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MAHGOUB TRANSFORM AND HYERS–ULAM STABILITY OF FIRST–ORDER LINEAR DIFFERENTIAL EQUATIONS

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Abstract. The main aim of this paper is to investigate various types of Hyers-Ulam stability of linear differential equations of first order with constant coefficients using the Mahgoub transform method. We also show the Hyers-Ulam constants of these differential equations and give some examples to better illustrate the main results.

1. Introduction

In 1940, Ulam [36] proposed a very general Hyers-Ulam stability problem: When is the statement of the theorem still true or nearly true, despite slight variations on the theorem's hypotheses? In the following year, Hyers [9] came up with the first positive answer to Ulam's question by proving the stability of the additive functional equation in Banach spaces. Since then, Hyers' result has been widely generalized in terms of the control conditions used to define the concept of an approximate solution (see [4, 5, 30, 32, 37]).

The generalization of Ulam's question has been relatively recently proposed by replacing functional equations with differential equations: Let I be a subinterval of $\mathbb{R}$, let $\mathbb{K}$ denote either $\mathbb{R}$ or $\mathbb{C}$, and let n be a positive integer. The differential equation $\psi\big(f,x,x',x'',\ldots,x^{(n)}\big)=0$ has the Hyers-Ulam stability if there exists a constant $K>0$ such that the following statement is true for any $\varepsilon>0$: If an n times continuously differentiable function $z:I\to\mathbb{K}$ satisfies the inequality

$$\left|\psi\big(f,z,z',z'',\ldots,z^{(n)}\big)\right|\leqslant\varepsilon$$

for all $t\in I$, then there exists a solution $y:I\to\mathbb{K}$ of the differential equation that satisfies the inequality $|z(t)-y(t)|\leqslant K\varepsilon$ for all $t\in I$.

Obłoza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [26, 27]). Then, in 1998, Alsina and Ger continued the study of Obłoza's Hyers-Ulam stability of differential equations. Indeed, they proved in [3] the following theorem.

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THEOREM 1. *Let $I \neq \emptyset$ be an open subinterval of $\mathbb{R}$. If a differentiable function $x : I \to \mathbb{R}$ satisfies the differential inequality $\|x'(t) - x(t)\| \leqslant \varepsilon$ for any $t \in I$ and for some $\varepsilon > 0$, then there exists a differentiable function $y : I \to \mathbb{R}$ satisfying $y'(t) = y(t)$ and $\|x(t) - y(t)\| \leqslant 3\varepsilon$ for any $t \in I$.*

This result of Alsina and Ger has been generalized by Takahashi *et al.* They proved in [35] that the Hyers-Ulam stability holds true for the Banach space valued differential equation $x'(t) = \lambda x(t)$. Indeed, the Hyers-Ulam stability has been proved for the first-order linear differential equations in more general settings (see [10, 11, 12, 13, 17]).

In 2006, Jung [13] investigated the Hyers-Ulam stability of a system of first-order linear differential equations with constant coefficients by using matrix method. Then, in 2008, Wang *et al.* [38] studied the Hyers-Ulam stability of linear differential equations of first order using the integral factor method. Meanwhile, Rus [34] discussed various types of Hyers-Ulam stability of the ordinary differential equation of the form $x'(t) = Ax(t) + f(t,x(t))$. In 2014, Alqifiary and Jung [2] proved the generalized Hyers-Ulam stability of linear differential equation of the form

$$x^{(n)}(t) + \sum_{k=0}^{n-1} \alpha_k x^{(k)}(t) = f(t)$$

by using the Laplace transform method, where α_k are scalars and $x(t)$ is an n times continuously differentiable function and of the exponential order (see also [33]).

In recent years, many authors are studying the Hyers-Ulam stability of differential equations, and a number of mathematicians are paying attention to the new results of the Hyers-Ulam stability of differential equations (see [6, 7, 8, 15, 16, 18, 19, 20, 21, 22, 24, 28, 29]). Recently, Murali *et al.* [25] have investigated the Hyers-Ulam stability of the linear differential equation using Fourier transform method (see also [23, 31]).

