, j \leqslant n$,

$$\frac{\partial}{\diamond_{\alpha_j} v_j} f_i^{\diamond_{\alpha_i}}(v_1, v_2, \cdots, v_n) = f_{i,j}^{\diamond_{\alpha_i}, \diamond_{\alpha_j}}(v_1, v_2, \cdots, v_n),$$

and higher order partial Diamond-Alpha dynamic derivatives also can be defined in the same way.

It is worth noting that the above two denotations are equivalent. And both of them are used in this paper.

If $\alpha = 0$ or 1, partial Diamond-Alpha dynamic derivatives change into the combinations of partial delta and nabla derivatives, such as

$$\frac{\partial^k f(v_1, v_2, \cdots, v_n)}{\Delta_{j_1} v_{j_1} \nabla_{j_2} v_{j_2} \cdots \Delta_{j_k} v_{j_k}},$$

for convenience we call it partial Delta-Nabla derivatives.

According to the definition of diamond-alpha differential, the formula next point out the link between partial Diamond-Alpha and Delta-Nabla dynamic derivatives.

$$\begin{aligned} &\frac{\partial^n \phi(v_1, v_2, \cdots, v_n)}{\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n} \\ &= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n) \frac{\partial^n \phi(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n}, \end{aligned} \tag{2.1}$$

where $\omega_1, \cdots, \omega_n = 0$ or 1 means all non-repetitive possibilities in binary and functions $\psi_i : \{0,1\} \to [0,1] (i = 1, 2, \cdots, n)$ defined as follows,

$$\psi_i(x) = \begin{cases} \alpha_i & \text{if } x = 1, \\ 1 - \alpha_i & \text{if } x = 0. \end{cases}$$

In the same way, [12] gave the following theorem.

THEOREM 2.1. (see [12]) *Suppose $\phi(v_1, v_2, \cdots, v_n)$ is diamond-α_i $(i = 1, \cdots, n)$ integrable on $\mathbb{T}_1 \times \mathbb{T}_2 \times \cdots \times \mathbb{T}_n$, then*

$$\begin{aligned} &\int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \phi(v_1, \cdots, v_n) \diamond_{\alpha_1} v_1 \cdots \diamond_{\alpha_n} v_n \\ &= \sum_{\omega_1, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \cdots \psi_n(\omega_n) \int_{a_n}^{b_n} \cdots \int_{a_1}^{b_1} \phi(v_1, \cdots, v_n) \diamond_{\omega_1} v_1 \cdots \diamond_{\omega_n} v_n, \end{aligned}$$

where $\omega_1, \cdots, \omega_n = 0$ or 1 means all non-repetitive possibilities in binary and functions $\psi_i : \{0,1\} \to [0,1] (i = 1, 2, \cdots, n)$ defined as follows,

$$\psi_i(x) = \begin{cases} \alpha_i & \text{if } x = 1, \\ 1 - \alpha_i & \text{if } x = 0. \end{cases}$$

3. Left-Opial Diamond-Alpha inequality of one variable

In this section, one of our aims is giving the proof of Theorem 1.10. Meantime we will give a special case as well.

Proof of Theorem 1.10. Noting that

$$\begin{aligned}\frac{\partial f^2(t)}{\diamond_\alpha t} &= \alpha\frac{\partial f^2(t)}{\Delta t} + (1-\alpha)\frac{\partial f^2(t)}{\nabla t}\\ &= \alpha f^\Delta(t)(f^\sigma(t)+f(t)) + (1-\alpha)f^\nabla(t)(f^\rho(t)+f(t)).\end{aligned}$$

Left hand side equal to

$$\begin{aligned}&\int_0^h \left|\frac{\partial f^2(t)}{\diamond_\alpha t}\right| \diamond_\alpha t\\ &= \alpha\int_0^h \left|\frac{\partial f^2(t)}{\diamond_\alpha t}\right|\Delta t + (1-\alpha)\int_0^h \left|\frac{\partial f^2(t)}{\diamond_\alpha t}\right|\nabla t\\ &= \alpha\int_0^h \Big|\alpha f^\Delta(t)(f^\sigma(t)+f(t)) + (1-\alpha)f^\nabla(t)(f^\rho(t)+f(t))\Big|\Delta t\\ &\quad +(1-\alpha)\int_0^h \Big|\alpha f^\Delta(t)(f^\sigma(t)+f(t)) + (1-\alpha)f^\nabla(t)(f^\rho(t)+f(t))\Big|\nabla t\\ &\leqslant \alpha^2\int_0^h \Big|f^\Delta(t)(f^\sigma(t)+f(t))\Big|\Delta t + \alpha(1-\alpha)\int_0^h \Big|f^\nabla(t)(f^\rho(t)+f(t))\Big|\Delta t\\ &\quad +\alpha(1-\alpha)\int_0^h \Big|f^\Delta(t)(f^\sigma(t)+f(t))\Big|\nabla t + (1-\alpha)^2\int_0^h \Big|f^\nabla(t)(f^\rho(t)+f(t))\Big|\nabla t.\end{aligned}$$

Based on the Theorems 1.1 and 1.2, the following two inequalities have been proved.

$$\int_0^h |f^\Delta(t)(f^\sigma(t)+f(t))|\Delta t \leqslant h\int_0^h |f^\Delta(t)|^2\Delta t, \tag{3.1}$$

$$\int_0^h |f^\nabla(t)(f^\rho(t)+f(t))|\nabla t \leqslant h\int_0^h |f^\nabla(t)|^2\nabla t. \tag{3.2}$$

Since f^Δ, f^∇ are continuous functions, we have

$$f^\Delta(x) = f^\nabla(\sigma(x)), \qquad f^\nabla(x) = f^\Delta(\rho(x)) \qquad \forall x\in\mathbb{T}.$$

Hence we get

$$\int_0^h |f^\nabla(t)(f^\rho(t)+f(t))|\Delta t = \int_0^h \left|\frac{\partial f^2(t)}{\nabla t}\right|\Delta t = \int_0^h \left|\frac{\partial f^2(\rho(t))}{\Delta t}\right|\Delta t.$$

Noting that $f(\rho(0)) = f(0) = 0$, then we use Theorem 1.1

$$\int_0^h \left|\frac{\partial f^2(\rho(t))}{\Delta t}\right|\Delta t \leqslant h\int_0^h \left|\frac{\partial f(\rho(t))}{\Delta t}\right|^2\Delta t.$$

So we have

$$\int_0^h |f^\nabla(t)(f^\rho(t)+f(t))|\Delta t \leqslant h\int_0^h \left|\frac{\partial f(\rho(t))}{\Delta t}\right|^2 \Delta t. \tag{3.3}$$

In the same manner, we can get

$$\int_0^h |f^\Delta(t)(f^\sigma(t)+f(t))|\nabla t \leqslant h\int_0^h \left|\frac{\partial f(\sigma(t))}{\nabla t}\right|^2 \nabla t. \tag{3.4}$$

