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LIMIT BEHAVIORS FOR ANA RANDOM VARIABLES UNDER $R-h$–INTEGRABILITY AND $SR-h$–INTEGRABILITY

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Abstract. In this work, the L_r convergence for maximum weighted sums of ANA R-h-integrable random variables as well as the complete moment convergence for maximum weighted sums of ANA SR-h-integrable random variables with respect to the array $\{a_{ni}\}$ of constants are established, which improve and generalize the results of Sung et al. [30], Wu et al. [33], and Wang et al. [34].

1. Introduction

The independence assumption is usually assumed in plenty of statistical models, even if it has turned out to be unrealistic. Fortunately, more and more statisticians are inclined to use the dependence assumption. In the past decades, many dependence structures were introduced by scholars. Among them the negative association structure received considerable attention recently on account of its extensive applications in systems reliability and multivariate statistical analysis. Alam and Saxena [1] firstly proposed the following concept of negatively associated (NA) random variables:

DEFINITION 1.1. Random variables $\{X_i, 1 \leqslant i \leqslant n\}$ are called to be NA if for each pair of subsets $\mathbb{A}$ and $\mathbb{B}$ of $\{1,2,\cdots,n\}$ satisfying $\mathbb{A}\bigcap\mathbb{B}=\emptyset$,

$$\mathrm{Cov}(f_1(X_i, i\in\mathbb{A}), f_2(X_j, j\in\mathbb{B})) \leqslant 0,$$

where f_1 and f_2 are both nondecreasing functions defined on $\mathbf{R}^{\mathbb{A}}$ and $\mathbf{R}^{\mathbb{B}}$ respectively such that the covariance above exists. A random sequence $\{X_n, n \geqslant 1\}$ is called to be NA if for each fixed n, $\{X_i, 1 \leqslant i \leqslant n\}$ are NA.

After this dependence structure was presented, plenty of studies appeared subsequently. For example, Joag-Dev and Proschan [2] further studied its essential properties; Shao [3] established some moment equalities for it; Roussas [4] got the central limit theorem in random fields; Cai and Roussas [5] investigated the convergence rate

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of asymptotic normality for a nonparametric estimator; Liang [6] studied the complete convergence; Wu et al. [7] obtained the result on complete moment convergence, etc.

Bradley [8] introduced the following concept of ρ^*-mixing ($\tilde{\rho}$-mixing, or weakly mixing) random variables.

DEFINITION 1.2. A random sequence $\{X_i, i \geqslant 1\}$ is said to be ρ^*-mixing if the mixing coefficients

$$\rho^*(t) = \sup\{\rho(\mathbb{U},\mathbb{V}); \mathbb{U},\mathbb{V} \subset \mathbb{N}, dist(\mathbb{U},\mathbb{V}) \geqslant t\} \to 0$$

as $t \to \infty$, where

$$\rho(\mathbb{U},\mathbb{V}) = \sup\left\{ \frac{|\mathrm{Cov}(\xi,\eta)|}{\sqrt{\mathrm{Var}(\xi)\mathrm{Var}(\eta)}} : \xi \in L_2(\sigma(X_i, i \in \mathbb{U})), \eta \in L_2(\sigma(X_j, j \in \mathbb{V}))\right\}.$$

It is known that the moving average processes and some Markov chains with regular conditions satisfy the ρ^*-mixing structure. Hence, there are also many researches on the ρ^*-mixing random sequences. For more works on ρ^*-mixing random variables, one can see in Utev and Peligrad [9], Wu and Jiang [10], Sung [11], Wu et al. [12], Shen et al. [13], Wu et al. [14], Chen and Sung [15] and so on.

Zhang and Wang [16] put forwarded the concept of asymptotically negatively associated (ANA), or ρ^--mixing random variables as follows.

DEFINITION 1.3. A random sequence $\{X_n, n \geqslant 1\}$ is said to be ANA if the mixing coefficients

$$\rho^-(t) = \sup\{\rho^-(\mathbb{U},\mathbb{V}) : \mathbb{U},\mathbb{V} \subset \mathbb{N}, dist(\mathbb{U},\mathbb{V}) \geqslant t\} \to 0$$

as $t \to \infty$, where

$$\rho^-(S,T) = 0 \vee \left\{ \frac{\mathrm{Cov}[f(X_i, i \in \mathbb{U}), g(X_j, j \in \mathbb{V})]}{\sqrt{\mathrm{Var}[f(X_i, i \in \mathbb{U})]\mathrm{Var}[g(X_j, j \in \mathbb{V})]}} : f, g \in \mathscr{C} \right\}$$

where $\mathscr{C}$ stands for the set of nondecreasing functions.

An array $\{X_{ni}, i \geqslant 1, n \geqslant 1\}$ is said to be rowwise ANA if for every $n \geqslant 1$, $\{X_{ni}, i \geqslant 1\}$ is a sequence of ANA random variables.

It is not difficult for us to verify that ANA degenerates to NA if and only if $\rho^-(1) = 0$, and the mixing coefficients $\rho^-(s) \leqslant \rho^*(s)$. Hence, ρ^*-mixing structure and NA structure are both special cases of ANA structure. It exemplified in Zhang and Wang [16] that the inverse is not true. Therefore, it is of relative interest to investigate the large sample properties under ANA assumption. For more works on ANA random variables, we refer to Zhang [17–18] for the central limit theorems, Wang and Lu [19] for the Rosenthal-type inequalities of the maximum partial sums, Wang and Zhang [20] for the law of the iterated logarithm, Huang et al. [21] for some results on strong convergence, and so on.