Based on the above results, our main goal is to more efficiently prove the Hyers-Ulam stability of the first-order linear differential equations

$$x'(t) + \lambda x(t) = 0 \tag{1}$$

and

$$x'(t) + \lambda x(t) = r(t) \tag{2}$$

by using the Mahgoub integral transform method, where λ is a scalar and $x(t)$ is a continuously differentiable function of exponential order.

2. Preliminaries and basic definitions

In this section, we introduce some standard notations and definitions which will be useful to prove our main results.

Throughout this paper, $\mathbb{K}$ denotes either the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. A function $f : [0,\infty) \to \mathbb{K}$ is of exponential order if there exist constants $A, B \in \mathbb{R}$ such that $|f(t)| \leqslant Ae^{Bt}$ for all $t \geqslant 0$. Similarly, a function $g : (-\infty, 0] \to \mathbb{K}$ is of exponential order if there exist constants $A, B \in \mathbb{R}$ such that $|g(t)| \leqslant Ae^{Bt}$ for all $t \leqslant 0$.

DEFINITION 1. ([1]) The Mahgoub integral transform of the function $f:[0,\infty)\to\mathbb{K}$ is defined by

$$\mathscr{M}\{f(t)\}=u\int_0^\infty f(s)e^{-us}ds=F(u),$$

where $\mathscr{M}$ is the Mahgoub integral transform operator.

The Mahgoub integral transform for the function $f:[0,\infty)\to\mathbb{K}$ exists if $f(t)$ is piecewise continuous and of exponential order. These conditions are the only sufficient conditions for the existence of Mahgoub transform of the function $f(t)$.

DEFINITION 2. (Convolution of two functions) ([1]). The convolution of two functions $f(t)$ and $g(t)$ is denoted by $f(t)*g(t)$ and is defined by

$$f(t)*g(t)=(f*g)(t)=\int_0^t f(s)g(t-s)ds=\int_0^t f(t-s)g(s)ds.$$

THEOREM 2. (Convolution theorem for Mahgoub transform) ([1]) *Assume that $f(t)$ and $g(t)$ are given functions defined for $t\geqslant 0$. If $\mathscr{M}\{f(t)\}=F(u)$ and $\mathscr{M}\{g(t)\}=G(u)$, then*

$$\mathscr{M}\{f(t)*g(t)\}=\frac{1}{u}F(u)G(u).$$

DEFINITION 3. (Inverse Mahgoub transform) ([1]) If $\mathscr{M}\{f(t)\}=F(u)$, then $f(t)$ is called the inverse Mahgoub transform of $F(u)$ and is denoted as $f(t)=\mathscr{M}^{-1}\{F(u)\}$, where $\mathscr{M}^{-1}$ is the inverse Mahgoub transform operator.

DEFINITION 4. ([14]) The Mittag-Leffler function of one parameter is denoted by $E_\nu(t)$ and defined as

$$E_\nu(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\nu k+1)},$$

where $t,\nu\in\mathbb{C}$ and $\Re(\nu)>0$. If we put $\nu=1$, then the above equation becomes

$$E_1(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k+1)}=\sum_{k=0}^{\infty}\frac{t^k}{k!}=e^t.$$

Now we give the definitions of Hyers-Ulam stability and Hyers-Ulam ϕ-stability of the differential equations (1) and (2).

Throughout this section, we set

$$\mathscr{F}:=\left\{f:[0,\infty)\to\mathbb{K}\mid f \text{ is a continuously differentiable function of exponential order}\right\}.$$

DEFINITION 5. (i) The linear differential equation (1) is said to have the Hyers-Ulam stability (for the class $\mathscr{F}$) when there exists a constant $K > 0$ such that the following statement is true for any $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t)| \leqslant \varepsilon \tag{3}$$

for all $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (1) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\varepsilon$$

for all $t \geqslant 0$.

(ii) We say that the non-homogeneous linear differential equation (2) has the Hyers-Ulam stability (for the class $\mathscr{F}$), if there exists a constant $K > 0$ such that the following statement is true for each $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t) - r(t)| \leqslant \varepsilon \tag{4}$$

for all $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (2) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\varepsilon$$

for any $t \geqslant 0$. Then the constant K is called a Hyers-Ulam constant.

