Journal of Mathematical Inequalities
Volume 15, Number 3 (2021), 1173–1183 doi:10.7153/jmi-2021-15-78

A NOTE ON KANTOROVICH TYPE BERNSTEIN CHLODOVSKY OPERATORS WHICH PRESERVE EXPONENTIAL FUNCTION

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(*Communicated by M. Mursaleen*)

Abstract. This paper is mainly focused on the integral extension of Bernstein-Chlodovsky operators which preserve exponential function. Inspire of the Bernstein-Chlodovsky operators which preserve exponential function, we define the integral extension of these operators by using a different technique. We give weighted approximation properties including a weighted uniform convergence and a weighted quantitative theorem in terms of exponential weighted modulus of continuity. Furthermore, we give the Voronovskaya type theorem.

1. Introduction

In approximation theory, studies on linear positive operators have continued to be important for many years. The positive approximation processes discovered by Korovkin play an important role and arise in a natural way in many problems related to many areas of mathematics like harmonic analysis, measure theory, partial differential equations etc.

In 1932, n-th Bernstein-Chlodovsky operator was defined by Chlodovsky [1] as

$$C_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{b_n}\right)^k \left(1-\frac{x}{b_n}\right)^{n-k} f\left(\frac{k}{n}b_n\right) \quad (n \in \mathbb{N}). \tag{1.1}$$

for $x \in [0,b_n]$ where b_n, $n \in \mathbb{N}$, is a strictly positive sequence increasing to $+\infty$, also $\lim\limits_{n\to\infty} \frac{b_n}{n} = 0$. In this setting, f is therefore defined on the infinite interval $[0,\infty)$ such that the series in (1.1) is convergent.

The convergency of the sequence of Bernstein-Chlodovsky operators for bounded and continuous functions on the infinite interval can be seen in [1].

Many researchers have studied intensively Kantorovich type generalization of Szàsz, Baskakov and Bernstein operators (see e.g. [15, 16, 17, 18, 19]). The Kantorovich version of Bernstein operators defined by replacing the sample values $f\left(\frac{k}{n}\right)$ with the mean values of f in the interval $\left[\frac{k}{n}, \frac{k+1}{n}\right]$, namely

$$K_n(f)(x) = (n+1)\sum_{k=0}^{n} \binom{n}{k} P_{n,k}(x) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt,\ x \in [0,1],\ n \in \mathbb{N}, \tag{1.2}$$

Mathematics subject classification (2020): 41A25, 41A36.

Keywords and phrases: Voronovskaja type theorem, weighted modulus of continuity, rate of convergence.

where $f:[0,1]\to\mathbb{R}$ is a locally integrable function (see [2]). We note that K_n reproduce only 1.These operators allow us to switch a Lebesgue integrable function by means of its mean values on the sets $\left[\frac{k}{n},\frac{k+1}{n}\right]$. A connection between B_n and K_n operator is in the following form

$$K_n = D\circ B_{n+1}\circ I,$$

where D is the differential operator where $D(f)=f'$, $f\in C^1[0,\infty)$ and I is the antiderivative operator $I(f;x)=\int\limits_0^x f(t)dt$, $f\in C[0,\infty)$ and $x\in[0,\infty)$.

Recently many researchers have been studied some operators which preserve exponentials (see [8], [9], [10], [11], [12], [13], [14], ...). Aral et al. studied Bernstein Chlodovsky type operators preserving exponential functions. They gave the rate of convergence and quantitative results for this operator.

To have a better approximation, we define $\tilde{G}_n^{\mu}$ operator and give some approximation results. This paper is organized as follows. In section 2, we construct the $\tilde{G}_n^{\mu}$ operator. In section 3, we evaluate some moments of $\tilde{G}_n^{\mu}$. In section 4, we give some approximation properties of these operators in the polynomial weighted space and in Section 5, we give a quantitative Voronovskaya- type asymptotic formula.

2. Construction of the operator

Inspire of the Bernstein-Chlodowsky operators which preserve exponential function, we define integral extension of these operators by using different technique.