This work will mainly investigate the L_r convergence for maximum weighted sums of ANA residually-h-integrable (R-h-integrable) random variables and complete moment convergence for maximum weighted sums of ANA strongly residually-h-integrable (SR-h-integrable) random variables. Therefore, some relevant definitions should be recalled in what follows.

We first recall the concept of complete convergence, which was raised by Hsu and Robbins [22] as follows:

DEFINITION 1.4. Random sequence $\{X_n, n \geqslant 1\}$ is known as to converge completely to some constant c if for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - c| > \varepsilon) < \infty.$$

The statement above together with the the Borel-Cantelli lemma concludes $X_n \to c$ a.s.

The earliest result concerning the complete moment convergence originates to Chow [23] as follows:

DEFINITION 1.5. Let $q > 0$, $\{a_n, n \geqslant 1\}$ and $\{b_n, n \geqslant 1\}$ be two sequences of positive numbers. If for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n E\{b_n^{-1}|X_n| - \varepsilon\}_+^q < \infty,$$

then the random sequence $\{X_n, n \geqslant 1\}$ is said to be complete moment convergent.

The complete convergence can be easily obtained by the complete moment convergence, which can also be seen in the proof of Theorem 3.3. For more details about the two convergence properties mentioned above, we refer the reader to Liang et al. [24], Guo and Zhu [25], Wu et al. [26], Wang et al. [27], Shen et al. [28], and so on.

As a weaker concept than uniform integrability, the Cesàro uniform integrability was first proposed by Chandra [29] as follows: *Random sequence $\{X_n, n \geqslant 1\}$ is called to be Cesàro uniformly integrable if*

$$\lim_{c\to\infty} \sup_{n\geqslant 1} \frac{1}{m_n} \sum_{i=1}^{m_n} E|X_i| I(|X_i| > c) = 0,$$

where $\{m_n\}$ are positive integers diverging to infinity as $n \to \infty$.

Later on, many concepts on the integrability, the conditions of which are weaker and weaker, were introduced subsequently. For example, Sung et al. [30] introduced the concept of h-integrability, which is much weaker than the integrability above.

Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of random variables and r be some positive number. Suppose that $\{h_n, n \geqslant 1\}$ is a sequence of increasing positive numbers such that $h_n \to \infty$ and $\{k_n, n \geqslant 1\}$ is a sequence of positive numbers such that $k_n \to \infty$ as $n \to \infty$. Then $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is said to be h-integrable with exponent r if

$$\sup_{n\geqslant 1} \frac{1}{k_n} \sum_{i=u_n}^{v_n} E|X_{ni}|^r < \infty,$$

and

$$\lim_{n\to\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|^rI(|X_{ni}|^r>h_n)=0.$$

Cabrera and Volodin [31] put forwarded the notion of h-integrability with respect to an array $\{a_{ni}\}$ of weights as follows.

Let $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *be an array of random variables. Suppose that* $\{a_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *is an array of real numbers. Assume that the positive sequence* $\{h_n,n\geqslant 1\}$ *is increasing and* $h_n\to\infty$ *as* $n\to\infty$*. Then* $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *is said to be* h*-integrable with respect to* $\{a_{ni}\}$ *if*

$$\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n}|a_{ni}|E|X_{ni}|<\infty,$$

and

$$\lim_{n\to\infty}\sum_{i=u_n}^{v_n}|a_{ni}|E|X_{ni}|I(|X_{ni}|>h_n)=0.$$

Sung et al. [30] obtained the result on L_r convergence for NA h-integrable random sequence as follows.

THEOREM 1.1. *Let* $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *be an array of NA* h*-integrable random variables with exponent* r *satisfying* $1\leqslant r<2$*. Suppose that the positive sequence* $\{h_n,n\geqslant 1\}$ *is increasing and* $h_n\to\infty$*,* $k_n\to\infty$*, and* $h_n/k_n\to 0$ *as* $n\to\infty$*. Then* $\frac{1}{k_n^{1/r}}\sum_{i=u_n}^{v_n}(X_{ni}-EX_{ni})\to 0$ *in* L_r *and therefore in probability as* $n\to\infty$*.*

Sung et al. [30] also established the following result dealing with the weighted sums.

THEOREM 1.2. *Let* $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *be an array of NA random variables and* $\{a_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *be an array of real numbers. If*

$$\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n}|a_{ni}|^rE|X_{ni}|^r<\infty,\quad \lim_{n\to\infty}\sum_{i=u_n}^{v_n}|a_{ni}|^rE|X_{ni}|^rI(|X_{ni}|^r>h_n)=0$$

for some $1\leqslant r<2$*, and*

$$\lim_{n\to\infty}h_n\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|=0,$$

then $\sum_{i=u_n}^{v_n}a_{ni}(X_{ni}-EX_{ni})\to 0$ *in* L_r *and therefore in probability as* $n\to\infty$*.*

Wang and Hu [32] raised the following concept of R-h-integrability, which is much weaker than h-integrability.