DEFINITION 6. Let $\phi : [0,\infty) \to (0,\infty)$ be a function.

(i) We say that the homogeneous linear differential equation (1) has the Hyers-Ulam ϕ-stability (for the class $\mathscr{F}$), if there exists a constant $K > 0$ such that the following statement is true for every $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t)| \leqslant \phi(t)\varepsilon \tag{5}$$

for any $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (1) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\phi(t)\varepsilon$$

for any $t \geqslant 0$.

(ii) The differential equation (2) is said to have the Hyers-Ulam ϕ-stability (for the class $\mathscr{F}$) when there exists a constant $K > 0$ such that the following statement is true for all $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t) - r(t)| \leqslant \phi(t)\varepsilon \tag{6}$$

for all $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (2) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\phi(t)\varepsilon$$

for all $t \geqslant 0$. For the case, we call the constant K a Hyers-Ulam ϕ-constant.

Finally, we introduce the definitions of Mittag-Leffler-Hyers-Ulam stability and Mittag-Leffler-Hyers-Ulam ϕ-stability of differential equations (1) and (2).

DEFINITION 7. Let $E_\nu(t)$ be the Mittag-Leffler function.

(i) We say that the differential equation (1) has the Mittag-Leffler-Hyers-Ulam stability (for the class $\mathscr{F}$), if there exists a constant $K > 0$ such that the following statement holds true for any $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t)| \leqslant \varepsilon E_\nu(t) \tag{7}$$

for all $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (1) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\varepsilon E_\nu(t)$$

for any $t \geqslant 0$.

(ii) We say that the non-homogeneous differential equation (2) has the Mittag-Leffler-Hyers-Ulam stability (for the class $\mathscr{F}$) when there exists a constant $K > 0$ such that the following statement is true for each $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t) - r(t)| \leqslant \varepsilon E_\nu(t) \tag{8}$$

for every $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (2) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\varepsilon E_\nu(t)$$

for any $t \geqslant 0$. We call the constant K a Mittag-Leffler-Hyers-Ulam constant.

DEFINITION 8. Let $E_\nu(t)$ be the Mittag-Leffler function and let $\phi : [0,\infty) \to (0,\infty)$ be a function.

(i) We say that the differential equation (1) has the Mittag-Leffler-Hyers-Ulam ϕ-stability (for the class $\mathscr{F}$), if there exists a constant $K > 0$ such that the following statement is true for any $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t)| \leqslant \phi(t)\varepsilon E_\nu(t) \tag{9}$$

for each $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (1) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\phi(t)\varepsilon E_\nu(t)$$

for any $t \geqslant 0$.

(ii) We say that the differential equation (2) has the Mittag-Leffler-Hyers-Ulam ϕ-stability (for the class $\mathscr{F}$) when there exists a constant $K > 0$ such that the following statement is true for any $\varepsilon > 0$: If a function $x \in \mathscr{F}$ satisfies the inequality

$$|x'(t) + \lambda x(t) - r(t)| \leqslant \phi(t)\varepsilon E_\nu(t) \tag{10}$$

for any $t \geqslant 0$, then there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (2) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\phi(t)\varepsilon E_\nu(t)$$

for any $t \geqslant 0$. For this case, we call K a Mittag-Leffler-Hyers-Ulam ϕ-constant.

3. Hyers-Ulam stability of (1)

In this section, we prove several types of Hyers-Ulam stability of homogeneous first-order linear differential equation (1) using Mahgoub transform.

It should be noted that in this and the next section we investigate various types of Hyers-Ulam stability for the class $\mathscr{F}$, where $\mathscr{F}$ is the class of all continuously differentiable functions $f : [0,\infty) \to \mathbb{K}$ of exponential order. For any $\lambda \in \mathbb{K}$, we use the notation $\Re(\lambda)$ to denote the real part of λ.

