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A GENERALIZATION OF S–NEKRASOV MATRICES

ZHEN-HUA LYU, LIXIN ZHOU AND JIANZHOU LIU *

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Abstract. The class of H-matrices plays an important role in various scientific disciplines. In this paper, we introduce a new subclass of H-matrices, called generalized S-Nekrasov matrices. We prove that this class contains the class of S-Nekrasov matrices. We also present a sufficient condition for a weak Nekrasov matrix to be an H-matrix.

1. Introduction

H-matrices and its subclasses play a significant role in many fields of science such as computational mathematics, mathematical physics and control theory; see [6, 7, 10, 11, 12] and the references therein. In 2009, Cvetković, Kostić, and Rauški [2] introduced a new subclass of H-matrices: S-Nekrasov matrices. This matrix class has been extensively studied; see [1, 3, 4, 5, 8, 9]. In this paper, we introduce a new subclass of H-matrices, called generalized S-Nekrasov matrices. In addition, we find that every S-Nekrasov matrix belongs to this new matrix class.

To present our result, we need the following notations and definitions. Let $\langle n\rangle = \{1,2,\cdots,n\}$ and let M_n be the set of all $n\times n$ complex matrices. For $A=(a_{ij})\in M_n$, denote

$$P_i(A)=\sum_{j\in\langle n\rangle, j\neq i}|a_{ij}|, \quad \forall\, i\in\langle n\rangle;$$

$$R_1(A)=P_1(A), \quad R_i(A)=\sum_{j=1}^{i-1}|a_{ij}|\frac{R_j(A)}{|a_{jj}|}+\sum_{j=i+1}^{n}|a_{ij}|, \ 2\leqslant i\leqslant n;$$

$$l_1(A)=0, \ l_i(A)=\sum_{j=1}^{i-1}|a_{ij}|\frac{R_j(A)}{|a_{jj}|}, \ 2\leqslant i\leqslant n.$$

DEFINITION 1.1. Let $A=(a_{ij})\in M_n$. Then A is a (row) diagonally dominant matrix (D_n) if

$$|a_{ii}|\geqslant P_i(A), \quad \forall\, i\in\langle n\rangle. \tag{1.1}$$

A is a strictly diagonally dominant matrix (SD_n) if all representative inequalities in (1.1) are strict. If there exists a positive diagonal matrix X such that $AX\in SD_n$, A is said to be a generalized strictly diagonally dominant matrix (i.e., nonsingular H-matrix).

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* Corresponding author.

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DEFINITION 1.2. Let $A = (a_{ij}) \in M_n$. Then A is a weak Nekrasov matrix if

$$|a_{ii}| \geqslant R_i(A), \quad \forall\, i \in \langle n \rangle. \tag{1.2}$$

A is a Nekrasov matrix (N_n) if all the inequalities in (1.2) are strict.

A matrix A is a nonsingular H-matrix if there exists a diagonal matrix D such that AD is strictly diagonally dominant. For each Nekrasov matrix B, it is a nonsingular H-matrix and then there is a diagonal matrix D such that BD is strictly diagonally dominant. Hence, if a matrix can be scaled to a Nekrasov matrix by a diagonal matrix from the right side, this matrix is a nonsingular H-matrix. The initial purpose of this paper is to find some practical and efficient criteria for H-matrices in this way. To our surprise, every S-Nekrasov matrix satisfies the sufficient condition we get.

Denote by S a nonempty subset of $\langle n \rangle$, and $\overline{S}$ the complement set of S in $\langle n \rangle$. We also need the following notations.