Let $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *be an array of random variables. Let positive sequence* $\{h_n,n\geqslant 1\}$ *be increasing and* $h_n\to\infty$*,* $\{k_n,n\geqslant 1\}$ *be another positive sequence satisfying* $k_n\to\infty$ *as* $n\to\infty$*. The array* $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *is called to be* R*-*h*-integrable with exponent* $r>0$ *if*

$$\sup_{n\geqslant 1}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|^r<\infty,$$

and

$$\lim_{n\to\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E(|X_{ni}|-h_n^{1/r})^r I(|X_{ni}|^r>h_n)=0.$$

Wang and Hu [32] improved Theorem 1.1 from NA h-integrable random variables to NOD R-h-integrable random variables; Wu et al. [33] extended Theorem 1.1 from NA structure to END structure. Moreover, Wu et al. [33] also obtained the result on complete convergence as follows.

THEOREM 1.3. *Let $1\leqslant r<2$. Suppose that $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ is an array of END random variables such that*

$$\sup_{n\geqslant 1}\frac{1}{k_n}\sum_{k=u_n}^{v_n}E|X_{ni}|^r<\infty,$$

and

$$\sum_{n=1}^{\infty}\frac{1}{k_n}\sum_{i=u_n}^{v_n}E|X_{ni}|^r I(|X_{ni}|^r>h_n)<\infty.$$

Assume that the positive sequences $\{k_n,n\geqslant 1\}$ and $\{h_n,n\geqslant 1\}$ satisfy $k_n\to\infty$, $h_n\to\infty$ as $n\to\infty$ and $\sum_{n=1}^{\infty}(h_n/k_n)^{\lambda(2-r)/r}<\infty$ for some $\lambda>r$. Then for every $\varepsilon>0$,

$$\sum_{n=1}^{\infty}P\left(\left|\sum_{i=u_n}^{v_n}(X_{ni}-EX_{ni})\right|>\varepsilon k_n^{1/r}\right)<\infty,$$

and

$$\sum_{n=1}^{\infty}E\left(\left|\sum_{i=u_n}^{v_n}\frac{1}{k_n^{1/r}}(X_{ni}-EX_{ni})\right|-\varepsilon\right)_+^r<\infty.$$

Recently, Wang et al. [34] proposed the concepts of R-h-integrability and SR-h-integrability with respect to the array $\{a_{ni}\}$ of constants with exponent $r>0$.

DEFINITION 1.6. Let $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ be an array of random variables and r be a positive constant. Assume that $\{a_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ is an array of real numbers, positive sequence $\{h_n,n\geqslant 1\}$ is increasing and $h_n\to\infty$ as $n\to\infty$. Then $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ is called to be R-h-integrable with respect to the array $\{a_{ni}\}$ of constants with exponent r if

$$\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n}|a_{ni}|^r E|X_{ni}|^r<\infty, \tag{1.1}$$

and

$$\lim_{n\to\infty}\sum_{i=u_n}^{v_n}|a_{ni}|^r E(|X_{ni}|-h_n^{1/r})^r I(|X_{ni}|^r>h_n)=0. \tag{1.2}$$

DEFINITION 1.7. Let $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ be an array of random variables and r be a positive constant. Assume that $\{a_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ is an array of real numbers, positive sequence $\{h_n, n \geqslant 1\}$ is increasing and $h_n \to \infty$ as $n \to \infty$. Then $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is called to be SR-h-integrable with respect to the array $\{a_{ni}\}$ of constants with exponent r if (1.1) holds and

$$\sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n} |a_{ni}|^r E(|X_{ni}| - h_n^{1/r})^r I(|X_{ni}|^r > h_n) < \infty. \tag{1.3}$$

Wang et al. [34] improved Theorems 1.1 and 1.2 from NA h-integrable random variables to m-NOD R-h-integrable random variables. In addition, Wang et al. [34] obtained the result on strong convergence for m-NOD SR-h-integrable random variables with respect to the array $\{a_{ni}\}$ of constants.

THEOREM 1.4. *Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of rowwise m-NOD SR-h-integrable random variables with respect to $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ with exponent $1 \leqslant r < 2$. If $\sum_{n=1}^{\infty}\left(h_n \sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|^r\right)^{(2-r)/r} < \infty$. Then $\sum_{i=u_n}^{v_n} a_{ni}(X_{ni} - EX_{ni}) \to 0$ a.s. as $n \to \infty$.*

In this work, we study the L_r convergence for maximum weighted sums of ANA R-h-integrable random variables and complete moment convergence for maximum weighted sums of ANA SR-h-integrable random variables with respect to the array $\{a_{ni}\}$ of constants with exponent $1 \leqslant r < 2$. As the best of our knowledge, there was no result concerning the L_r convergence and complete moment convergence for maximum weighted sums under the condition of R-h-integrability. The results established in this work improve and extend the results mentioned above. In what follows, C always means a generic positive constant which may differ in different lines. $x_+ = \max\{x, 0\}$ and $x_- = \max\{-x, 0\}$. $I(\cdot)$ means the indicator function.

2. Some lemmas

We are now at a position to recall some essential conclusions as follows.