THEOREM 3. *Assume that λ is a constant with $\Re(\lambda) > 0$. The homogeneous linear differential equation* (1) *is Hyers-Ulam stable in the class $\mathscr{F}$.*

Proof. Assume that $x \in \mathscr{F}$ and $x(t)$ satisfies the inequality (3) for all $t \geqslant 0$. Let us define a function $p : [0,\infty) \to \mathbb{K}$ by $p(t) := x'(t) + \lambda x(t)$ for each $t \geqslant 0$. In view of (3), the inequality $|p(t)| \leqslant \varepsilon$ holds for each $t \geqslant 0$. Mahgoub transform of $p(t)$ gives the following result:

$$\begin{aligned} P(u) &:= \mathscr{M}\{p(t)\} = \mathscr{M}\{x'(t) + \lambda x(t)\} = \mathscr{M}\{x'(t)\} + \lambda \mathscr{M}\{x(t)\} \\ &= uX(u) - ux(0) + \lambda X(u), \end{aligned}$$

where $X(u) = \mathscr{M}\{x(t)\}$ and since $\mathscr{M}\{x'(t)\} = u\mathscr{M}\{x(t)\} - ux(0)$. Thus, we have

$$\mathscr{M}\{x(t)\} = X(u) = \frac{ux(0) + P(u)}{\lambda + u}. \tag{11}$$

If we put $y(t) = e^{-\lambda t}x(0)$, then $y(0) = x(0)$ and $y \in \mathscr{F}$. Mahgoub transform of $y(t)$ gives the following result:

$$\mathscr{M}\{y(t)\} = Y(u) = \frac{ux(0)}{\lambda + u}. \tag{12}$$

Thus,

$$\mathscr{M}\{y'(t) + \lambda y(t)\} = \mathscr{M}\{y'(t)\} + \lambda \mathscr{M}\{y(t)\} = uY(u) - uy(0) + \lambda Y(u).$$

Using (12), we have $\mathscr{M}\{y'(t) + \lambda y(t)\} = 0$. Since $\mathscr{M}$ is one-to-one operator, $y'(t) + \lambda y(t) = 0$. Hence, $y(t)$ is a solution of the differential equation (1).

Plugging (11) into (12), we obtain

$$\begin{aligned} \mathscr{M}\{x(t)\} - \mathscr{M}\{y(t)\} = X(u) - Y(u) &= \frac{P(u)}{\lambda + u} = \frac{1}{u}P(u) \cdot \frac{u}{\lambda + u} \\ &= \frac{1}{u}P(u)Q(u) = \mathscr{M}\{p(t) * q(t)\}, \end{aligned}$$

where $Q(u) = \frac{u}{\lambda+u}$ which gives $q(t) = \mathscr{M}^{-1}\{\frac{u}{\lambda+u}\} = e^{-\lambda t}$.

Consequently, $\mathscr{M}\{x(t)-y(t)\} = \mathscr{M}\{p(t)*q(t)\}$ and thus $x(t)-y(t) = p(t)*q(t)$. Taking modulus on both sides, we have

$$\begin{aligned}|x(t)-y(t)| &= |p(t)*q(t)| = \left|\int_0^t p(s)q(t-s)ds\right| \leqslant \int_0^t |p(s)||q(t-s)|ds \\ &\leqslant \varepsilon \int_0^t |q(t-s)|ds = \varepsilon e^{-\Re(\lambda)t}\int_0^t e^{\Re(\lambda)s}ds = \frac{\varepsilon}{\Re(\lambda)}\left(1-e^{-\Re(\lambda)t}\right) \\ &\leqslant K\varepsilon\end{aligned}$$

for all $t \geqslant 0$, where we set $K = \frac{1}{\Re(\lambda)}$, which implies that the homogeneous linear differential equation (1) has the Hyers-Ulam stability for the class $\mathscr{F}$. □

We note that if $\Re(\lambda) < 0$, then $\frac{\varepsilon}{\Re(\lambda)}\left(1-e^{-\Re(\lambda)t}\right)$ diverges to infinity as t grows to infinity. Hence, in the case of $\Re(\lambda) < 0$, we notice that we cannot prove the Hyers-Ulam stability by applying the Mahgoub transform method.