According to (3.1), (3.2), (3.3) and (3.4), we can complete the proof

$$\begin{aligned}
&\int_0^h \left|\frac{\partial f^2(t)}{\diamond_\alpha t}\right| \diamond_\alpha t \\
&= \alpha^2 \int_0^h \left|f^\Delta(t)(f^\sigma(t)+f(t))\right|\Delta t + \alpha(1-\alpha)\int_0^h \left|f^\nabla(t)(f^\rho(t)+f(t))\right|\Delta t \\
&\quad +\alpha(1-\alpha)\int_0^h \left|f^\Delta(t)(f^\sigma(t)+f(t))\right|\nabla t + (1-\alpha)^2\int_0^h \left|f^\nabla(t)(f^\rho(t)+f(t))\right|\nabla t \\
&\leqslant \alpha^2 h\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2 \Delta t + \alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(\rho(t))}{\Delta t}\right|^2 \Delta t \\
&\quad +\alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(\sigma(t))}{\nabla t}\right|^2 \nabla t + (1-\alpha)^2 h\int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2 \nabla t \\
&= \alpha^2 h\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2 \Delta t + \alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2 \Delta t \\
&\quad +\alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2 \nabla t + (1-\alpha)^2 h\int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2 \nabla t. \quad \square
\end{aligned}$$

We can find that if $\alpha = 0$ or 1, Theorem 1.10 reduces to Theorems 1.1 and 1.2. The right hand side of (1.9) equivalent to

$$\begin{aligned}
&\alpha^2 h\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2 \Delta t + \alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2 \Delta t \\
&\quad +\alpha(1-\alpha)h\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2 \nabla t + (1-\alpha)^2 h\int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2 \nabla t \\
&= \alpha h\int_0^h |f^\Delta(t)|^2 \diamond_\alpha t + (1-\alpha)h\int_0^h |f^\nabla(t)|^2 \diamond_\alpha t \\
&= h\int_0^h \alpha|f^\Delta(t)|^2 + (1-\alpha)|f^\nabla(t)|^2 \diamond_\alpha t.
\end{aligned} \tag{3.5}$$

If taking $\alpha = \frac{1}{2}$, then we have the following corollary.

COROLLARY 3.1. *Suppose* $0, h \in \mathbb{T}$ *and* $f : \mathbb{T} \to \mathbb{R}$ *with* $f(\sigma(0)) = f(0) =$

$f(\rho(0)) = 0$, *and* f^Δ, f^∇ *are continuous functions, then*

$$\begin{aligned}\int_0^h \left|\frac{\partial f^2(t)}{\diamond_{\frac{1}{2}} t}\right| \diamond_{\frac{1}{2}} t \leqslant \frac{1}{4} h\Bigg(&\int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2 \Delta t + \int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2 \Delta t \\ &+ \int_0^h \left|\frac{\partial f(t)}{\Delta t}\right|^2 \nabla t + \int_0^h \left|\frac{\partial f(t)}{\nabla t}\right|^2 \nabla t\Bigg) \\ &= \frac{h}{2}\int_0^h |f^\Delta(t)|^2 + |f^\nabla(t)|^2 \diamond_{\frac{1}{2}} t. \end{aligned} \tag{3.6}$$

4. Opial inequalities of n variables on time scales

In [13, Theorem 6.80, 6.94], the authors give Delta and nabla dynamic derivatives Leibniz Formula. They also point Delta-Nabla dynamic derivatives Leibniz Formula can be defined in the same manner in the end of chapter 6, that's means, they explore the form of

$$(f(u_1,u_2,\cdots,u_n)g(u_1,u_2,\cdots,u_n))_{i^m}^{\diamond_{\omega_i}^m}, \tag{4.1}$$

where $\omega_i = 0$ or 1.

And we will explore the following formula in the next,

$$(f(u_1,u_2,\cdots,u_n)g(u_1,u_2,\cdots,u_n))_{i_1,i_2,\cdots,i_n}^{\diamond_{\omega_{i_1}},\diamond_{\omega_{i_2}},\cdots,\diamond_{\omega_{i_n}}},$$

where $\omega_i = 0$ or 1 for all $1 \leqslant i \leqslant n$ and $i_j \neq i_k$ when $j \neq k$ for all $1 \leqslant j,k \leqslant n$. In this paper, we only take

$$(f(u_1,u_2,\cdots,u_n)g(u_1,u_2,\cdots,u_n))_{1,2,\cdots,n}^{\diamond_{\omega_1},\diamond_{\omega_2},\cdots,\diamond_{\omega_n}} \tag{4.2}$$

as an example.

To simplify statement, we introduce the notation

$$f_{1,2,\cdots,n}^{\tau_1,\tau_2,\cdots,\tau_n}(u_1,u_2,\cdots,u_n), \tag{4.3}$$

where τ_i $(i = 1,2,\cdots n) \in \{\Delta,\nabla,\sigma,\rho,O\}$. And if $\tau_i \in \{\Delta,\nabla\}$, then we differentiate with respect to the u_i, else if $\tau_i \in \{\sigma,\rho\}$, then replace u_i with $\tau_i(u_i)$, and we do nothing to u_i when $\tau_i = O$. For examples, we set

$$f_{1,2,3,4,5}^{\Delta,\sigma,\sigma,\Delta,O}(u_1,u_2,u_3,u_4,u_5) := \frac{\partial^2 f(u_1,\sigma(u_2),\sigma(u_3),u_4,u_5)}{\Delta_1 u_1 \Delta_4 u_4},$$

and

$$f_{1,2,3,4}^{\rho,\nabla,O,\rho,O,\rho}(u_1,u_2,u_3,u_4,u_5,u_6) := \frac{\partial f(\rho(u_1),u_2,u_3,\rho(u_4),u_5,\rho(u_6))}{\nabla_2 u_2}.$$

One thing we should emphasize is that every variable only correspond to one τ in this notation.