LEMMA 2.1. (Zhang and Wang [16]) *Increasing functions defined on disjoint subsets of a ρ^--mixing field $\{X_i, i \in \mathbb{N}^d\}$ with mixing coefficients $\rho^-(s)$ are also ρ^--mixing with mixing coefficients not greater than $\rho^-(s)$.*

LEMMA 2.2. (Wang and Lu [19]) *Assume that random sequence $\{X_i, i \geqslant 1\}$ is ANA with $EX_i = 0$ and $E|X_i|^p < \infty$ for some $p \geqslant 2$. Then for each $n \geqslant 1$,*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|\right)^p \leqslant C_{p,\rho^-(\cdot)}\left\{\sum_{i=1}^{n} E|X_i|^p + \left(\sum_{i=1}^{n} EX_i^2\right)^{p/2}\right\},$$

where $C_{p,\rho^-(\cdot)}$ is a positive number depending only on p and $\rho^-(\cdot)$.

LEMMA 2.3. *Assume that random sequence $\{X_i, i \geqslant 1\}$ is ANA with $EX_i = 0$ and $E|X_i|^p < \infty$ for some $1 \leqslant p < 2$. Then for each $n \geqslant 1$,*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|\right)^p \leqslant C_{p,\rho^-(\cdot)} \sum_{i=1}^{n} E|X_i|^p,$$

where $C_{p,\rho^-(\cdot)}$ is a positive number depending only on p and $\rho^-(\cdot)$.

Proof. By the same method of Chen et al. [35] for pairwise independent random variables, we can obtain the same conclusion for ANA random variables. The details are omitted. □

LEMMA 2.4. (Wu et al. [14]) *Let $q > r > 0$. Assume that $\{X_n, n \geqslant 1\}$ and $\{Y_n, n \geqslant 1\}$ are two random sequences, then for any $\varepsilon > 0$ and $a > 0$,*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j}(X_i+Y_i)\right| - \varepsilon a\right)_+^r \leqslant C_r\left(\varepsilon^{-q} + \frac{r}{q-r}\right) a^{r-q} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} X_i\right|^q\right) + C_r E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{i=1}^{j} Y_i\right|^r\right),$$

where $C_r = 1$ if $0 < r \leqslant 1$ or $C_r = 2^{r-1}$ if $r > 1$.

3. Main results

We are ready to present our main results. The first one is the L_r convergence and weak law of large numbers for maximum weighted sums.

THEOREM 3.1. *Suppose that $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is an array of rowwise ANA R-h-integrable random variables with respect to the array $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ of constants with exponent $1 \leqslant r < 2$. If $\lim_{n\to\infty} h_n \sup_{u_n\leqslant i\leqslant v_n} |a_{ni}|^r = 0$, then*

$$\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni} - EX_{ni})\right| \to 0$$

in L_r and therefore in probability as $n \to \infty$.

Via choosing $a_{ni} = \frac{1}{k_n^{1/r}}$ for each $u_n \leqslant i \leqslant v_n$ and $n \geqslant 1$, we can get the following result.

COROLLARY 3.1. *Asssume that $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ is an array of rowwise ANA R-h-integrable random variables with exponent $1 \leqslant r < 2$. Suppose that $k_n \to \infty$, $h_n \to \infty$, and $h_n/k_n \to 0$ as $n \to \infty$. Then*

$$\frac{1}{k_n^{1/r}} \max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}(X_{ni} - EX_{ni})\right| \to 0$$

in L_r and therefore in probability as $n \to \infty$.

REMARK 3.1. Comparing to Theorem 1.1 and Theorem 1.2, Theorem 3.1 and Corollary 3.1 not only extend the dependence structure from NA to ANA, but also improve the condition of h-integrability to R-h-integrability. In addition, noting that

$$\left|\sum_{i=u_n}^{v_n} a_{ni}(X_{ni}-EX_{ni})\right| \leqslant \max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni}-EX_{ni})\right|$$

and

$$\left|\sum_{i=u_n}^{v_n} (X_{ni}-EX_{ni})\right| \leqslant \max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} (X_{ni}-EX_{ni})\right|,$$

our results are stronger. By the way, the condition $\lim_{n\to\infty} h_n \sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r = 0$ in Theorem 3.1 is weaker than $\lim_{n\to\infty} h_n \sup_{u_n\leqslant i\leqslant v_n}|a_{ni}| = 0$ in Theorem 1.2.

The next one is the complete moment convergence for maximum weighted sums, which is much stronger than L_r convergence and complete convergence.

THEOREM 3.2. *Assume that $\{X_{ni}, u_n\leqslant i\leqslant v_n, n\geqslant 1\}$ is an array of rowwise ANA SR-h-integrable random variables with respect to the array $\{a_{ni}, u_n\leqslant i\leqslant v_n, n\geqslant 1\}$ of constants with exponent $1\leqslant r<2$. If $\sum_{n=1}^{\infty}\left(h_n \sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}<\infty$ for some $\lambda\geqslant 1$, then for any $\varepsilon>0$,*

$$\sum_{n=1}^{\infty} E\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni}-EX_{ni})\right|-\varepsilon\right)_+^r<\infty. \tag{3.1}$$

REMARK 3.2. Note that if we choose $a_{ni}=k_n^{-1/r}$ for each $u_n\leqslant i\leqslant v_n$ and $n\geqslant 1$, the conclusion (3.1) is still stronger than Theorem 1.3 since it considers the maximum sums and the condition of strong h-integrability is weakened to SR-h-integrability. In addition, it is deserved to mention that the proofs of Theorem 1.3 and Theorem 3.2 are quite different. Actually, one can easily find that the proof of Theorem 3.2 is is much simpler than that of Theorem 1.3.