$$R_1^S(A) = \sum_{j \in S, j \neq 1} |a_{1j}|, \;\; R_i^S(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j^S(A)}{|a_{jj}|} + \sum_{j=i+1, j \in S}^{n} |a_{ij}|, \; 2 \leqslant i \leqslant n;$$

$$l_1^S(A) = 0, \;\; l_i^S(A) = \sum_{j=1}^{i-1} |a_{ij}| \frac{R_j^S(A)}{|a_{jj}|}, \;\; 2 \leqslant i \leqslant n;$$

$$r_S = \max_{i \in S} \frac{R_i^{\overline{S}}(A)}{|a_{ii}| - R_i^S(A)}, \quad \delta_i^S(A) = \frac{R_i^{\overline{S}} + r_S R_i^S(A)}{|a_{ii}|}, \quad \forall\, i \in S;$$

$$N_1(A) = \{i \in \langle n \rangle \mid |a_{ii}| \leqslant R_i(A)\}, \quad N_2(A) = \{i \in \langle n \rangle \mid |a_{ii}| > R_i(A)\}.$$

It is easy to conclude that the following equality holds for $i \in \langle n \rangle$.

$$f_i(A) = f_i^S(A) + f_i^{\overline{S}}(A), \tag{1.3}$$

where $f \in \{l, R\}$. Now we introduce a new matrix class as follows.

DEFINITION 1.3. Let $A = (a_{ij}) \in M_n$ with $n \geqslant 2$. Given an arbitrary $S \subseteq N_2(A)$, A is said to be a generalized S-Nekrasov matrix (GSN) if the following inequalities hold.

$$|a_{ii}| > R_i^{\overline{S}}(A) + r_S l_i^S(A) + \sum_{j > i+1, j \in S} |a_{ij}| \delta_j^S(A), \quad \forall\, i \in N_1(A). \tag{1.4}$$

In the second section, we will prove that GSN is a subclass of H-matrices. Moreover, GSN contains the class of S-Nekrasov matrices.

2. A generalization of S-Nekrasov matrices

We need the following lemmas.

LEMMA 2.1. *Let $A=(a_{ij})\in M_n$. If $S\subseteq N_2(A)$, then*

$$r_S \geqslant \max_{i\in S} \delta_i^S(A). \tag{2.1}$$

Proof. By the definition of $N_2(A)$, $i\in N_2(A)$ leads to $|a_{ii}|>0$. To the contrary, suppose that there exists $i\in S$ such that $r_S<\delta_i^S(A)$. By the definition of $\delta_i^S(A)$, we get

$$r_S < \frac{R_i^{\overline{S}}(A)+r_S R_i^S(A)}{|a_{ii}|}.$$

It is equivalent with

$$r_S|a_{ii}| < R_i^{\overline{S}}(A)+r_S R_i^S(A).$$

Collect the terms of r_S and then we get

$$r_S < R_i^{\overline{S}}(A)/(|a_{ii}|-R_i^S(A)),$$

which contradicts the definition of r_S. □

LEMMA 2.2. *Let $A=(a_{ij})\in M_n$, and let X be a positive diagonal matrix with all its entries less than or equal to 1. Given any $S\subseteq\langle n\rangle$, then*

$$R_i^S(AX)\leqslant R_i^S(A), \quad \forall\, i\in\langle n\rangle. \tag{2.2}$$

Proof. Let $B=AX$. Let $X=diag(x_1,\cdots,x_n)$ with $x_i\leqslant 1$ for $i\in\langle n\rangle$. For $i=1$, we have

$$R_1^S(B)=\sum_{j>1,j\in S}|b_{1j}|\leqslant \sum_{j>1,j\in S}|a_{1j}|=R_1^S(A).$$

Assume that for $i=2,3,\ldots,k$, the inequalities in (2.2) hold. Now consider the case $i=k+1$. We have

$$\begin{aligned} R_{k+1}^S(B) &= \sum_{j=1}^{k}|b_{k+1,j}|\frac{R_j^S(B)}{|b_{jj}|}+\sum_{j\geqslant k+2,j\in S}|b_{k+1,j}| \\ &\leqslant \sum_{j=1}^{k}|b_{k+1,j}|\frac{R_j^S(A)}{|b_{jj}|}+\sum_{j=k+2,j\in S}|b_{k+1,j}| \\ &\leqslant \sum_{j=1}^{k}|a_{k+1,j}|\frac{|R_j^S(A)|}{|a_{jj}|}+\sum_{j=k+2,j\in S}|a_{k+1,j}|=R_{k+1}^S(A). \end{aligned}$$

Hence, we get (2.2). □

The proof of the following lemma follows the same idea as in the proofs of [2, Theorem 2]. We state the details for completeness.