LEMMA 4.1. *If $f(u_1,u_2,\cdots,u_n)$, $g(u_1,u_2,\cdots,u_n)$ are Δ_i-differential ($i=1,2,\cdots,n$) on $\mathbb{T}_1\times\mathbb{T}_2\times\cdots\times\mathbb{T}_n$, then*

$$\begin{aligned}&\frac{\partial^n(f(u_1,u_2,\cdots,u_n)g(u_1,u_2,\cdots,u_n))}{\Delta_1u_1\Delta_2u_2\cdots\Delta_nu_n}\\&=\frac{1}{2^n}\sum_{\langle\tau_1,\tau_2,\cdots,\tau_{2n}\rangle\in L_0^{(n)}}f_{1,2,\cdots,n}^{\tau_1,\tau_2,\cdots,\tau_n}(u_1,u_2,\cdots,u_n)g_{1,2,\cdots,n}^{\tau_{n+1},\tau_{n+2},\cdots,\tau_{2n}}(u_1,u_2,\cdots,u_n),\end{aligned}\tag{4.4}$$

where $L_0^{(n)}$ defined as follows,

$$L_0^{(n)}=\{\langle\tau_1,\tau_2,\cdots,\tau_{2n}\rangle:\ \text{one of } \tau_i \text{ and } \tau_{n+i} \text{ is } \Delta,\ \text{another is } O \text{ or } \sigma\}.\tag{4.5}$$

Proof. We will use induction. Clearly, we have

$$\begin{aligned}(f(s)g(s))^{\Delta}&=f^{\Delta}(s)g(s)+f^{\sigma}(s)g^{\Delta}(s)\\&=f^{\Delta}(s)g^{\sigma}(s)+f(s)g^{\Delta}(s)\\&=\frac{1}{2}f^{\Delta}(s)g(s)+f^{\sigma}(s)g^{\Delta}(s)+\frac{1}{2}f^{\Delta}(s)g^{\sigma}(s)+f(s)g^{\Delta}(s)\\&=\frac{1}{2}\Big(f(s)g^{\Delta}(s)+f^{\sigma}(s)g^{\Delta}(s)+f^{\Delta}(s)g(s)+f^{\Delta}(s)g^{\sigma}(s)\Big).\end{aligned}\tag{4.6}$$

On the other hand, when $n=1$, $L_0^{(1)}=\{\langle O,\Delta\rangle,\langle\sigma,\Delta\rangle,\langle\Delta,O\rangle,\langle\Delta,\sigma\rangle\}$. So (4.4) holds when $n=1$.

Then we suppose it is true for $n=k$, that's means

$$\begin{aligned}&\frac{\partial^k(f(u_1,u_2,\cdots,u_k)g(u_1,u_2,\cdots,u_k))}{\Delta_1u_1\Delta_2u_2\cdots\Delta_ku_k}\\&=\frac{1}{2^k}\sum_{\langle\tau_1,\tau_2,\cdots,\tau_{2k}\rangle\in L_0^{(k)}}f_{1,2,\cdots,k}^{\tau_1,\tau_2,\cdots,\tau_k}(u_1,u_2,\cdots,u_k)g_{1,2,\cdots,k}^{\tau_{k+1},\tau_{k+2},\cdots,\tau_{2k}}(u_1,u_2,\cdots,u_k).\end{aligned}\tag{4.7}$$

When $n=k+1$, we have

$$\begin{aligned}&\frac{\partial^{k+1}(f(u_1,\cdots,u_k,u_{k+1})g(u_1,\cdots,u_k,u_{k+1}))}{\Delta_1u_1\cdots\Delta_ku_k\Delta_{k+1}u_{k+1}}\\&=\frac{\partial}{\Delta_{k+1}u_{k+1}}\Big(\frac{\partial^k(f(u_1,\cdots,u_k,u_{k+1})g(u_1,\cdots,u_k,u_{k+1}))}{\Delta_1u_1\cdots\Delta_ku_k}\Big)\\&=\frac{1}{2^k}\frac{\partial}{\Delta_{k+1}u_{k+1}}\Big(\sum_{\langle\tau_1,\cdots,\tau_{2k}\rangle\in L_0^{(k)}}f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k,O}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k},O}(u_1,\cdots,u_k,u_{k+1})\Big)\\&=\frac{1}{2^{k+1}}\sum_{\langle\tau_1,\cdots,\tau_{2k}\rangle\in L_0^{(k)}}\Big(f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k,\Delta}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k},O}(u_1,\cdots,u_k,u_{k+1})\\&+f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k,\Delta}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k},\sigma}(u_1,\cdots,u_k,u_{k+1})\end{aligned}$$

$$+f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k,O}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k},\Delta}(u_1,\cdots,u_k,u_{k+1})$$
$$+f_{1,2,\cdots,k+1}^{\tau_1,\cdots,\tau_k,\sigma}(u_1,\cdots,u_k,u_{k+1})g_{1,2,\cdots,k+1}^{\tau_{k+1},\cdots,\tau_{2k},\Delta}(u_1,\cdots,u_k,u_{k+1})\Big)$$
$$=\frac{1}{2^{k+1}}\sum_{\langle\tau_1,\cdots,\tau_{2k+2}\rangle\in L_0^{(k+1)}} f_{1,\cdots,k+1}^{\tau_1,\cdots,\tau_{k+1}}(u_1,\cdots,u_{k+1})g_{1,\cdots,k+1}^{\tau_{k+2},\cdots,\tau_{2k+2}}(u_1,\cdots,u_{k+1}), \tag{4.8}$$

where using (4.6). □

When ω_i $(i=1,2,\cdots,n)=0$ or 1, then $\diamond_{\omega_i}=\nabla$ or Δ. Noting that Δ and ∇ have the same algorithms, so if replace Δ_i with $\diamond_{\omega_i}$, we can get the following lemma. And in order to simplify notation, we define

$$\Psi(x)=\begin{cases}\sigma, & \text{if } x=1,\\ \rho, & \text{if } x=0,\end{cases}$$

and

$$p=\sum_{i=1}^{n}\omega_i 2^{i-1}.$$

LEMMA 4.2. *If $f(u_1,u_2,\cdots,u_n)$, $g(u_1,u_2,\cdots,u_n)$ are Δ_i-differential and ∇_i-differential $(i=1,2,\cdots,n)$ on $\mathbb{T}_1\times\mathbb{T}_2\times\cdots\times\mathbb{T}_n$, then*

$$\frac{\partial^n(f(u_1,u_2,\cdots,u_n)g(u_1,u_2,\cdots,u_n))}{\diamond_{\omega_1}u_1\diamond_{\omega_2}u_2\cdots\diamond_{\omega_n}u_n}$$
$$=\frac{1}{2^n}\sum_{\langle\tau_1,\tau_2,\cdots,\tau_{2n}\rangle\in L_p^{(n)}} f_{1,2,\cdots,n}^{\tau_1,\tau_2,\cdots,\tau_n}(u_1,u_2,\cdots,u_n)g_{1,2,\cdots,n}^{\tau_{n+1},\tau_{n+2},\cdots,\tau_{2n}}(u_1,u_2,\cdots,u_n), \tag{4.9}$$

where $L_p^{(n)}$ defined as follows,

$$L_p^{(n)}=\{\langle\tau_1,\tau_2,\cdots,\tau_{2n}\rangle:\ \text{one of } \tau_i \text{ and } \tau_{n+i} \text{ is } \diamond_{\omega_i},\ \text{another is } O \text{ or } \Psi(\omega_i)\}. \tag{4.10}$$

REMARK 4.1. If $p=0$, then every $\omega_i=1(i=1,2,\cdots,n)$, lemma 4.2 reduces to lemma 4.1.