The generalization of the Bernstein-Chlodovsky operators of the form in [12]

$$U_nf(x)=\sum_{k=0}^{n}\alpha_{n,k}(x)f\left(\frac{kb_n}{n}\right)P_{n,k}(a_n(x)),\ x\in[0,b_n]$$

where $a_n(x)=b_n\frac{e^{\frac{\mu x}{n}}-1}{e^{\frac{\mu b_n}{n}}-1}$ and $\alpha_{n,k}(x)=e^{\mu x}e^{-\frac{\mu k}{bn}}$. Inspire of this operator, we construct Kantorovich type operator as similar as given in [6]. The connection of this operator with classical Bernstein-Chlodovsky operator can be written as

$$U_n(f;x)=\exp_{\mu}(x)C_n\left(\frac{f}{\exp_{\mu}};a_n(x)\right).$$

for a fixed real parameter $\mu>0$ and the exponential function $\exp_{\mu}(x)=e^{\mu x}$. For giving the generalization of our operator U_n, we will use similar tecnique in [6] satisfies the followings:

$$D_{\mu}:C^1[0,\infty)\to C[0,\infty)\ \text{and defined by}$$

$$D_{\mu}(f,x)=f'(x)-\mu f(x),\ f'\in C[0,\infty). \tag{2.1}$$

Also

$$I_\mu : C[0,\infty) \to C^1[0,\infty) \text{ and defined by}$$

$$I_\mu(f,x) = e^{\mu x}\int_0^x e^{-\mu t} f(t)dt,\ f \in C[0,\infty). \tag{2.2}$$

We note that $(D_\mu \circ I_\mu)(f) = f$, for $f \in C[0,\infty)$ and $(I_\mu \circ D_\mu)(f) = f$, for $f \in C^1[0,\infty)$ and $f(0) = 0$.

Now we give the Kantorovich generalization of the operator, we define $\tilde{G}_n^\mu$ as

$$\tilde{G}_n^\mu = D_\mu \circ U_{n+1} \circ I_\mu,$$

where D_μ and I_μ is respectively given by (2.1) and (2.2).

DEFINITION 1. Let $\mu > 0$. The operator $\tilde{G}_n^\mu : C[0,b_n) \to C[0,\infty)$ defined by

$$\tilde{G}_n^\mu(f;x) = a_{n+1}'(x)\frac{n+1}{b_{n+1}}e^{\mu x}\sum_{k=0}^n P_{n,k}(a_{n+1}(x))\int_{\frac{kb_{n+1}}{n+1}}^{\frac{(k+1)b_{n+1}}{n+1}} e^{-\mu t} f(t)dt,\ x \in [0,b_n] \tag{2.3}$$

where $a_n(x) = b_n\frac{e^{\frac{\mu x}{n}}-1}{e^{\frac{\mu b_n}{n}}-1}$. Also we consider $F_\mu(x)$ as

$$F_\mu(x) = \int_0^x e^{-\mu t} f(t)dt,\ \text{for } f \in L_1[0,\infty). \tag{2.4}$$

2.1. Auxiliary results

We first present some results which will be used in the proofs of our theorems.

THEOREM 1. *Let $n \in \mathbb{N}$ and $x \in [0,b_n]$. Then*

$$\tilde{G}_n^\mu = D_\mu \circ U_{n+1} \circ I_\mu.$$

Proof. Let $x \in [0,b_n]$ and $f \in C[0,\infty)$.

$$\begin{aligned}
\tilde{G}_n^\mu &= (D_\mu \circ U_{n+1} \circ I_\mu)(f)(x) \\
&= U_{n+1}'(I_\mu(f);x) - \mu U_{n+1}(I_\mu(f);x) \\
&= \left(\exp_\mu(x)C_{n+1}\left(F_\mu;a_{n+1}(x)\right)\right)' - \mu\exp_\mu(x)C_{n+1}\left(F_\mu;a_{n+1}(x)\right) \\
&= \exp_\mu(x)C_{n+1}'(F_\mu;a_{n+1}(x))
\end{aligned}$$

where F_μ is given in (2.4). Now we are starting by derivative of C_{n+1},

$$
\begin{aligned}
&C'_{n+1}(F_\mu;a_{n+1}(x))\\
&=\sum_{k=0}^{n+1} f\left(\frac{kb_{n+1}}{n+1}\right)\binom{n+1}{k}\left[\left(\frac{a_{n+1}(x)}{b_{n+1}}\right)^k\left(1-\frac{a_{n+1}(x)}{b_{n+1}}\right)^{n+1-k}\right]'\\
&=\sum_{k=1}^{n+1} f\left(\frac{kb_{n+1}}{n+1}\right)\binom{n+1}{k}\left[k\left(\frac{a_{n+1}(x)}{b_{n+1}}\right)^{k-1}\frac{a'_{n+1}(x)}{b_{n+1}}\left(1-\frac{a_{n+1}(x)}{b_{n+1}}\right)^{n+1-k}\right]\\
&+\sum_{k=0}^{n} f\left(\frac{kb_{n+1}}{n+1}\right)\binom{n+1}{k}\left[(n+1-k)\left(1-\frac{a_{n+1}(x)}{b_{n+1}}\right)^{n-k}\frac{a'_{n+1}(x)}{b_{n+1}}\left(\frac{a_{n+1}(x)}{b_{n+1}}\right)^k\right]
\end{aligned}
$$