Similar to Theorem 3, we will prove the Hyers-Ulam ϕ-stability for the differential equation (1). For the sake of the completeness of this paper, the proof is introduced here in detail.

THEOREM 4. *Assume that $\phi : [0,\infty) \to (0,\infty)$ is an increasing function and λ is a constant with $\Re(\lambda) > 0$. Then the differential equation (1) has the Hyers-Ulam ϕ-stability for the class $\mathscr{F}$.*

Proof. Assume that $x \in \mathscr{F}$ and $\phi : [0,\infty) \to (0,\infty)$ is an increasing function satisfying the inequality (5) for all $t \geqslant 0$. If we define a function $p : [0,\infty) \to \mathbb{K}$ by $p(t) := x'(t) + \lambda x(t)$ for each $t \geqslant 0$, then we have $|p(t)| \leqslant \phi(t)\varepsilon$ for any $t \geqslant 0$.

As we did in the first part of the proof of Theorem 3, we can prove that $y(t) = e^{-\lambda t}x(0)$ is a solution of the differential equation (1). Of course, $y \in \mathscr{F}$. On the other hand, $Q(u) = \frac{u}{\lambda+u}$ gives $q(t) = \mathscr{M}^{-1}\{\frac{u}{\lambda+u}\} = e^{-\lambda t}$. Moreover, it follows from (11) and (12) that

$$\begin{aligned}\mathscr{M}\{x(t)\} - \mathscr{M}\{y(t)\} &= X(u) - Y(u) = \frac{P(u)}{\lambda+u} = \frac{1}{u}P(u)Q(u) = \frac{1}{u}\mathscr{M}\{p(t)\}\mathscr{M}\{q(t)\} \\ &= \mathscr{M}\{p(t)*q(t)\},\end{aligned}$$

which yields that $\mathscr{M}\{x(t)-y(t)\} = \mathscr{M}\{p(t)*e^{-\lambda t}\}$. Therefore, $x(t)-y(t) = p(t)*e^{-\lambda t}$.

Similar to the proof of Theorem 3, we can show that

$$\begin{aligned}|x(t)-y(t)| &= \left|p(t)*e^{-\lambda t}\right| = \left|\int_0^t p(s)e^{-\lambda(t-s)}ds\right| \leqslant \int_0^t |p(s)|\left|e^{-\lambda(t-s)}\right|ds \\ &\leqslant \phi(t)\varepsilon e^{-\Re(\lambda)t}\int_0^t e^{\Re(\lambda)s}ds = \frac{\phi(t)\varepsilon}{\Re(\lambda)}\left(1-e^{-\Re(\lambda)t}\right) \\ &\leqslant K\phi(t)\varepsilon\end{aligned}$$

for all $t \geqslant 0$, where we set $K = \frac{1}{\Re(\lambda)}$. □

Now, we are going to establish the Mittag-Leffler-Hyers-Ulam stability of differential equation (1) using Mahgoub transform.

THEOREM 5. *Let λ and ν be constants satisfying $\Re(\lambda) > 0$ and $\nu > 0$. Then the homogeneous differential equation (1) has the Mittag-Leffler-Hyers-Ulam stability for the class $\mathscr{F}$.*

Proof. Assume that $x \in \mathscr{F}$ and it satisfies the inequality (7) for any $t \geqslant 0$. Let $p : [0,\infty) \to \mathbb{K}$ be a function defined by $p(t) := x'(t) + \lambda x(t)$ for each $t \geqslant 0$. In view of (7), we have $|p(t)| \leqslant \varepsilon E_\nu(t)$ for all $t \geqslant 0$. Mahgoub transform of $p(t)$ yields the following result:

$$P(u) := \mathscr{M}\{p(t)\} = \mathscr{M}\{x'(t) + \lambda x(t)\} = uX(u) - ux(0) + \lambda X(u).$$