REMARK 4.2. We give an example when $n=2,p=1$, then

$$\begin{aligned}L_1^{(2)}=\{&\langle\Delta,\rho,\sigma,\nabla\rangle,\langle\Delta,\nabla,\sigma,\rho\rangle,\langle\sigma,\nabla,\Delta,\rho\rangle,\langle\sigma,\rho,\Delta,\nabla\rangle,\\ &\langle\Delta,O,O,\nabla\rangle,\langle\Delta,\nabla,O,O\rangle,\langle O,\nabla,\Delta,O\rangle,\langle O,O,\Delta,\nabla\rangle,\\ &\langle\Delta,\rho,O,\nabla\rangle,\langle\Delta,\nabla,\sigma,O\rangle,\langle\sigma,\nabla,\Delta,O\rangle,\langle\sigma,O,\Delta,\nabla\rangle,\\ &\langle\Delta,O,\sigma,\nabla\rangle,\langle\Delta,\nabla,O,\rho\rangle,\langle O,\nabla,\Delta,\rho\rangle,\langle O,\rho,\Delta,\nabla\rangle\}.\end{aligned}$$

So we can get

$$\frac{\partial^2(f(s_1,s_2)g(s_1,s_2))}{\Delta_1 s_1\nabla_2 s_2}$$
$$=\frac{1}{4}\Big(f_{1,2}^{\Delta,\rho}(s_1,s_2)g_{1,2}^{\sigma,\nabla}(s_1,s_2)+f_{1,2}^{\sigma,\nabla}(s_1,s_2)g_{1,2}^{\Delta,\rho}(s_1,s_2)+f_{1,2}^{\Delta,\nabla}(s_1,s_2)g_{1,2}^{\sigma,\rho}(s_1,s_2)$$

$$+f_{1,2}^{\sigma,\rho}(s_1,s_2)g_{1,2}^{\Delta,\nabla}(s_1,s_2)+f_{1,2}^{\Delta,O}(s_1,s_2)g_{1,2}^{O,\nabla}(s_1,s_2)+f_{1,2}^{O,\nabla}(s_1,s_2)g_{1,2}^{\Delta,O}(s_1,s_2)$$
$$+f_{1,2}^{\Delta,\nabla}(s_1,s_2)g_{1,2}^{O,O}(s_1,s_2)+f_{1,2}^{O,O}(s_1,s_2)g_{1,2}^{\Delta,\nabla}(s_1,s_2)+f_{1,2}^{\Delta,\rho}(s_1,s_2)g_{1,2}^{O,\nabla}(s_1,s_2)$$
$$+f_{1,2}^{\sigma,\nabla}(s_1,s_2)g_{1,2}^{\Delta,O}(s_1,s_2)+f_{1,2}^{\Delta,\nabla}(s_1,s_2)g_{1,2}^{\sigma,O}(s_1,s_2)+f_{1,2}^{\sigma,O}(s_1,s_2)g_{1,2}^{\Delta,\nabla}(s_1,s_2)$$
$$+f_{1,2}^{\Delta,O}(s_1,s_2)g_{1,2}^{\sigma,\nabla}(s_1,s_2)+f_{1,2}^{O,\nabla}(s_1,s_2)g_{1,2}^{\Delta,\rho}(s_1,s_2)+f_{1,2}^{\Delta,\nabla}(s_1,s_2)g_{1,2}^{O,\rho}(s_1,s_2)$$
$$+f_{1,2}^{O,\rho}(s_1,s_2)g_{1,2}^{\Delta,\nabla}(s_1,s_2)\Big).$$

Noting that if $g=f$, then lemma 4.2 reduces to

COROLLARY 4.1. *If $f(u_1,u_2,\cdots,u_n)$ is Δ_i-differential and ∇_i-differential $(i=1,2,\cdots,n)$ on $\mathbb{T}_1\times\mathbb{T}_2\times\cdots\times\mathbb{T}_n$, then*

$$\begin{aligned}&\frac{\partial^n(f^2(u_1,u_2,\cdots,u_n))}{\diamond_{\omega_1}u_1\diamond_{\omega_2}u_2\cdots\diamond_{\omega_n}u_n}\\&=\frac{1}{2^n}\sum_{\langle\tau_1,\tau_2,\cdots,\tau_{2n}\rangle\in L_p^{(n)}}f_{1,2,\cdots,n}^{\tau_1,\tau_2,\cdots,\tau_n}(u_1,u_2,\cdots,u_n)f_{1,2,\cdots,n}^{\tau_{n+1},\tau_{n+2},\cdots,\tau_{2n}}(u_1,u_2,\cdots,u_n),\end{aligned}\tag{4.11}$$

where $L_p^{(n)}$ defined as follows,

$$L_p^{(n)}=\{\langle\tau_1,\tau_2,\cdots,\tau_{2n}\rangle:\ \textit{one of } \tau_i \textit{ and } \tau_{n+i} \textit{ is } \diamond_{\omega_i},\ \textit{another is } O \textit{ or } \Psi(\omega_i)\}.\tag{4.12}$$

In order to prove Theorem 1.12, we give the proof of following theorem first.

Proof of Theorem 1.11. Set

$$m(u_1,u_2,\cdots,u_n)=\int_0^{u_n}\int_0^{u_{n-1}}\cdots\int_0^{u_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n}\right|\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n.$$

Hence we get

$$\frac{\partial^n m(u_1,u_2,\cdots,u_n)}{\diamond_{\omega_1}u_1\diamond_{\omega_2}u_2\cdots\diamond_{\omega_n}u_n}=\left|\frac{\partial^n f(u_1,u_2,\cdots,u_n)}{\diamond_{\omega_1}u_1\diamond_{\omega_2}u_2\cdots\diamond_{\omega_n}u_n}\right|.$$

And then we suppose set $S=\{j_1,j_2,\cdots,j_s\}\subset N=\{1,2,\cdots,n\}$, that's means $j_1,j_2,\cdots,j_s$ are arbitrary s unequal numbers in the set N where $s<n$. We also set $N\setminus S=\{j_{s+1},j_{s+2},\cdots,j_n\}$.