by making necessary arrangements, we have

$$
\begin{aligned}
&\tilde{G}_n^\mu(f)(x)\\
&=\tilde{G}_n^\mu(f;x)\\
&=(D_\mu\circ U_{n+1}\circ I_\mu)(f)(x)\\
&=(n+1)\frac{a'_{n+1}(x)}{b_{n+1}}e^{\mu x}\sum_{k=0}^{n}\left(\frac{a_{n+1}(x)}{b_{n+1}}\right)^k\left(1-\frac{a_{n+1}(x)}{b_{n+1}}\right)^{n-k}\\
&\quad\times\left[F_\mu\left(\frac{(k+1)b_{n+1}}{n+1}\right)-F_\mu\left(\frac{kb_{n+1}}{n+1}\right)\right]\\
&=(n+1)\frac{a'_{n+1}(x)}{b_{n+1}}e^{\mu x}\sum_{k=0}^{n}\left(\frac{a_{n+1}(x)}{b_{n+1}}\right)^k\left(1-\frac{a_{n+1}(x)}{b_{n+1}}\right)^{n-k}\int_{\frac{kb_{n+1}}{n+1}}^{\frac{(k+1)b_{n+1}}{n+1}} e^{-\mu t}f(t)\,dt. \quad\square
\end{aligned}
$$

LEMMA 1. *The operators $\tilde{G}_n^\mu(f;x)$ defined by* (2.3) *satisfy the following equalities for $\mu>0$, $n\in\mathbb{N}$ and $x\in[0,b_n]$:*

$$\tilde{G}_n^\mu(e_0;x)=\frac{e^{\frac{\mu x}{n+1}}e^{\mu x}}{e^{\mu b_{n+1}}}\left(e^{\mu\frac{b_{n+1}}{n+1}}-e^{\frac{\mu x}{n+1}}+1\right)^n, \tag{2.5}$$

$$\tilde{G}_n^\mu(\exp_\mu;x)=\mu\frac{b_{n+1}}{n+1}\frac{e^{\frac{\mu x}{n+1}}}{e^{\mu\frac{b_{n+1}}{n+1}}-1}e^{\mu x}, \tag{2.6}$$

$$\tilde{G}_n^\mu(\exp_\mu^2;x)=e^{2\mu x}. \tag{2.7}$$

$$\tilde{G}_n^\mu(\exp_\mu^3;x)=\frac{1}{2}e^{\mu x}e^{\frac{\mu x}{n+1}}\left(e^{\mu\frac{b_{n+1}}{n+1}}+1\right)\left(e^{\frac{\mu x}{n+1}}+e^{\frac{\mu x}{n+1}}e^{\mu\frac{b_{n+1}}{n+1}}-e^{\mu\frac{b_{n+1}}{n+1}}\right)^n \tag{2.8}$$

and

$$
\begin{aligned}
\tilde{G}_n^\mu(\exp_\mu^4;x)&=\frac{1}{3}e^{\mu x}e^{\frac{\mu x}{n+1}}\left(e^{2\mu\frac{b_{n+1}}{n+1}}+e^{\mu\frac{b_{n+1}}{n+1}}+1\right)\\
&\times\left(e^{\frac{\mu x}{n+1}}e^{2\mu\frac{b_{n+1}}{n+1}}-e^{\frac{\mu x}{n+1}}e^{\mu\frac{b_{n+1}}{n+1}}+e^{\frac{\mu x}{n+1}}-e^{2\mu\frac{b_{n+1}}{n+1}}-e^{\mu\frac{b_{n+1}}{n+1}}\right)^n.
\end{aligned} \tag{2.9}
$$