REMARK 3.3. We will show that Theorem 3.2 is stronger than Theorem 3.1. Actually, it can be verified by (3.1) that

$$E\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni}-EX_{ni})\right|-\varepsilon\right)_+^r \to 0 \text{ as } n\to\infty.$$

Hence, it follows by Cr-inequality that

$$\begin{aligned}
&E\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|\right)^r\\
&=E\left\{\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|^r I\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|\leqslant\varepsilon\right)\right\}\\
&\quad+E\left\{\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|^r I\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|>\varepsilon\right)\right\}\\
&\leqslant\varepsilon^r+E\left\{\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|^r I\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|>\varepsilon\right)\right\}\\
&\leqslant 2^{r-1}E\left\{\left[\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|-\varepsilon\right]^r I\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|>\varepsilon\right)\right\}\\
&\quad+2^{r-1}\varepsilon^r+\varepsilon^r\\
&=2^{r-1}E\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|-\varepsilon\right)_+^r+2^{r-1}\varepsilon^r+\varepsilon^r\to 0 \text{ by letting } \varepsilon\to 0.
\end{aligned}$$

THEOREM 3.3. *Assume that* $\{X_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *is an array of rowwise ANA* SR-h*-integrable random variables with respect to the array* $\{a_{ni},u_n\leqslant i\leqslant v_n,n\geqslant 1\}$ *of constants with exponent* $1\leqslant r<2$*. If* $\sum_{n=1}^{\infty}\left(h_n\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}<\infty$ *for some* $\lambda\geqslant 1$*, then for any* $\varepsilon>0$*,*

$$\sum_{n=1}^{\infty}P\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|>\varepsilon\right)<\infty \tag{3.2}$$

and thus by the Borel-Cantelli lemma,

$$\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j}a_{ni}(X_{ni}-EX_{ni})\right|\to 0 \ a.s. \text{ as } n\to\infty. \tag{3.3}$$

REMARK 3.4. Comparing Theorem 3.3 to Theorem 1.4, we not only extend the dependence structure from m-NOD to ANA, but also improve condition

$$\sum_{n=1}^{\infty}\left(h_n\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{(2-r)/r}<\infty$$

to

$$\sum_{n=1}^{\infty}\left(h_n\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}<\infty$$

for some $\lambda\geqslant 1$. Moreover, the weighted sums are improved to maximum weighted sums in Theorem 3.3.

By Theorem 3.2 and Theorem 3.3, we can conclude the following corollary.

COROLLARY 3.2. *Let $\{X_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ be an array of rowwise ANA SR-h-integrable random variables with respect to the array $\{a_{ni}, u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ of constants with exponent $1 \leqslant r < 2$. If $h_n \sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|^r = O(n^{-\delta})$ for some $\delta > 0$, then for any $\varepsilon > 0$, (3.1), (3.2), and (3.3) hold.*

4. Proofs

Proof of Theorem 3.1. Noticing that $a_{ni} = a_{ni+} - a_{ni-}$, without loss of generality, we can assume that $a_{ni} \geqslant 0$. Denote for given $u_n \leqslant i \leqslant v_n$, $n \geqslant 1$ that

$$Y_{ni} = -h_n^{1/r} I(X_{ni} < -h_n^{1/r}) + X_{ni} I(|X_{ni}| \leqslant h_n^{1/r}) + h_n^{1/r} I(X_{ni} > h_n^{1/r});$$
$$Z_{ni} = X_{ni} - Y_{ni} = (X_{ni} + h_n^{1/r}) I(X_{ni} < -h_n^{1/r}) + (X_{ni} - h_n^{1/r}) I(X_{ni} > h_n^{1/r}).$$

From Lemma 2.1 one can easily check that both $\{a_{ni}(Y_{ni} - EY_{ni}), u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ and $\{a_{ni}(Z_{ni} - EZ_{ni}), u_n \leqslant i \leqslant v_n, n \geqslant 1\}$ are still ANA. In addition, by Cr-inequality we have that

$$\begin{aligned}
&E\left(\max_{u_n \leqslant j \leqslant v_n} \left|\sum_{i=u_n}^{j} a_{ni}(X_{ni} - EX_{ni})\right|\right)^r \\
&\leqslant 2^{r-1} E\left(\max_{u_n \leqslant j \leqslant v_n} \left|\sum_{i=u_n}^{j} a_{ni}(Y_{ni} - EY_{ni})\right|\right)^r + 2^{r-1} E\left(\max_{u_n \leqslant j \leqslant v_n} \left|\sum_{i=u_n}^{j} a_{ni}(Z_{ni} - EZ_{ni})\right|\right)^r \\
&= 2^{r-1} ES_n^r + 2^{r-1} ET_n^r,
\end{aligned}$$

where

$$S_n = \max_{u_n \leqslant j \leqslant v_n} \left|\sum_{i=u_n}^{j} a_{ni}(Y_{ni} - EY_{ni})\right|$$

and

$$T_n = \max_{u_n \leqslant j \leqslant v_n} \left|\sum_{i=u_n}^{j} a_{ni}(Z_{ni} - EZ_{ni})\right|.$$

To prove $\max_{u_n \leqslant j \leqslant v_n} \left|\sum_{i=u_n}^{j} a_{ni}(X_{ni} - EX_{ni})\right| \to 0$ in L^r, we only need to prove $ES_n^r \to 0$ and $ET_n^r \to 0$ as $n \to \infty$.