Thus, we get

$$\mathscr{M}\{x(t)\} = X(u) = \frac{ux(0) + P(u)}{\lambda + u}. \tag{13}$$

If we put $y(t) = e^{-\lambda t}x(0)$, then $y(0) = x(0)$ and $y \in \mathscr{F}$. Moreover, Mahgoub transform of $y(t)$ yields

$$\mathscr{M}\{y(t)\} = Y(u) = \frac{ux(0)}{\lambda + u}. \tag{14}$$

Thus, it follows from (14) that

$$\mathscr{M}\{y'(t) + \lambda y(t)\} = \mathscr{M}\{y'(t)\} + \lambda \mathscr{M}\{y(t)\} = uY(u) - uy(0) + \lambda Y(u) = 0.$$

Since $\mathscr{M}$ is one-to-one operator, $y'(t) + \lambda y(t) = 0$. Hence, $y(t)$ is a solution of the differential equation (1).

If we set $Q(u) = \frac{u}{\lambda+u}$, then the equality $\mathscr{M}\{q(t)\} = \frac{u}{\lambda+u}$ implies that $q(t) = e^{-\lambda t}$. Plugging (13) into (14), we obtain

$$\mathscr{M}\{x(t)\} - \mathscr{M}\{y(t)\} = X(u) - Y(u) = \frac{P(u)}{\lambda + u} = \frac{1}{u}P(u)Q(u).$$

Consequently,

$$\mathscr{M}\{x(t) - y(t)\} = \mathscr{M}\left\{p(t) * e^{-\lambda t}\right\},$$

which gives $x(t) - y(t) = p(t) * e^{-\lambda t}$. Taking modulus on both sides and using the fact that $|p(t)| \leqslant \varepsilon E_\nu(t)$ for $t \geqslant 0$ and since $E_\nu(t)$ is increasing for $t \geqslant 0$, we have

$$\begin{aligned} |x(t) - y(t)| &= \left|p(t) * e^{-\lambda t}\right| = \left|\int_0^t p(s)e^{-\lambda(t-s)}ds\right| \leqslant \int_0^t |p(s)|\left|e^{-\lambda(t-s)}\right|ds \\ &\leqslant \varepsilon E_\nu(t)e^{-\Re(\lambda)t}\int_0^t e^{\Re(\lambda)s}ds = \varepsilon E_\nu(t)\frac{1}{\Re(\lambda)}\left(1 - e^{-\Re(\lambda)t}\right) \\ &= K\varepsilon E_\nu(t) \end{aligned}$$

for all $t \geqslant 0$, where we choose $K = \frac{1}{\Re(\lambda)}$. Then, referring to Definition 7, we can confirm that the homogeneous linear differential equation (1) has the Mittag-Leffler-Hyers-Ulam stability for the class $\mathscr{F}$. □

Similar to the case of Theorem 5, the Mittag-Leffler-Hyers-Ulam ϕ-stability of the linear differential equation (1) is proved. For the sake of this paper's completeness, we present the whole proof.

THEOREM 6. *Assume that $\phi : [0,\infty) \to (0,\infty)$ is an increasing function and that λ and ν are constants which satisfy $\Re(\lambda) > 0$ and $\nu > 0$. Then the differential equation (1) has the Mittag-Leffler-Hyers-Ulam ϕ-stability for the class $\mathscr{F}$.*

Proof. Assume that $x \in \mathscr{F}$, $\phi : [0,\infty) \to (0,\infty)$ is a function, and that $x(t)$ and $\phi(t)$ satisfy the inequality (9) for all $t \geqslant 0$. We will prove that there exist a positive constant $K > 0$ (independent of ε) and a solution $y : [0,\infty) \to \mathbb{K}$ of the differential equation (1) such that $y \in \mathscr{F}$ and

$$|x(t) - y(t)| \leqslant K\phi(t)\varepsilon E_\nu(t)$$

for any $t \geqslant 0$.