LEMMA 2. *Let* $\exp_{\mu,x}(t) = e^{\mu t} - e^{\mu x}$. *For the operator* $\tilde{G}_n^{\mu}$ *given by* (2.3), *we have the following limits with mathematical software:*

$$\lim_{n\to\infty}\frac{n}{b_n}(\tilde{G}_n^{\mu}(e_0;x)-1) = \mu(\mu x - 1),$$

$$\lim_{n\to\infty}\frac{n}{b_n}(\tilde{G}_n^{\mu}(\exp_{\mu};x)-e^{\mu x}) = -\frac{1}{2}\mu e^{\mu x}.$$

Using the above limits, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\frac{n}{b_n}(\tilde{G}_n^{\mu}(\exp_{\mu,x};x)\\
&= \lim_{n\to\infty}\frac{n}{b_n}(\tilde{G}_n^{\mu}(\exp_{\mu};x) - e^{\mu x}\tilde{G}_n^{\mu}(e_0;x))\\
&= \lim_{n\to\infty}\frac{n}{b_n}(\tilde{G}_n^{\mu}(\exp_{\mu};x) - e^{\mu x}) - \lim_{n\to\infty}\frac{n}{b_n}e^{\mu x}(\tilde{G}_n^{\mu}(e_0;x)-1)\\
&= e^{\mu x}\mu\left(\frac{1}{2}-\mu\right)
\end{aligned}$$

and

$$\lim_{n\to\infty}\frac{n}{b_n}(\tilde{G}_n^{\mu}(\exp_{\mu,x}^4;x) = \frac{1}{24}e^{4\mu x}(84-120\mu x-12\mu^3 x+72\mu^4x^2-\mu^2(7+24x^2)).$$

LEMMA 3. *Let* $\gamma_{n,\mu} = \tilde{G}_n^{\mu}(\exp_{\mu,x}^2;x)$ *and* $\alpha_n = \tilde{G}_n^{\mu}(e_0;x)-1$. *Then* $\gamma_{n,\mu} \to 0$.

Proof. If we take the derivative of $\tilde{G}_n^{\mu}(e_0)(x)$, we have

$$\begin{aligned}
&\left(\tilde{G}_n^{\mu}e_0\right)'(x)\\
&= \frac{e^{\frac{\mu x}{n+1}}e^{\mu x}}{e^{\mu b_{n+1}}}\frac{\mu}{n+1}\left(e^{\mu\frac{b_{n+1}}{n+1}} - e^{\frac{\mu x}{n+1}}+1\right)^{n-1}\left[(n+2)\left(e^{\mu\frac{b_{n+1}}{n+1}} - e^{\frac{\mu x}{n+1}}+1\right) - ne^{\frac{\mu x}{n+1}}\right].
\end{aligned}$$

If $\left(\tilde{G}_n^{\mu}e_0\right)'(x) = 0$, then we have $x_o = \frac{n+1}{\mu}\ln\left[\frac{n+2}{2(n+1)}\left(e^{\mu\frac{b_{n+1}}{n+1}}+1\right)\right]$. Writing the critical point in the $\tilde{G}_n^{\mu}(e_0;x)$, we obtain

$$\tilde{G}_n^{\mu}(e_0;x_0) = \left[\frac{n+2}{2n+2}\left(e^{\mu\frac{b_{n+1}}{n+1}}+1\right)\right]^{n+2}\frac{1}{2^n}\left(e^{\mu\frac{b_{n+1}}{n+1}}+1\right)^n\left(\frac{n}{n+1}\right)^n e^{-\mu b_{n+1}}$$

and

$$\begin{aligned}
\tilde{G}_n^{\mu}(e_0;x_0)-1 &= \left[\frac{n+2}{2n+2}\left(e^{\mu\frac{b_{n+1}}{n+1}}+1\right)\right]^{n+2}\frac{1}{2^n}\left(e^{\mu\frac{b_{n+1}}{n+1}}+1\right)^n\left(\frac{n}{n+1}\right)^n e^{-\mu b_{n+1}}-1\\
&= \left(\frac{e^{\mu\frac{b_{n+1}}{n+1}}+1}{2}\right)^{2n+2}\left(\frac{n+2}{n+1}\right)^{n+2}\left(\frac{n}{n+1}\right)^n e^{-\mu b_{n+1}}-1.
\end{aligned}$$

$$
\begin{aligned}
\tilde{G}_n^{\mu}(\exp_{\mu,x}^2;x) &= 2e^{\mu x}(e^{\mu x} - \tilde{G}_n^{\mu}(\exp_{\mu};x)) + e^{2\mu x}(\tilde{G}_n^{\mu}(e_0;x)-1) \\
&= 2e^{2\mu x}\left(1-\mu\frac{b_{n+1}}{n+1}\frac{e^{\frac{\mu x}{n+1}}}{e^{\mu\frac{b_{n+1}}{n+1}}-1}\right) + e^{2\mu x}(\tilde{G}_n^{\mu}(e_0;x)-1).
\end{aligned}
$$