We first show $ES_n^2 \to 0$ as $n \to \infty$. By Lemma 2.2, $|Y_{ni}| = \min\{|X_{ni}|, h_n^{1/r}\}$, condition (1.1), and $\lim_{n\to\infty} h_n \sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|^r = 0$ we have that

$$\begin{aligned}
ES_n^2 &= E\left(\max_{u_n \leqslant j \leqslant v_n} \left|\sum_{i=u_n}^{j} a_{ni}(Y_{ni} - EY_{ni})\right|\right)^2 \leqslant C \sum_{i=u_n}^{v_n} a_{ni}^2 E|Y_{ni}|^2 \\
&\leqslant C h_n^{(2-r)/r} \sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|^{2-r} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|Y_{ni}|^r \\
&\leqslant C \left(h_n \sup_{u_n \leqslant i \leqslant v_n} |a_{ni}|^r\right)^{(2-r)/r} \sup_{n \geqslant 1} \sum_{i=u_n}^{v_n} |a_{ni}|^r E|X_{ni}|^r \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

Since $1 \leqslant r < 2$, it follows by Jensen inequality that

$$ES_n^r \leqslant (ES_n^2)^{r/2} \to 0 \text{ as } n \to \infty.$$

Next, we will show $ET_n^r \to 0$ as $n \to \infty$. Noting that $|Z_{ni}| = (|X_{ni}| - h_n^{1/r})I(|X_{ni}| > h_n^{1/r})$, by condition (1.2) and Lemma 2.3 we have that as $n \to \infty$,

$$\begin{aligned} ET_n^r &= E\left(\max_{u_n \leqslant j \leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(Z_{ni} - EZ_{ni})\right|\right)^r \leqslant C\sum_{i=u_n}^{v_n} |a_{ni}|^r E|Z_{ni}|^r \\ &= C\sum_{i=u_n}^{v_n} |a_{ni}|^r E(|X_{ni}| - h_n^{1/r})^r I(|X_{ni}| > h_n^{1/r}) \to 0. \end{aligned}$$

Hence, Theorem 3.1 has been proved. □

Proof of Theorem 3.2. Without loss of generality, we also assume here that $a_{ni} \geqslant 0$. Decompose $X_{ni} - EX_{ni}$ by

$$X_{ni} - EX_{ni} = Y_{ni} - EY_{ni} + Z_{ni} - EZ_{ni},$$

where Y_{ni} and Z_{ni} are defined in the proof of Theorem 3.1. It can be obtained by Lemma 2.4 that for all $\lambda \geqslant 1$,

$$\begin{aligned} &\sum_{n=1}^{\infty} E\left(\max_{u_n \leqslant j \leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni} - EX_{ni})\right| - \varepsilon\right)_+^r \\ &\leqslant C\sum_{n=1}^{\infty} E\left(\max_{u_n \leqslant j \leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(Y_{ni} - EY_{ni})\right|^{2\lambda}\right) \\ &\quad + C\sum_{n=1}^{\infty} E\left(\max_{u_n \leqslant j \leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(Z_{ni} - EZ_{ni})\right|^{r}\right) \\ &= C\sum_{n=1}^{\infty} ES_n^{2\lambda} + C\sum_{n=1}^{\infty} ET_n^r, \end{aligned}$$

where

$$S_n = \max_{u_n \leqslant j \leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(Y_{ni} - EY_{ni})\right|$$

and

$$T_n = \max_{u_n \leqslant j \leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(Z_{ni} - EZ_{ni})\right|.$$

We first show $\sum_{n=1}^{\infty} ET_n^r < \infty$. We have by Lemma 2.3, $Z_{ni} = (|X_{ni}| - h_n^{1/r})I(|X_{ni}| >$

$h_n^{1/r}$) and condition (1.3) that

$$\begin{aligned}\sum_{n=1}^{\infty} ET_n^r &= \sum_{n=1}^{\infty} E\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(Z_{ni}-EZ_{ni})\right|^r\right)\\ &\leqslant C\sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n}|a_{ni}|^r E|Z_{ni}|^r\\ &= C\sum_{n=1}^{\infty}\sum_{i=u_n}^{v_n}|a_{ni}|^r E(|X_{ni}|-h_n^{1/r})^r I(|X_{ni}|>h_n^{1/r})<\infty.\end{aligned}$$