If we define a function $p : [0,\infty) \to \mathbb{K}$ by $p(t) := x'(t) + \lambda x(t)$ for each $t \geqslant 0$, then we have $|p(t)| \leqslant \phi(t)\varepsilon E_\nu(t)$ for each $t \geqslant 0$. Then by applying the same method as presented in the proof of Theorem 5, we can easily get

$$\begin{aligned} |x(t) - y(t)| &= \left|p(t) * e^{-\lambda t}\right| = \left|\int_0^t p(s)e^{-\lambda(t-s)}ds\right| \leqslant \int_0^t |p(s)|\left|e^{-\lambda(t-s)}\right|ds \\ &\leqslant \phi(t)\varepsilon E_\nu(t)e^{-\Re(\lambda)t}\int_0^t e^{\Re(\lambda)s}ds = \phi(t)\varepsilon E_\nu(t)\frac{1}{\Re(\lambda)}\left(1 - e^{-\Re(\lambda)t}\right) \\ &\leqslant K\phi(t)\varepsilon E_\nu(t) \end{aligned}$$

for all $t \geqslant 0$, where we set $K = \frac{1}{\Re(\lambda)}$. Then, referring to Definition 8, we confirm that the homogeneous linear differential equation (1) has the Mittag-Leffler-Hyers-Ulam ϕ-stability for the class $\mathscr{F}$. □

4. Hyers-Ulam stability of (2)

In this section, we prove several types of Hyers-Ulam stability of homogeneous first-order linear differential equation (2) using Mahgoub transform. We recall the following definition

$$\mathscr{F} = \left\{f : [0,\infty) \to \mathbb{K} \mid f \text{ is a continuously differentiable function of exponential order}\right\},$$

which was introduced in the previous section.

THEOREM 7. *Assume that $r:[0,\infty)\to\mathbb{K}$ is a continuous function of exponential order and λ is a constant with $\Re(\lambda)>0$. The differential equation (2) has the Hyers-Ulam stability for the class $\mathscr{F}$.*

Proof. Suppose that $x\in\mathscr{F}$ and it satisfies the inequality (4) for all $t\geqslant 0$. Consider the function $p:[0,\infty)\to\mathbb{K}$ defined by

$$p(t):=x'(t)+\lambda x(t)-r(t)$$

for all $t\geqslant 0$. Then it holds that $|p(t)|\leqslant\varepsilon$ for all $t\geqslant 0$.

Mahgoub transform of $p(t)$ gives the following result:

$$\mathscr{M}\{p(t)\}=\mathscr{M}\{x'(t)+\lambda x(t)-r(t)\}.$$

That is,

$$P(u):=\mathscr{M}\{x'(t)\}+\lambda\mathscr{M}\{x(t)\}-\mathscr{M}\{r(t)\}=uX(u)-ux(0)+\lambda X(u)-R(u),$$

which implies that

$$\mathscr{M}\{x(t)\}=X(u)=\frac{ux(0)+P(u)+R(u)}{\lambda+u}. \tag{15}$$

If we set $y(t)=e^{-\lambda t}x(0)+(r(t)*e^{-\lambda t})$, then $y\in\mathscr{F}$. Mahgoub transform of $y(t)$ yields the following result:

$$\mathscr{M}\{y(t)\}=Y(u)=\frac{ux(0)+R(u)}{\lambda+u}. \tag{16}$$

On the other hand,

$$\mathscr{M}\{y'(t)+\lambda y(t)\}=uY(u)-ux(0)+\lambda Y(u)=(\lambda+u)Y(u)-ux(0).$$

Then, by using (16), we have

$$\mathscr{M}\{y'(t)+\lambda y(t)\}=R(u)=\mathscr{M}\{r(t)\}$$

and thus, $y'(t)+\lambda y(t)=r(t)$. Hence, $y(t)$ is a solution of the differential equation (2).