LEMMA 2.3. *Let $A=(a_{ij})\in M_n$, let $S\subseteq\langle n\rangle$ and let $X=diag(x_1,\cdots,x_n)$, where*

$$x_i=\begin{cases}1, & i\in\bar{S}\\ \gamma, & i\in S\end{cases}.$$

The following equalities hold for all $i\in\langle n\rangle$.

(i) $f_i^S(AX)=\gamma f_i^S(A)$;

(ii) $f_i^{\bar{S}}(AX)=f_i^{\bar{S}}(A)$,

where $f\in\{l,R\}$.

Proof. Let $B=AX$. First we consider $f=R$. We use induction on the row index i. For $i=1$, we have

$$R_1^S(B)=\sum_{j>1,j\in S}|b_{1j}|=\sum_{j>1,j\in S}\gamma|a_{1j}|=\gamma R_1^S(A).$$

Assume that (i) holds for $i=2,3,\dots,k$, then we consider the case $i=k+1$.

$$\begin{aligned}R_{k+1}^S(B)&=\sum_{j=1}^{k}|b_{k+1,j}|\frac{R_j^S(B)}{|b_{jj}|}+\sum_{j=k+2,j\in S}^{n}|b_{k+1,j}|\\&=\sum_{j=1}^{k}|a_{k+1,j}|\frac{\gamma R_j^S(A)}{|a_{jj}|}+\sum_{j=k+2,j\in S}^{n}\gamma|a_{k+1,j}|\\&=\gamma R_{k+1}^S(A)\end{aligned}$$

Hence we get (i). Similarly, we can get the equality in (ii).

Now assume $f=l$. For $i=1$, (i) holds trivially. For $i\geqslant 2$, we have

$$l_i^S(B)=\sum_{j=1}^{i-1}|b_{ij}|\frac{R_j^S(B)}{|b_{jj}|}=\sum_{j=1}^{i-1}|a_{ij}|\frac{\gamma R_j^S(A)}{|a_{jj}|}=\gamma l_i^S(A).$$

Applying the same argument we get $l_i^{\bar{S}}(B)=l_i^{\bar{S}}(A)$. □

LEMMA 2.4. *Let $A\in M_n$, let $S\subseteq\langle n\rangle$, and let $X=diag(x_1,\cdots,x_n)$, where*

$$x_i=\begin{cases}1, & i\in\bar{S}\\ \gamma_i, & i\in S\end{cases}.$$

The following inequalities hold for all $i\in\langle n\rangle$.

(i) $R_i(AX)\leqslant\gamma R_i^S(A)+R_i^{\bar{S}}(A)$;

(ii) $l_i^S(AX)\leqslant\gamma l_i^S(A)$,

where $\gamma = \max_{i \in S} \gamma_i$.

Proof. Let $C = AX$. Let B be the same matrix as defined in Lemma 2.3. By (1.3) and Lemma 2.3, we have

$$R_i(B) = \gamma R_i^S(A) + R_i^{\bar{S}}(A), \quad \forall\, i \in \langle n \rangle.$$

Let $X^* = diag(x_1^*, x_2^*, \ldots, x_n^*)$, where

$$x_i^* = \begin{cases} 1, & i \in \bar{S} \\ \frac{\gamma_i}{\gamma}, & i \in S \end{cases}.$$

It is clear that $C = BX^*$. Since $\gamma_i/\gamma \leqslant 1$, by Lemma 2.2 we get $R_i(C) \leqslant R_i(B)$, which leads to (i).