Now we turn to prove $\sum_{n=1}^{\infty} ES_n^{2\lambda}<\infty$. Noting from $\sum_{n=1}^{\infty}\left(h_n \sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}<\infty$ that

$$h_n \sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r \to 0 \text{ as } n\to\infty,$$

and $(2\lambda-r)/r\geqslant\lambda(2-r)/r$ for $\lambda\geqslant 1$, we have that

$$\left(h_n \sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{(2\lambda-r)/r}\leqslant\left(h_n \sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}$$

for all n large enough, which together with Lemma 2.2 and condition (1.1) obtains that

$$\begin{aligned}\sum_{n=1}^{\infty} ES_n^{2\lambda} &= \sum_{n=1}^{\infty} E\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(Y_{ni}-EY_{ni})\right|^{2\lambda}\right)\\ &\leqslant C\sum_{n=1}^{\infty}\left\{\sum_{i=u_n}^{v_n}|a_{ni}|^{2\lambda}E|Y_{ni}|^{2\lambda}+\left(\sum_{i=u_n}^{v_n}a_{ni}^2EY_{ni}^2\right)^{\lambda}\right\}\\ &\leqslant C\sum_{n=1}^{\infty}\left\{h_n^{(2\lambda-r)/r}\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^{2\lambda-r}\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n}|a_{ni}|^r E|Y_{ni}|^r\right\}\\ &\quad+\left\{C\sum_{n=1}^{\infty}\left(h_n^{(2-r)/r}\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^{2-r}\sup_{n\geqslant 1}\sum_{i=u_n}^{v_n}|a_{ni}|^r E|Y_{ni}|^r\right)^{\lambda}\right\}\\ &\leqslant C\sum_{n=1}^{\infty}\left\{\left(h_n\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{(2\lambda-r)/r}+\left(h_n\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}\right\}\\ &\leqslant C\sum_{n=1}^{\infty}\left(h_n\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}<\infty.\end{aligned}$$

Thus, according to the statements above, the proof of the theorem is complete. □

Proof of Theorem 3.3. Actually, it is not difficult to verify by Theorem 3.2 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} P\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni}-EX_{ni})\right|>\varepsilon\right)\\
&\leqslant \left(\frac{\varepsilon}{2}\right)^{-r}\sum_{n=1}^{\infty}\int_0^{\left(\frac{\varepsilon}{2}\right)^r} P\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni}-EX_{ni})\right|-\frac{\varepsilon}{2}>t^{1/r}\right)dt\\
&\leqslant \left(\frac{\varepsilon}{2}\right)^{-r}\sum_{n=1}^{\infty}\int_0^{\infty} P\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni}-EX_{ni})\right|-\frac{\varepsilon}{2}>t^{1/r}\right)dt\\
&= \left(\frac{\varepsilon}{2}\right)^{-r}\sum_{n=1}^{\infty} E\left(\max_{u_n\leqslant j\leqslant v_n}\left|\sum_{i=u_n}^{j} a_{ni}(X_{ni}-EX_{ni})\right|-\frac{\varepsilon}{2}\right)_+^r<\infty.
\end{aligned}$$

The proof is thus finished. □

Proof of Corollary 3.2. Choosing $\lambda>\max\{1,\frac{r}{(2-r)\delta}\}$, we can easily check that

$$\sum_{n=1}^{\infty}\left(h(n)\sup_{u_n\leqslant i\leqslant v_n}|a_{ni}|^r\right)^{\lambda(2-r)/r}\leqslant C\sum_{n=1}^{\infty} n^{-\lambda(2-r)\delta/r}<\infty.$$

Then the conclusions (3.1), (3.2), and (3.3) follow by Theorems 3.2 and 3.3 immediately. □

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REFERENCES

[1] K. ALAM, K. M. L. SAXENA, *Positive dependence in multivariate distributions*, Communication in Statistics-Theory and Methods, 10 (12), (1981), 1183–1196.

[2] K. JOAG-DEV, F. PROSCHAN, *Negative association of random variables with applications*, The Annals of Statistics, 11 (1), (1983), 286–295.

[3] Q. M. SHAO, *A comparison theorem on moment inequalities between negatively associated and independent random variables*, Journal of Theoretical Probability, 13 (2), (2000), 343–356.

[4] G. G. ROUSSAS, *Asymptotic normality of random fields of positively or negatively associated processes*, Journal of Multivariate Analysis, 50 (1), (1994), 152–173.

[5] Z. W. CAI, G. G. ROUSSAS, *Berry-Esseen bounds for smooth estimator of a distribution function under association*, Journal of Nonparametric Statistics, 11 (1–3), (1999), 79–106.

[6] H. Y. LIANG, *Complete convergence for weighted sums of negatively associated random variables*, Statistics & Probability Letters, 48 (4), (2000), 317–325.

[7] Y. WU, X. J. WANG, S. H. SUNG, *Complete moment convergence for arrays of rowwise negatively associated random variables and its application in nonparametric regression model*, Probability in the Engineering and Informational Sciences, 32 (1), (2018), 37–57.

[8] R. C. BRADLEY, *On the spectral density and asymptotic normality of weakly dependent random fields*, Journal of Theoretical Probability, 5, (1992), 355–373.

[9] S. UTEV, M. PELIGRAD, *Maximal inequalities and an invariance principle for a class of weakly dependent random variables*, Journal of Theoretical Probability, 16, (2003), 101–115.

[10] Q. Y. WU, Y. Y. JIANG, *Chover-type laws of the k-iterated logarithm for $\tilde{\rho}$-mixing sequences of random variables*, Journal of Mathematical Analysis and Applications, 366, (2010), 435–443.

[11] S. H. SUNG, *On the strong convergence for weighted sums of ρ^*-mixing random variables*, Statistical Papers, 54, (2013), 773–781.