Using mathematical software, we have $\lim\limits_{n\to\infty} \gamma_{n,\mu} = 0.$ $\square$

3. Approximation in a weighted space

We know that if we take a function on $[0,\infty)$, then the uniform norm is not valid to evaluate the rate of convergence for unbounded functions. We can not find a rate of convergence in terms of usual modulus of continuity $\omega(f;\ \delta)$ of a function f. Because on the infinite interval, the modulus of continuity $\omega(f;\ \delta)$ does not tend to zero as δ tends to zero. For this reason, we use a weighted modulus of continuity for unbounded functions. In [3] and [4], weighted Korovkin type theorems have been proved by Gadjiev et al.

We give approximation properties of the operators L_n of the weighted spaces of continuous functions with exponential growth on $\mathbb{R}^+ = [0,\infty)$ with the help of the weighted Korovkin type theorem proved by Gadjiev in [3,4]. Therefore we consider the following weighted spaces of functions which are defined on the $\mathbb{R}^+$.

Let $\rho(x) = 1 + e^{2\mu x}$ weight function and M_f be a positive constant depending of f, we define

$$B_\rho\left(\mathbb{R}^+\right) = \left\{f : \mathbb{R}^+ \to \mathbb{R} : |f(x)| \leqslant M_f\ \rho(x)\right\}$$

and

$$C_\rho\left(\mathbb{R}^+\right) = C\left(\mathbb{R}^+\right) \cap B_\rho\left(\mathbb{R}^+\right).$$

We also consider the space of functions

$$C_\rho^k\left(\mathbb{R}^+\right) = \left\{f \in C_\rho\left(\mathbb{R}^+\right) : \lim_{x\to\infty}\frac{f(x)}{\rho(x)} = K_f < \infty\right\}.$$

It is obvious that $C_\rho^k\left(\mathbb{R}^+\right) \subset C_\rho\left(\mathbb{R}^+\right) \subset B_\rho\left(\mathbb{R}^+\right)$. These spaces are normed spaces with the norm

$$\|f\|_\rho = \sup_{x\in\mathbb{R}^+}\frac{|f(x)|}{\rho(x)}.$$

If $f \in C_\rho^k\left(\mathbb{R}^+\right)$, then $\|L_n(f)\|_\rho \leqslant \|f\|_\rho$. The following results on the sequence of positive linear operators in these spaces and Korovkin type theorems are given in [3,4].

By using the above expressions, we can give the following theorem.

THEOREM 2. *If $f \in C_\rho^k\left(\mathbb{R}^+\right)$, then* $\lim\limits_{n\to\infty}\left\|\tilde{G}_n^{\mu}(f) - f\right\|_\rho = 0.$

Proof. By using the well known result in [9], we have

$$\lim_{n\to\infty}\left\|\tilde{G}_n^{\mu}(\exp_{\mu}) - e^{t\mu}\right\|_\rho = 0,\ t = 0,1,2. \tag{3.1}$$

Now we consider (2.5).

$$\left\|\tilde{G}_n^{\mu}(e_0)-1\right\|_{\rho}=\sup_{x\in\mathbb{R}^+}\frac{\left|\frac{e^{\frac{\mu x}{n+1}}e^{\mu x}}{e^{\mu b_{n+1}}}\left(e^{\mu\frac{b_{n+1}}{n+1}}-e^{\frac{\mu x}{n+1}}+1\right)^n-1\right|}{1+e^{2\mu x}}=0.$$

Now we are passing to limit condition,we have $\lim_{n\to\infty}\left\|\tilde{G}_n^{\mu}(e_0)-1\right\|_{\rho}=0$ from lemma 3. Similarly by (2.6), we get

$$\begin{aligned}\left\|\tilde{G}_n^{\mu}(\exp_{\mu},x)-e^{\mu x}\right\|_{\rho}&=\sup_{x\in\mathbb{R}^+}\frac{\left|\mu\frac{b_{n+1}}{n+1}\frac{e^{\frac{\mu x}{n+1}}}{e^{\mu\frac{b_{n+1}}{n+1}}-1}e^{\mu x}-e^{\mu x}\right|}{1+e^{2\mu x}}\\&\leqslant\frac{e^{\mu x}}{1+e^{2\mu x}}\left[\frac{e^{\frac{\mu x}{n+1}}\mu\frac{b_{n+1}}{n+1}}{e^{\mu\frac{b_{n+1}}{n+1}}-1}-1\right]\\&\leqslant\left[\frac{e^{\frac{\mu x}{n+1}}\mu\frac{b_{n+1}}{n+1}}{e^{\mu\frac{b_{n+1}}{n+1}}-1}-1\right]\end{aligned}$$