For the second part, by Lemma 2.2 and Lemma 2.3 (i) we have

$$\begin{aligned} l_i^S(C) &= \sum_{j=1, j \in S}^{i-1} |c_{ij}| \frac{R_j^S(C)}{|c_{jj}|} \leqslant \sum_{j=1, j \in S}^{i-1} |a_{ij}| \frac{R_j^S(B)}{|a_{jj}|} \\ &= \sum_{j=1, j \in S}^{i-1} |a_{ij}| \frac{\gamma R_j^S(A)}{|a_{jj}|} \\ &= \gamma l_i^S(A). \quad \square \end{aligned}$$

Now we are ready to prove that GSN is a subclass of H-matrices.

THEOREM 2.5. *Let $A \in M_n$. Given any $S \subseteq N_2(A)$, if $A \in GSN$, then A is a nonsingular H-matrix.*

Proof. Let

$$\varepsilon_i = \frac{|a_{ii}| - R_i^{\bar{S}}(A) - r_S l_i^S(A) - \sum_{j > i+1, j \in S} |a_{ij}| \delta_j^S(A)}{R_i(A)}.$$

If $R_i(A) = 0$, we let $\varepsilon_i = \infty$. By (1.4), we get $\varepsilon_i > 0$ for all $i \in N_1(A)$. Since $S \subseteq N_2(A)$, we get $r_S < 1$. It follows from Lemma 2.1 that $\delta_i^S(A) < 1$ for all $i \in S$. Let ε be a positive number satisfying

$$\varepsilon < \min\{\min_{i \in N_1(A)} \varepsilon_i, \min_{i \in S}(1 - \delta_i^S(A))\}.$$

Then we have

$$\delta_i^S + \varepsilon < 1, \quad \forall\, i \in S, \tag{2.3}$$

and

$$|a_{ii}| > R_i^{\overline{S}}(A) + (r_S+\varepsilon) l_i^S(A) + \sum_{j>i+1, j\in S} |a_{ij}|(\delta_j^S(A)+\varepsilon), \quad \forall\, i \in N_1(A). \tag{2.4}$$

Let $X = diag(x_1, \cdots, x_n)$, where

$$x_i = \begin{cases} 1, & i \in \overline{S} \\ \delta_i^S(A)+\varepsilon, & i \in S \end{cases}.$$

Let $B = (b_{ij}) = AX$. We consider the following cases.

Case 1. $i \in N_1(A)$. Combining (2.4), Lemma 2.1, Lemma 2.2 and Lemma 2.4 (ii), we have

$$\begin{aligned} |b_{ii}| = |a_{ii}| &> R_i^{\overline{S}}(A) + (r_S+\varepsilon) l_i^S(A) + \sum_{j>i+1, j\in S} |a_{ij}|(\delta_i^S(A)+\varepsilon) \\ &\geqslant R_i^{\overline{S}}(B) + l_i^S(B) + \sum_{j>i+1, j\in S} |b_{ij}| \\ &= R_i^{\overline{S}}(B) + R_i^S(B) = R_i(B). \end{aligned}$$

Case 2. $i \in N_2(A) \setminus S$. By Lemma 2.2 and the definition of $N_2(A)$, we have

$$|b_{ii}| = |a_{ii}| > R_i(A) \geqslant R_i(B).$$

Case 3. $i \in S$. By the definition of $\delta_i^S(A)$ and Lemma 2.4 (i), we have

$$\begin{aligned} |b_{ii}| &= |a_{ii}|(\delta_i^S(A)+\varepsilon) \\ &= R_i^{\overline{S}}(A) + r_S R_i^S(A) + |a_{ii}|\varepsilon \\ &= R_i^{\overline{S}}(A) + (r_S+\varepsilon) R_i^S(A) + (|a_{ii}| - R_i^S(A))\varepsilon \\ &> R_i^{\overline{S}}(A) + (r_S+\varepsilon) R_i^S(A) \geqslant R_i(B) \end{aligned}$$

Hence $B \in N_n$. It follows that A is a nonsingular H-matrix. This completes the proof. □

By Theorem 2.5, we can get the following corollary immediately.

COROLLARY 2.6. *If A is a weak Nekrasov matrix of order n and*

$$R_i^{N_2(A)}(A) \neq 0, \quad \forall\, i \in N_1(A),$$

then A is a nonsingular H-matrix.