[12] Y. F. WU, S. H. SUNG, A. VOLODIN, *A note on the rates of convergence for weighted sums of ρ^*-mixing random variables*, Lithuanian Mathematical Journal, 54, (2014), 220–228.

[13] A. T. SHEN, H. Y. ZHU, Y. ZHANG, *Exponential inequality for $\tilde{\rho}$-mixing sequences and its applications*, Filomat, 28 (4), (2014), 859–870.

[14] Y. WU, X. J. WANG, S. H. HU, *Complete moment convergence for weighted sums of weakly dependent random variables and its application in nonparametric regression model*, Statistics & Probability Letters, 127, (2017), 56–66.

[15] P. Y. CHEN, S. H. SUNG, *On complete convergence and complete moment convergence for weighted sums of ρ^*-mixing random variables*, Journal of Inequalities and Applications, 1, (2018), doi:10.1186/s13660-018-1710-2.

[16] L. X. ZHANG, X. Y. WANG, *Convergence rates in the strong laws of asymptotically negatively associated random fields*, Applied Mathematics-A Journal of China Universities, Series B, 14 (4), (1999), 406–416.

[17] L. X. ZHANG, *A functional central limit theorem for asymptotically negatively dependent random fields*, Acta Mathematica Hungarica, 86 (3), (2000), 237–259.

[18] L. X. ZHANG, *Central limit theorems for asymptotically negatively associated random fields*, Acta Mathematica Sinica, English Series, 16 (4), (2000), 691–710.

[19] J. F. WANG, F. B. LU, *Inequalities of maximum partial sums and weak convergence for a class of weak dependent random variables*, Acta Mathematica Sinica, English Series, 22 (3), (2006), 693–700.

[20] J. F. WANG, L. X. ZHANG, *A Berry-Esseen theorem and a law of the iterated logarithm for asymptotically negatively associated sequences*, Acta Mathematica Sinica, English Series, 23 (1), (2007), 127–136.

[21] H. W. HUANG, J. Y. PENG, X. T. WU, B. WANG, *Complete convergence and complete moment convergence for arrays of rowwise ANA random variables*, Journal of Inequalities and Applications, 1, (2016), doi:10.1186/s13660-016-1016-1.

[22] P. L. HSU, H. ROBBINS, *Complete convergence and the law of large numbers*, Proceedings of the National Academy of Sciences U.S.A., 33, (1947), 25–31.

[23] Y. S. CHOW, *On the rate of moment convergence of sample sums and extremes*, Bulletin of the Institute of Mathematics, Academia Sinica, 16 (3), (1988), 177–201.

[24] H. Y. LIANG, D. L. LI, A. ROSALSKY, *Complete moment and integral convergence for sums of negatively associated random variables*, Acta Mathematica Sinica, English Series, 26 (3), (2010), 419–432.

[25] M. L. GUO, D. J. ZHU, *Equivalent conditions of complete moment convergence of weighted sums for ρ^*-mixing sequence of random variables*, Statistics & Probability Letters, 83, (2013), 13–20.

[26] Y. F. WU, M. O. CABREA, A. VOLODIN, *Complete convergence and complete moment convergence for arrays of rowwise END random variables*, Glasnik Matematički, 49 (69), (2014), 449–468.

[27] X. J. WANG, A. T. SHEN, Z. Y. CHEN, S. H. HU, *Complete convergence for weighted sums of NSD random variables and its application in the EV regression model*, TEST, 24 (1), (2015), 166–184.

[28] A. T. SHEN, M. YAO, W. J. WANG, A. VOLODIN, *Exponential probability inequalities for WNOD random variables and their applications*, RACSAM, 110 (1), (2016), 251–268.

[29] T. K. CHANDRA, *Uniform integrability in the Cesàro sense and the weak law of large numbers*, Sankhyā: The Indian Journal of Statistics, Series A, 51, (1989), 309–317.

[30] S. H. SUNG, S. LISAWADI, A. VOLODIN, *Weak laws of large numbers for arrays under a condition of uniform integrability*, Journal of the Korean Mathematical Society, 45, (2008), 289–300.

[31] M. O. CABRERA, A. VOLODIN, *Mean convergence theorems and weak laws of large numbers for weighted sums of random variables under a condition of weighted integrability*, Journal of Mathematical Analysis & Applications, 305 (2), (2005), 644–658.

[32] X. H. WANG, S. H. HU, *Weak laws of large numbers for arrays of dependent random variables*, Stochastics: An International Journal of Probability and Stochastic Processes, 86 (5), (2014), 759–775.

[33] Y. F. WU, J. Y. PENG, T.-C. HU, *Limiting behaviour for arrays of rowwise END random variables under conditions of h-integrability*, Stochastics: An International Journal of Probability and Stochastic Processes, 87 (3), (2014), 409–423.
[34] X. J. WANG, S. H. HU, A. VOLODIN, *Moment inequalities for m-NOD random variables and their applications*, Theory of Probability and Its Applications, 62 (3), (2018), 471–490.
[35] P. Y. CHEN, P. BAI, S. H. SUNG, *The von Bahr-Esseen moment inequality for pairwise independent random variables and applications*, Journal of Mathematical Analysis and Applications, 419 (2), (2014), 1290–1302.

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