$$\begin{aligned}&\frac{\partial^s m(u_1,u_2,\cdots,u_n)}{\diamond_{\omega_{j_1}}u_{j_1}\diamond_{\omega_{j_2}}u_{j_2}\cdots\diamond_{\omega_{j_s}}u_{j_s}}\\&=\frac{\partial^s}{\diamond_{\omega_{j_1}}u_{j_1}\cdots\diamond_{\omega_{j_n}}u_{j_s}}\int_0^{u_n}\cdots\int_0^{u_1}\left|\frac{\partial^n f(v_1,\cdots,v_n)}{\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n}\right|\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n\\&=\frac{\partial^s}{\diamond_{\omega_{j_1}}u_{j_1}\diamond_{\omega_{j_2}}u_{j_2}\cdots\diamond_{\omega_{j_s}}u_{j_s}}\end{aligned}$$

$$\begin{aligned}
&\int_0^{u_{j_1}}\int_0^{u_{j_2}}\cdots\int_0^{u_{j_n}}\left|\frac{\partial^n f(v_1,\cdots,v_n)}{\diamond_{\omega_1} v_1\cdots\diamond_{\omega_n} v_n}\right|\diamond_{\omega_{j_n}} v_{j_n}\diamond_{\omega_{j_{n-1}}} v_{j_{n-1}}\cdots\diamond_{\omega_{j_1}} v_{j_1}\\
=&\ \frac{\partial^{s-1}}{\diamond_{\omega_{j_2}} u_{j_2}\diamond_{\omega_{j_3}} u_{j_3}\cdots\diamond_{\omega_{j_s}} u_{j_s}}\\
&\int_0^{u_{j_2}}\int_0^{u_{j_3}}\cdots\int_0^{u_{j_n}}\left|\frac{\partial^n f(v_1,\cdots,v_n)}{\diamond_{\omega_{j_1}} v_{j_1}\cdots\diamond_{\omega_{j_n}} v_{j_n}}\Big|_{v_{j_1}=u_{j_1}}\right|\diamond_{\omega_{j_n}} v_{j_n}\diamond_{\omega_{j_{n-1}}} v_{j_{n-1}}\cdots\diamond_{\omega_{j_2}} v_{j_2}\\
=&\ \vdots\\
=&\int_0^{u_{j_{s+1}}}\cdots\int_0^{u_{j_n}}\left|\frac{\partial^n f(v_1,\cdots,v_n)}{\diamond_{\omega_{j_1}} v_{j_1}\cdots\diamond_{\omega_{j_n}} v_{j_n}}\Big|_{v_{j_i}=u_{j_i}(i=1,2,\cdots,s)}\right|\diamond_{\omega_{j_n}} v_{j_n}\cdots\diamond_{\omega_{j_{s+1}}} v_{j_{s+1}}\\
\geqslant&\left|\int_0^{u_{j_{s+1}}}\cdots\int_0^{u_{j_n}}\frac{\partial^n f(v_1,\cdots,v_n)}{\diamond_{\omega_{j_1}} v_{j_1}\cdots\diamond_{\omega_{j_n}} v_{j_n}}\Big|_{v_{j_i}=u_{j_i}(i=1,2,\cdots,s)}\diamond_{\omega_{j_n}} v_{j_n}\cdots\diamond_{\omega_{j_{s+1}}} v_{j_{s+1}}\right|\\
=&\left|\frac{\partial^s f(u_1,u_2,\cdots,u_n)}{\diamond_{\omega_{j_1}} u_{j_1}\diamond_{\omega_{j_2}} u_{j_2}\cdots\diamond_{\omega_{j_s}} u_{j_s}}\right| \qquad (4.13)
\end{aligned}$$

In particular, if we let the set $S=\varnothing$, then

$$m(u_1,u_2,\cdots,u_n)\geqslant|f(u_1,u_2,\cdots,u_n)|. \qquad (4.14)$$

Then using corollary 4.1, we have

$$\begin{aligned}
&\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n(f^2(v_1,v_2,\cdots,v_n))}{\diamond_{\omega_1} v_1\diamond_{\omega_2} v_2\cdots\diamond_{\omega_n} v_n}\right|\diamond_{\omega_1} v_1\diamond_{\omega_2} v_2\cdots\diamond_{\omega_n} v_n\\
=&\ \frac{1}{2^n}\int_0^{h_n}\cdots\int_0^{h_1}\\
&\left|\sum_{\langle\tau_1,\cdots,\tau_{2n}\rangle\in L_p^{(n)}} f_{1,\cdots,n}^{\tau_1,\cdots,\tau_n}(v_1,\cdots,v_n)f_{1,\cdots,n}^{\tau_{n+1},\cdots,\tau_{2n}}(v_1,\cdots,v_n)\right|\diamond_{\omega_1} v_1\cdots\diamond_{\omega_n} v_n\\
\leqslant&\ \frac{1}{2^n}\int_0^{h_n}\cdots\int_0^{h_1}\\
&\sum_{\langle\tau_1,\cdots,\tau_{2n}\rangle\in L_p^{(n)}}\left|f_{1,\cdots,n}^{\tau_1,\cdots,\tau_n}(v_1,\cdots,v_n)\right|\left|f_{1,\cdots,n}^{\tau_{n+1},\cdots,\tau_{2n}}(v_1,\cdots,v_n)\right|\diamond_{\omega_1} v_1\cdots\diamond_{\omega_n} v_n\\
\leqslant&\ \frac{1}{2^n}\int_0^{h_n}\cdots\int_0^{h_1}\\
&\sum_{\langle\tau_1,\cdots,\tau_{2n}\rangle\in L_p^{(n)}} m_{1,\cdots,n}^{\tau_1,\cdots,\tau_n}(v_1,\cdots,v_n)m_{1,\cdots,n}^{\tau_{n+1},\cdots,\tau_{2n}}(v_1,\cdots,v_n)\diamond_{\omega_1} v_1\cdots\diamond_{\omega_n} v_n\\
=&\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\frac{\partial^n(m^2(v_1,v_2,\cdots,v_n))}{\diamond_{\omega_1} v_1\diamond_{\omega_2} v_2\cdots\diamond_{\omega_n} v_n}\diamond_{\omega_1} v_1\diamond_{\omega_2} v_2\cdots\diamond_{\omega_n} v_n\\
=&\ (m^2(h_1,h_2,\cdots,h_n))-0\\
=&\ (m^2(h_1,h_2,\cdots,h_n))
\end{aligned}$$

$$
\begin{aligned}
&= \left(\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n} \right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \right)^2 \\
&\leqslant \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\
&\quad \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n} \right|^2 \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n \\
&= \prod_{i=1}^n h_i \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n} \right|^2 \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n. \qquad (4.15)
\end{aligned}
$$

□

In particular, if we set $\omega_i \ (i = 1, 2, \cdots, n) = 1$ in Theorem 1.11, hence

COROLLARY 4.2. *Suppose for all $i \in \mathbb{N}$, $0, h_i \in \mathbb{T}_i$, diamond-α_i integral and differentiable function $f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \times \cdots \times [0, h_n]_{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ satisfies that $f(v_1, v_2, \cdots, v_n) = 0$ when one of $v_i \ (i = 1, 2, \cdots, n) = 0$ and Δ-integrable, then*

$$
\begin{aligned}
&\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n (f^2(v_1, v_2, \cdots, v_n))}{\Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n} \right| \Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n \\
&\leqslant \prod_{i=1}^n h_i \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n} \right|^2 \Delta_1 v_1 \Delta_2 v_2 \cdots \Delta_n v_n. \qquad (4.16)
\end{aligned}
$$