which leads to

$$\lim_{n\to\infty}\left\|\tilde{G}_n^{\mu}(\exp_{\mu},x)-e^{\mu x}\right\|_{\rho}=0$$

and from (2.7), we have

$$\lim_{n\to\infty}\left\|\tilde{G}_n^{\mu}(\exp_{\mu}^2;x)-e^{2\mu x}\right\|_{\rho}=0.\quad\square$$

4. Rate of convergence

In this part, we give the rate of convergence of the $\tilde{G}_n^{\mu}$ to the identity operator by using weighted modulus of continuity.

Now we consider exponential weighted space $C_{\mu}(\mathbb{R}^+)$ with a fixed $\mu>0$, which is the set of all real valued functions continuous on $\mathbb{R}^+$ satisfying $|f(x)|\leqslant Me^{\mu x}$ where M is a positive constant. $C_{\mu}(\mathbb{R}^+)$ is a normed space with the norm $\|f\|_{\mu}=\sup_{x\in\mathbb{R}^+}\frac{|f(x)|}{e^{\mu x}}$.

Let $C_{\mu}^{k}(\mathbb{R}^+)$ be the subspace of all the functions $f\in C_{\mu}(\mathbb{R}^+)$ such that $\lim_{x\to\infty}\frac{|f(x)|}{e^{\mu x}}=k$, where k is a positive constant. We are using a weighted modulus of continuity for $f\in C_{\mu}^{k}(\mathbb{R}^+)$,

$$\tilde{w}(f;\delta)=\sup_{|t-x|\leqslant\delta,x\in R^+}\frac{|f(t)-f(x)|}{e^{\mu t}+e^{\mu x}}.$$

This function has the following properties (see in [8]):

i. For $f\in C_{\mu}^{k}(\mathbb{R}^+)$,

$$\lim_{\delta\to 0}\tilde{w}(f;\delta)=0.$$

ii. For $f \in C_\mu^k(\mathbb{R}^+)$ and for any integer λ, we have

$$\tilde{w}(f;\lambda\delta) \leqslant 2\lambda\tilde{w}(f;\delta).$$

THEOREM 3. *Let $f \in C_\mu^k(\mathbb{R}^+)$. We have $\left\|\tilde{G}_n^\mu(f)-f\right\|_{\frac{3\mu}{2}} \leqslant \|f\|\,\alpha_n + M\tilde{w}(f;u_n)$, where M is a constant.*

Proof. From property (ii), we can write $\tilde{w}(f;\lambda\delta) \leqslant 2(1+\lambda)\tilde{w}(f;\delta)$ for positive λ. By the definition of $\tilde{w}(f;\delta)$ for $f \in C_\mu^k(\mathbb{R}^+)$ and $x,\ t \in \mathbb{R}^+$ and $\delta > 0$, we have the following inequality:

$$\begin{aligned}|f(t)-f(x)| &\leqslant \left(e^{\mu t}+e^{\mu x}\right)\tilde{w}(f;|t-x|)\\ &\leqslant 2\left(e^{\mu t}+e^{\mu x}\right)\left(1+\frac{|t-x|}{\delta}\right)\tilde{w}(f;\delta).\end{aligned}$$

If we apply the Mean Value Theorem to the function $e^{\mu t}$ on $[x,t]$, then we have $\mu\,|t-x| \leqslant |e^{\mu t}-e^{\mu x}|$. Now we edit the above inequality by

$$|f(t)-f(x)| \leqslant 2\left(e^{\mu t}+e^{\mu x}\right)\left(1+\left|e^{\mu t}-e^{\mu x}\right|\right)\tilde{w}(f;\delta). \tag{4.1}$$

and by using Lemma 1 and the following inequality, we have

$$\left|\tilde{G}_n^\mu(f;x)-f(x)\right| \leqslant f(x)\left|1-\tilde{G}_n^\mu(1;x)\right| + \tilde{G}_n^\mu(|f(t)-f(x)|;x).$$