Next we will show the relationship between GSN and the S-Nekrasov matrices. Let recall the definition of the S-Nekrasov matrices.

DEFINITION 2.7. Let $A=(a_{ij})\in M_n$ with $n\geqslant 2$. Given any nonempty subset S of $\langle n\rangle$, A is an S-Nekrasov matrix if

$$|a_{ii}|>R_i^S(A),\ \ |a_{jj}|>R_j^{\bar{S}}(A)$$

and

$$\left[|a_{ii}|-R_i^S(A)\right]\left[|a_{jj}|-R_j^{\bar{S}}(A)\right]>R_i^{\bar{S}}(A)R_j^S(A) \tag{2.5}$$

for all $i\in S$ and $j\in\bar{S}$ hold.

REMARK 2.8. Given an arbitrary S-Nekrasov matrix A, then either S or $\bar{S}$ is a subset of $N_2(A)$. Otherwise, $N_1(A)\cap S\neq\emptyset$ and $N_1(A)\cap\bar{S}\neq\emptyset$. There exist $i\in S$ and $j\in\bar{S}$ such that $i\in N_1(A)$ and $j\in N_1(A)$. By the definition of $N_1(A)$, we get

$$\left[|a_{ii}|-R_i^S(A)\right]\left[|a_{jj}|-R_j^{\bar{S}}(A)\right]\leqslant R_i^{\bar{S}}(A)R_j^S(A),$$

which contradicts (2.5). Without loss of generality, we assume $S\subseteq N_2(A)$. We turn (2.5) into

$$|a_{jj}|-R_j^{\bar{S}}(A)>\frac{R_i^{\bar{S}}(A)}{|a_{ii}|-R_i^S(A)}R_j^S(A),\quad \forall\, i\in S,\ j\in\bar{S}.$$

By the definition of r_S, we have

$$|a_{jj}|-R_j^{\bar{S}}(A)>r_SR_j^S(A),\quad \forall\ j\in\bar{S}.$$

On the other hand, by Lemma 2.1 we obtain

$$\begin{aligned} r_SR_j^S(A)&=r_Sl_j^S(A)+r_S\sum_{k>j+1,k\in S}|a_{jk}|\\ &\geqslant r_Sl_j^S(A)+\sum_{k>j+1,k\in S}|a_{jk}|\delta_k^S(A),\quad \forall\ j\in\bar{S}. \end{aligned}$$

Note that $N_1(A)\subseteq\bar{S}$. Hence A is a GSN. We have proved that every S-Nekrasov matrix is in GSN. But not vice versa.

EXAMPLE 2.9. Consider the matrix as follows.

$$A=\begin{bmatrix} 40 & 1 & 3 & 2 & 2\\ 0 & 8 & 4 & 4 & 6\\ 20 & 2 & 15 & 4 & 8\\ 0 & 4 & 5 & 18 & 2\\ 40 & 4 & 0 & 0 & 40 \end{bmatrix}.$$

We get $N_1(A)=\{2,3\}$ and $N_2(A)=\{1,4,5\}$. For $S\in\{\{1\},\{4\},\{5\},\{1,4\},\{1,5\},$ $\{4,5\},\{1,4,5\}\}$, A is not an S-Nekrasov matrix and hence it is not an S-Nekrasov matrix for any nonempty subset S of $\langle n\rangle$. But for $S=\{1,4,5\}$, we can get $A\in GSN$ and hence it is a nonsingular H-matrix.

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Zhen-Hua Lyu
School of Science
Shenyang Aerospace University
Shenyang, Liaoning, 110136, P.R. China
e-mail: Lyuzhh@hnu.edu.cn

Lixin Zhou
College of Science
Guilin University of Aerospace Technology
Guilin, Guangxi, 541004, P.R. China
e-mail: zhoulixin@guat.edu.cn

Jianzhou Liu
Department of Mathematics and Computational Science
Xiangtan University
Xiangtan, Hunan, 411105, P.R. China
e-mail: liujz@xtu.edu.cn