Taking $\omega_i \ (i = 1, 2, \cdots, n) = 0$, hence we get

COROLLARY 4.3. *Suppose $0, h_i \in \mathbb{T}_i$, diamond-$\alpha_i \ (i = 1, 2, \cdots, n)$ integral and differentiable function $f : [0, h_1]_{\mathbb{T}_1} \times [0, h_2]_{\mathbb{T}_2} \times \cdots \times [0, h_n]_{\mathbb{T}_n} \to \mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$ satisfies that $f(v_1, v_2, \cdots, v_n) = 0$ when one of $v_i \ (i = 1, 2, \cdots, n) = 0$ and ∇-integrable, then*

$$
\begin{aligned}
&\int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n (f^2(v_1, v_2, \cdots, v_n))}{\nabla_1 v_1 \nabla_2 v_2 \cdots \nabla_n v_n} \right| \nabla_1 v_1 \nabla_2 v_2 \cdots \nabla_n v_n \\
&\leqslant \prod_{i=1}^n h_i \int_0^{h_n} \int_0^{h_{n-1}} \cdots \int_0^{h_1} \left| \frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\nabla_1 v_1 \nabla_2 v_2 \cdots \nabla_n v_n} \right|^2 \nabla_1 v_1 \nabla_2 v_2 \cdots \nabla_n v_n. \qquad (4.17)
\end{aligned}
$$

Further, we can find that Theorem 1.1 and 1.2 is the cases of $n = 1$ in Corollary 4.2 and 4.3.

Based on Theorem 1.11, we can give the proof of Theorem 1.12.

Proof of Theorem 1.12. Firstly, based on the condition (ii), we have

$$
\begin{aligned}
&\left(\psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) f_{1,2,\cdots,n}^{\diamond_{\omega_1}, \diamond_{\omega_2}, \cdots, \diamond_{\omega_n}} \right)^2 \\
&\leqslant \left(\sum_{\omega_1, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1) \psi_2(\omega_2) \cdots \psi_n(\omega_n) f_{1,2,\cdots,n}^{\diamond_{\omega_1}, \diamond_{\omega_2}, \cdots, \diamond_{\omega_n}} \right)^2 \\
&= \left(\frac{\partial^n f(v_1, v_2, \cdots, v_n)}{\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n} \right)^2. \qquad (4.18)
\end{aligned}
$$

And then using Theorem 1.11, we have

$$\begin{aligned}
&\sum_{\omega_1,\cdots,\omega_n=0 \text{ or } 1} \psi_1^3(\omega_1)\cdots\psi_n^3(\omega_n)\int_0^{h_n}\cdots\int_0^{h_1}\left|\frac{\partial^n(f^2(v_1,\cdots,v_n))}{\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n}\right|\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n\\
\leqslant&\sum_{\omega_1,\cdots,\omega_n=0 \text{ or } 1} \psi_1^3(\omega_1)\cdots\psi_n^3(\omega_n)\prod_{i=1}^n h_i\int_0^{h_n}\cdots\int_0^{h_1}\Big(\frac{\partial^n f(v_1,\cdots,v_n)}{\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n}\Big)^2\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n\\
=&\sum_{\omega_1,\cdots,\omega_n=0 \text{ or } 1} \psi_1(\omega_1)\cdots\psi_n(\omega_n)\\
&\prod_{i=1}^n h_i\int_0^{h_n}\cdots\int_0^{h_1}\psi_1^2(\omega_1)\cdots\psi_n^2(\omega_n)\Big(\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n}\Big)^2\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n\\
\leqslant&\sum_{\omega_1,\cdots,\omega_n=0 \text{ or } 1} \psi_1(\omega_1)\cdots\psi_n(\omega_n)\\
&\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left(\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n}\right)^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n\\
=&\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\Big(\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n}\Big)^2\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n. \qquad (4.19)
\end{aligned}$$

□

We have the following Corollary when $\alpha=\frac{1}{2}$ in Theorem 1.12,

COROLLARY 4.4. *Suppose* $0,h_i\in\mathbb{T}_i(\forall i\in\mathbb{N})$, *diamond-*$\alpha_i$ $(i=1,2,\cdots,n)$ *integrable and differentiable function* $f:[0,h_1]_{\mathbb{T}_1}\times[0,h_2]_{\mathbb{T}_2}\times\cdots\times[0,h_n]_{\mathbb{T}_n}\to\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$ *satisfies*
(i) $f(v_1,v_2,\cdots,v_n)=0$ *when one of* v_i $(i=1,2,\cdots,n)=0$*;*
(ii) $f_{1,2,\cdots,n}^{\diamond_{\omega_1},\diamond_{\omega_2},\cdots,\diamond_{\omega_n}}\geqslant 0$ *for all* $\omega_i=0$ *or* 1*;*
then

$$\begin{aligned}
&\sum_{\omega_1,\cdots,\omega_n=0 \text{ or } 1}\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n(f^2(v_1,v_2,\cdots,v_n))}{\diamond_{\frac{1}{2}}v_1\diamond_{\frac{1}{2}}v_2\cdots\diamond_{\frac{1}{2}}v_n}\right|\diamond_{\frac{1}{2}}v_1\diamond_{\frac{1}{2}}v_2\cdots\diamond_{\frac{1}{2}}v_n\\
\leqslant& 8^n\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\Big|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\frac{1}{2}}v_1\diamond_{\frac{1}{2}}v_2\cdots\diamond_{\frac{1}{2}}v_n}\Big|^2\diamond_{\frac{1}{2}}v_1\diamond_{\frac{1}{2}}v_2\cdots\diamond_{\frac{1}{2}}v_n, \qquad (4.20)
\end{aligned}$$

where $\omega_i(i=1,2,\cdots,n)=0$ *or* 1.

Theorem 1.12 reduces to Theorem 1.8 when $n=1$, and if taking $n=2$, we have the following Corollary.