Now applying (4.1) to $\tilde{G}_n^\mu$;

$$\begin{aligned}&\tilde{G}_n^\mu(|f(t)-f(x)|;x)\\ &\leqslant 2\tilde{G}_n^\mu\left(\left(e^{\mu t}+e^{\mu x}\right)\left(1+\frac{|t-x|}{\delta}\right)\tilde{w}(f;\delta);x\right)\\ &\leqslant 2\tilde{w}(f;\delta)[\tilde{G}_n^\mu(\exp_\mu;x)+e^{\mu x}\tilde{G}_n^\mu(1;x)+\frac{1}{\delta}\tilde{G}_n^\mu\left(\left(e^{\mu t}+e^{\mu x}\right)|t-x|;x\right),\end{aligned}$$

by using Hölder's inequality, we have

$$\begin{aligned}&\tilde{G}_n^\mu(|f(t)-f(x)|;x)\\ &\leqslant 2\tilde{w}(f;\delta)[\tilde{G}_n^\mu(\exp_\mu;x)+e^{\mu x}\tilde{G}_n^\mu(1;x)+\left(\tilde{G}_n^\mu((t-x)^2;x)\right)^{\frac{1}{2}}\left(e^{\mu x}+e^{\mu x}(\tilde{G}_n^\mu(1;x))^{\frac{1}{2}}\right)\\ &\leqslant 2\tilde{w}(f;\delta)[\tilde{G}_n^\mu(\exp_\mu;x)+e^{\mu x}\tilde{G}_n^\mu(1;x)\\ &\quad+\frac{1}{\delta}\frac{1}{\mu}\left(\tilde{G}_n^\mu(\left(e^{\mu t}-e^{\mu x}\right)^2;x)\right)^{\frac{1}{2}}\left(e^{\mu x}+e^{\mu x}(\tilde{G}_n^\mu(1;x)^{\frac{1}{2}}\right),\end{aligned}$$

it follows that

$$\begin{aligned}&\left|\tilde{G}_n^\mu(f;x)-f(x)\right|\\ &\leqslant f(x)\left|1-\tilde{G}_n^\mu(1;x)\right|+\tilde{w}(f;\delta)[\tilde{G}_n^\mu(\exp_\mu;x)+e^{\mu x}\tilde{G}_n^\mu(1;x)\\ &\quad+\frac{1}{\delta}\frac{1}{\mu}\left(\tilde{G}_n^\mu(\left(e^{\mu t}-e^{\mu x}\right)^2;x)\right)^{\frac{1}{2}}\left(e^{\mu x}+e^{\mu x}(\tilde{G}_n^\mu(1;x)^{\frac{1}{2}}\right).\end{aligned}$$

We denote that $\xi_n := \sup_{x\in\mathbb{R}^+} \left|1-\tilde{G}_n^{\mu}(1;x)\right|$ and $\beta_n := \mu \frac{b_{n+1}}{n+1} \frac{e^{\frac{\mu b_{n+1}}{n+1}}}{e^{\mu\frac{b_{n+1}}{n+1}}-1}$ (from lemma 3,we get $x=b_{n+1}$) and passing to norm we obtain

$$\begin{aligned}
&\left\|\tilde{G}_n^{\mu}(f;x)-f(x)\right\|_{\frac{3}{2}\mu} \\
&\leqslant \|f\|_{\mu}\,\xi_n+\tilde{w}(f;\delta)\left[\mu\frac{b_{n+1}}{n+1}\frac{e^{\frac{\mu b_{n+1}}{n+1}}}{e^{\mu\frac{b_{n+1}}{n+1}}-1}+\frac{1}{\delta}(\xi_n+1)(2+\xi_n)\right],
\end{aligned}$$

since $\sup\left(\frac{\tilde{G}_n^{\mu}(\exp_{\mu,t}^2;x)}{e^{2\mu x}}\right)<\infty$, choosing $\delta := u_n^2 = 2(1-\beta_n)+\xi_n$, then we have desired result. □

5. Voronovskaya type theorem

In this part, we give the quantitative version of Voronovskaya type theorem to show the rate of pointwise convergence for the operator $\tilde{G}_n^{\mu}$ given by (2.3).