COROLLARY 4.5. *Suppose* $0,h_1\in\mathbb{T}_1, 0,h_2\in\mathbb{T}_2$, *diamond-*$\alpha_i$ $(i=1,2)$ *integrable and differentiable function* $f:[0,h_1]_{\mathbb{T}_1}\times[0,h_2]_{\mathbb{T}_2}\to\mathbb{R}\times\mathbb{R}$ *satisfies*
(i) $f(v_1,v_2)=0$ *when one of* v_i $(i=1,2)=0$*;*
(ii) $f_{1,2}^{\diamond_{\omega_1},\diamond_{\omega_2}}(v_1,v_2)\geqslant 0$ *for all* ω_i $(i=1,2)=0$ *or* 1*;*
then

$$
\begin{aligned}
&\sum_{\omega_1,\omega_2=0\ or\ 1} \psi_1^3(\omega_1)\psi_2^3(\omega_2)\int_0^{h_2}\int_0^{h_1}\left|\frac{\partial^2(f^2(v_1,v_2))}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2}\right|\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\\
&=\alpha_1^3\alpha_2^3\int_0^{h_2}\int_0^{h_1}\left|\frac{\partial^2(f^2(v_1,v_2))}{\Delta_1 v_1\Delta_2 v_2}\right|\Delta_1 v_1\Delta_2 v_2\\
&\quad+\alpha_1^3(1-\alpha_2)^3\int_0^{h_2}\int_0^{h_1}\left|\frac{\partial^2(f^2(v_1,v_2))}{\Delta_1 v_1\nabla_2 v_2}\right|\Delta_1 v_1\nabla_2 v_2\\
&\quad+(1-\alpha_1)^3\alpha_2^3\int_0^{h_2}\int_0^{h_1}\left|\frac{\partial^2(f^2(v_1,v_2))}{\nabla_1 v_1\Delta_2 v_2}\right|\nabla_1 v_1\Delta_2 v_2\\
&\quad+(1-\alpha_1)^3(1-\alpha_2)^3\int_0^{h_2}\int_0^{h_1}\left|\frac{\partial^2(f^2(v_1,v_2))}{\nabla_1 v_1\nabla_2 v_2}\right|\nabla_1 v_1\nabla_2 v_2\\
&\leqslant h_1h_2\int_0^{h_2}\int_0^{h_1}\left|\frac{\partial^2 f(v_1,v_2)}{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2}\right|^2\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2,
\end{aligned}
\tag{4.21}
$$

where $\omega_i\ \ (i=1,2)=0\ or\ 1$.

Next we explore the Left-Opial Diamond-Alpha Inequality of n variable. First of all, suppose $f(v_1,v_2,\cdots,v_n)$ satisfies

$$f_i^{\Delta_i}(v_1,v_2,\cdots,v_i,\cdots,v_n)=f_i^{\nabla_i}(v_1,v_2,\cdots,\sigma(v_i),\cdots,v_n),$$

and

$$f_i^{\nabla_i}(v_1,v_2,\cdots,v_i,\cdots,v_n)=f_i^{\Delta_i}(v_1,v_2,\cdots,\rho(v_i),\cdots,v_n),$$

for all $i=1,2,\cdots,n$.

And we set

$$\kappa_x=\begin{cases}\rho & \text{if } x=-1,\\ O & \text{if } x=0,\\ \sigma & \text{if } x=1,\end{cases}\tag{4.22}$$

and

$$f^*(v_1,v_2,\cdots,v_n):=f_{1,2,\cdots,n}^{\kappa_{\omega_1'-\omega_1},\kappa_{\omega_2'-\omega_2},\cdots,\kappa_{\omega_n'-\omega_n}}(v_1,v_2,\cdots,v_n).\tag{4.23}$$

Under the conditions and notations above, the following lemma holds.

LEMMA 4.3. *Suppose* $f:\mathbb{T}_1\times\mathbb{T}_2\times\cdots\times\mathbb{T}_n\to\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$, *and function* f *satisfies*

$$f_i^{\Delta_i}(v_1,v_2,\cdots,v_i,\cdots,v_n)=f_i^{\nabla_i}(v_1,v_2,\cdots,\sigma(v_i),\cdots,v_n),$$

and

$$f_i^{\nabla_i}(v_1,v_2,\cdots,v_i,\cdots,v_n)=f_i^{\Delta_i}(v_1,v_2,\cdots,\rho(v_i),\cdots,v_n),$$

for all $i=1,2,\cdots,n$, *then*

$$\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2\cdots\diamond_{\omega_n'}v_n}=\frac{\partial^n f^*(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n},\tag{4.24}$$

where both ω_i' *and* ω_i *equal to* 0 *or* 1 *for all* $i = 1, 2, \cdots, n$.

Hence we can give the proof of Theorem 1.13.

Proof of Theorem 1.13. Noting that

$$\begin{aligned}&\frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n}\\ &= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n) \frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n},\end{aligned} \tag{4.25}$$

then, the left hand side of (1.12) can be rewritten as follows,

$$\begin{aligned}&\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n}\right| \diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n\\
&= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)\\
&\quad \int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\diamond_{\alpha_1} v_1 \diamond_{\alpha_2} v_2 \cdots \diamond_{\alpha_n} v_n}\right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n\\
&= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\\
&\quad \left|\sum_{\omega_1', \omega_2', \cdots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')\frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1'} v_1 \diamond_{\omega_2'} v_2 \cdots \diamond_{\omega_n'} v_n}\right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n\\
&\leqslant \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\\
&\quad \sum_{\omega_1', \omega_2', \cdots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')\left|\frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1'} v_1 \diamond_{\omega_2'} v_2 \cdots \diamond_{\omega_n'} v_n}\right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n\\
&= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n) \sum_{\omega_1', \omega_2', \cdots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')\\
&\quad \int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f^2(v_1, v_2, \cdots, v_n)}{\diamond_{\omega_1'} v_1 \diamond_{\omega_2'} v_2 \cdots \diamond_{\omega_n'} v_n}\right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n\\
&= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n) \sum_{\omega_1', \omega_2', \cdots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')\\
&\quad \int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n\left(f_{12\cdots n}^{\kappa_{\omega_1'-\omega_1}\kappa_{\omega_2'-\omega_2}\cdots\kappa_{\omega_n'-\omega_n}}(v_1, v_2, \cdots, v_n)\right)^2}{\diamond_{\omega_1} v_n \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n}\right| \diamond_{\omega_1} v_1 \diamond_{\omega_2} v_2 \cdots \diamond_{\omega_n} v_n\\
&= \sum_{\omega_1, \omega_2, \cdots, \omega_n = 0 \text{ or } 1} \psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n) \sum_{\omega_1', \omega_2', \cdots, \omega_n' = 0 \text{ or } 1} \psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')\end{aligned}$$

$$\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n\Big(f^*(v_1,v_2,\cdots,v_n)\Big)^2}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n}\right|\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n$$
$$\leqslant \sum_{\omega_1,\omega_2,\cdots,\omega_n=0\text{ or }1}\psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)\sum_{\omega_1',\omega_2',\cdots,\omega_n'=0\text{ or }1}\psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')$$
$$\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f^*(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n}\right|^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n$$
$$= \sum_{\omega_1,\omega_2,\cdots,\omega_n=0\text{ or }1}\psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)\sum_{\omega_1',\omega_2',\cdots,\omega_n'=0\text{ or }1}\psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')$$
$$\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2\cdots\diamond_{\omega_n'}v_n}\right|^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n \tag{4.26}$$