THEOREM 4. *Let $f\in C_{\rho}^{k}(R^+)$. Then we have*

$$\lim_{n\to\infty}\frac{n}{b_n}2\mu\left(\tilde{G}_n^{\mu}f(x)-f(x)\right)=(\mu x-1)\left(2\mu^2 f(x)+f'(x)+f''(x)\right).$$

Proof. We consider the Taylor formula for $f\in C_{\rho}^{k}(R^+)$,

$$\begin{aligned}
f(t) &= (f\circ\log_{\mu})(e^{\mu x})+(f\circ\log_{\mu})'(e^{\mu x})\exp_{\mu,x}(t) \\
&\quad+\frac{1}{2}(f\circ\log_{\mu})''(e^{\mu x})\exp_{\mu,x}^2(t)+h_x(t)\exp_{\mu,x}^2(x),
\end{aligned}\tag{5.1}$$

where $\log_{\mu}$ is the inverse function of e^{μ} and $h_x(t)$ is the remainder term such that $h_x(t)=\frac{f(t)-(f\circ\log_{\mu})(e^{\mu x})-f\circ\log_{\mu})'(e^{\mu x})\exp_{\mu,x}(x)}{\exp_{\mu,x}^2(x)}-\frac{(f\circ\log_{\mu})''(e^{\mu\xi x})}{2!}$. Also we have the following derivatives

$$(f\circ\log_{\mu})'(e^{\mu x})=\frac{f'(x)}{\mu e^{\mu x}}$$

and

$$(f\circ\log_{\mu})''(e^{\mu x})=\frac{f''(x)}{\mu^2 e^{2\mu x}}-\frac{f'(x)\mu^2 e^{\mu x}}{\mu^3 e^{3\mu x}}=e^{-2\mu x}(\mu^2 f''(x)-\mu^{-1}f'(x)).$$

Then we apply the Taylor expansion to the operator, we can write

$$
\begin{aligned}
&\tilde{G}_n^{\mu}(f;x)\\
&= f(x)\,\tilde{G}_n^{\mu}(e_0;x) + (f\circ\log_\mu)'(e^{\mu x})\tilde{G}_n^{\mu}(\exp_{\mu,x};x)\\
&\quad + \frac{1}{2}(f\circ\log_\mu)''(e^{\mu x})\tilde{G}_n^{\mu}(\exp_{\mu,x}^2;x) + \tilde{G}_n^{\mu}(h_x\exp_{\mu,x}^2;x)\\
\\
&= f(x)\,\tilde{G}_n^{\mu}(e_0;x) + e^{-\mu x}\mu^{-1}f'(x)e^{\mu x}[1-\tilde{G}_n^{\mu}(e_0;x)]\\
&\quad + \frac{e^{-2\mu x}\left(\mu^{-2}f''(x)-\mu^{-1}f'(x)\right)}{2}e^{2\mu x}[\tilde{G}_n^{\mu}(e_0;x)-1] + \tilde{G}_n^{\mu}(\exp_{\mu,x}^2 h_x;x).
\end{aligned}
$$

$$
\tilde{G}_n^{\mu}(f;x) - f(x) = \left(\tilde{G}_n^{\mu}(e_0;x)-1\right)\left(f(x)+\frac{f'(x)}{2\mu}+\frac{f''(x)}{2\mu^2}\right) + \tilde{G}_n^{\mu}(\exp_{\mu,x}^2 h_x;x)
$$

We can write with Cauchy-Schwarz inequality

$$
\frac{n}{b_n}\left|\tilde{G}_n^{\mu}\left(\exp_{\mu,x}^2 h_x;x\right)\right| \leqslant \left(\tilde{G}_n^{\mu}(h_x^2;x)\right)^{\frac{1}{2}}\left(\frac{n^2}{b_n^2}\tilde{G}_n^{\mu}(\exp_{\mu,x}^4;x)\right)^{\frac{1}{2}}.
$$

In this way, we take a limit when n tends to infinity,

$$
\lim_{n\to\infty}\tilde{G}_n^{\mu}(h_x^2;x) = 0
$$

and with mathematical software, we have

$$
\lim_{n\to\infty}\frac{n^2}{b_n^2}\tilde{G}_n^{\mu}(\exp_{\mu,x}^4;x) = \frac{1}{24}e^{4\mu x}(84-120\mu x-12\mu^3 x+72\mu^4 x^2-\mu^2(7+24x^2)).
$$

So we arrive at the following limit,

$$
\lim_{n\to\infty}\frac{n}{b_n}2\mu\left(\tilde{G}_n^{\mu}f(x)-f(x)\right) = (\mu x-1)\left(2\mu^2 f(x)+f'(x)+f''(x)\right),
$$

we have the desired result. □

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(Received December 11, 2020)

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