□

The right hand side of formula (1.12) also has an equivalent form

$$\sum_{\omega_1,\omega_2,\cdots,\omega_n=0\text{ or }1}\psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)\sum_{\omega_1',\omega_2',\cdots,\omega_n'=0\text{ or }1}\psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')$$
$$\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2\cdots\diamond_{\omega_n'}v_n}\right|^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n$$
$$= \sum_{\omega_1',\omega_2',\cdots,\omega_n'=0\text{ or }1}\psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')\sum_{\omega_1,\omega_1,\cdots,\omega_n=0\text{ or }1}\psi_1(\omega_1)\psi_2(\omega_2)\cdots\psi_n(\omega_n)$$
$$\prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2\cdots\diamond_{\omega_n'}v_n}\right|^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\cdots\diamond_{\omega_n}v_n$$
$$= \prod_{i=1}^n h_i\sum_{\omega_1',\omega_2',\cdots,\omega_n'=0\text{ or }1}\psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')$$
$$\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2\cdots\diamond_{\omega_n'}v_n}\right|^2\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n$$
$$= \prod_{i=1}^n h_i\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\sum_{\omega_1',\omega_2',\cdots,\omega_n'=0\text{ or }1}\psi_1(\omega_1')\psi_2(\omega_2')\cdots\psi_n(\omega_n')$$
$$\left|\frac{\partial^n f(v_1,v_2,\cdots,v_n)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2\cdots\diamond_{\omega_n'}v_n}\right|^2\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\cdots\diamond_{\alpha_n}v_n. \tag{4.27}$$

In particular, if seting $\alpha=\frac{1}{2}$ in Theorem 1.13 and using (4.27), then we have the following corollary.

COROLLARY 4.6. *Suppose* $0,h_i\in\mathbb{T}_i$, *diamond-*α_i $(i=1,2,\cdots,n)$ *integrable and differentiable function* $f:\mathbb{T}_1\times\mathbb{T}_2\times\cdots\times\mathbb{T}_n\to\mathbb{R}\times\mathbb{R}\times\cdots\times\mathbb{R}$ *satisfies*
(i) $f(v_1,v_2,\cdots,v_n)=0$ *when one of* v_i $(i=1,2,\cdots,n)=0$*;*

(ii) $f(v_1,v_2,\cdots,v_n)=0$ *when one of* $v_i\ \ (i=1,2,\cdots,n)=\rho(0)$*;*
(iii) $f(v_1,v_2,\cdots,v_n)=0$ *when one of* $v_i\ \ (i=1,2,\cdots,n)=\sigma(0)$*;*
(iv) $f_i^{\Delta_i}(v_1,v_2,\cdots,v_i,\cdots,v_n)=f_i^{\nabla_i}(v_1,v_2,\cdots,\sigma(v_i),\cdots,v_n)$ *for all* $i=1,2,\cdots,n$*;*
(v) $f_i^{\nabla_i}(v_1,v_2,\cdots,v_i,\cdots,v_n)=f_i^{\Delta_i}(v_1,v_2,\cdots,\rho(v_i),\cdots,v_n)$ *for all* $i=1,2,\cdots,n$*;*
then

$$\begin{aligned}&\int_0^{h_n}\int_0^{h_{n-1}}\cdots\int_0^{h_1}\Big|\frac{\partial^n f^2(v_1,v_2,\cdots,v_n)}{\diamond_{\frac12}v_1\diamond_{\frac12}v_2\cdots\diamond_{\frac12}v_n}\Big|\diamond_{\frac12}v_1\diamond_{\frac12}v_2\cdots\diamond_{\frac12}v_n\\ &\leqslant\frac{1}{2^n}\prod_{i=1}^{n}h_i\int_0^{h_n}\cdots\int_0^{h_1}\sum_{\omega_1,\cdots,\omega_n=0\ or\ 1}\Big|\frac{\partial^n f(v_1,\cdots,v_n)}{\diamond_{\omega_1}v_1\cdots\diamond_{\omega_n}v_n}\Big|^2\diamond_{\frac12}v_1\cdots\diamond_{\frac12}v_n.\end{aligned}\tag{4.28}$$

It can be found that Theorem 1.13 reduces to Theorem 1.10 when $n=1$, and using (4.27) we have following corollary when $n=2$.

COROLLARY 4.7. *Suppose* $0,h_i\in\mathbb{T}_i\ \ (i=1,2)$, *diamond-*$\alpha_i\ \ (i=1,2)$ *integrable and differentiable function* $f:\mathbb{T}_1\times\mathbb{T}_2\to\mathbb{R}\times\mathbb{R}$ *satisfies*
(i) $f(v_1,v_2)=0$ *when one of* $v_i\ \ (i=1,2)=0$*;*
(ii) $f(v_1,v_2)=0$ *when one of* $v_i\ \ (i=1,2)=\rho(0)$*;*
(iii) $f(v_1,v_2)=0$ *when one of* $v_i\ \ (i=1,2)=\sigma(0)$*;*
(iv) $f_1^{\Delta_1}(v_1,v_2)=f_1^{\nabla_1}(\sigma(v_1),v_2), f_2^{\Delta_2}(v_1,v_2)=f_2^{\nabla_2}(v_1,\sigma(v_2))$*;*
(v) $f_1^{\nabla_1}(v_1,v_2)=f_1^{\Delta_1}(\rho(v_1),v_2), f_2^{\nabla_2}(v_1,v_2)=f_2^{\Delta_2}(v_1,\rho(v_2))$*;*
then

$$\begin{aligned}&\int_0^{h_2}\int_0^{h_1}\Big|\frac{\partial^2 f^2(v_1,v_2)}{\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2}\Big|\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2\\ &\leqslant h_1h_2\sum_{\omega_1,\omega_2=0\ or\ 1}\psi_1(\omega_1)\psi_2(\omega_2)\\ &\quad\times\sum_{\omega_1',\omega_2'=0\ or\ 1}\psi_1(\omega_1')\psi_2(\omega_2')\int_0^{h_2}\int_0^{h_1}\Big|\frac{\partial^2 f(v_1,v_2)}{\diamond_{\omega_1'}v_1\diamond_{\omega_2'}v_2}\Big|^2\diamond_{\omega_1}v_1\diamond_{\omega_2}v_2\\ &=h_1h_2\int_0^{h_2}\int_0^{h_1}\alpha_1\alpha_2\Big|\frac{\partial^2 f(v_1,v_2)}{\Delta_1v_1\Delta_2v_2}\Big|^2+\alpha_1(1-\alpha_2)\Big|\frac{\partial^2 f(v_1,v_2)}{\Delta_1v_1\nabla_2v_2}\Big|^2\\ &\quad+(1-\alpha_1)\alpha_2\Big|\frac{\partial^2 f(v_1,v_2)}{\nabla_1v_1\Delta_2v_2}\Big|^2+(1-\alpha_1)(1-\alpha_2)\Big|\frac{\partial^2 f(v_1,v_2)}{\nabla_1v_1\nabla_2v_2}\Big|^2\diamond_{\alpha_1}v_1\diamond_{\alpha_2}v_2.\end{aligned}\tag{4.29}$$

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