In addition, by applying (15) and (16), we can obtain

$$\mathscr{M}\{x(t)\}-\mathscr{M}\{y(t)\}=X(u)-Y(u)=\frac{P(u)}{\lambda+u}=\frac{1}{u}P(u)Q(u)=\frac{1}{u}\mathscr{M}\{p(t)\}\mathscr{M}\{q(t)\},$$

where we set $Q(u)=\frac{u}{\lambda+u}$ which gives $q(t)=\mathscr{M}^{-1}\{\frac{u}{\lambda+u}\}=e^{-\lambda t}$. Therefore, we have

$$\mathscr{M}\{x(t)-y(t)\}=\mathscr{M}\{p(t)*e^{-\lambda t}\},$$

which yields $x(t)-y(t)=p(t)*e^{-\lambda t}$. Furthermore,

$$|x(t)-y(t)|=\left|p(t)*e^{-\lambda t}\right|=\left|\int_0^t p(s)e^{-\lambda(t-s)}ds\right|\leqslant\int_0^t |p(s)|\left|e^{-\lambda(t-s)}\right|ds$$
$$\leqslant \varepsilon e^{-\Re(\lambda)t}\int_0^t e^{\Re(\lambda)s}ds\leqslant K\varepsilon$$

for each $t\geqslant 0$, where we set $K=\frac{1}{\Re(\lambda)}$. □

For the Hyers-Ulam ϕ-stability of non-homogeneous linear differential equation (2), we obtain the following theorem.

THEOREM 8. *Assume that* $r:[0,\infty)\to\mathbb{K}$ *is a continuous function of exponential order,* $\phi:[0,\infty)\to(0,\infty)$ *is an increasing function, and that* λ *is a constant with* $\Re(\lambda)>0$. *The differential equation* (2) *has the Hyers-Ulam* ϕ*-stability for the class* $\mathscr{F}$.

Proof. We consider an arbitrary function $x\in\mathscr{F}$ that satisfies the inequality (6) for all $t\geqslant 0$. Now we define a function $p:[0,\infty)\to\mathbb{K}$ by $p(t):=x'(t)+\lambda x(t)-r(t)$ for each $t\geqslant 0$. Then, $|p(t)|\leqslant\phi(t)\varepsilon$ for all $t\geqslant 0$. It is not difficult to check

$$\mathscr{M}\{x(t)\}=X(u)=\frac{ux(0)+P(u)+R(u)}{\lambda+u}. \tag{17}$$

If we set $y(t)=e^{-\lambda t}x(0)+(r(t)*e^{-\lambda t})$, then $y\in\mathscr{F}$. Further, we apply the Mahgoub transform on both sides to get

$$\mathscr{M}\{y(t)\}=Y(u)=\frac{ux(0)+R(u)}{\lambda+u}. \tag{18}$$

On the other hand,

$$\mathscr{M}\{y'(t)+\lambda y(t)\}=(\lambda+u)Y(u)-ux(0).$$

The relation (18) implies that

$$\mathscr{M}\{y'(t)+\lambda y(t)\}=R(u)=\mathscr{M}\{r(t)\}$$

and thus, $y'(t)+\lambda y(t)=r(t)$, that is, $y(t)$ is a solution of the differential equation (2).

Using (17) and (18), we obtain

$$\mathscr{M}\{x(t)-y(t)\}=X(u)-Y(u)=\frac{P(u)}{\lambda+u}=\frac{1}{u}\mathscr{M}\{p(t)\}\mathscr{M}\{q(t)\},$$

where $Q(u)=\frac{u}{\lambda+u}$ which gives $q(t)=e^{-\lambda t}$. Hence, $\mathscr{M}\{x(t)-y(t)\}=\mathscr{M}\{p(t)*q(t)\}$ which gives $x(t)-y(t)=p(t)*q(t)$.

Similar to the proof of Theorem 4, we have

$$\begin{aligned}|x(t)-y(t)| = \left|p(t)*q(t)\right| = \left|\int_0^t p(s)q(t-s)ds\right| &\leqslant \int_0^t |p(s)||q(t-s)|ds \\ &\leqslant \phi(t)\varepsilon e^{-\Re(\lambda)t}\int_0^t e^{\Re(\lambda)s}ds = \frac{\phi(t)\varepsilon}{\Re(\lambda)}\left(1-e^{-\Re(\lambda)t}\right) \\ &\leqslant K\phi(t)\varepsilon\end{aligned}$$

for all $t \geqslant 0$, where we set $K = \frac{1}{\Re(\lambda)}$. □

We now prove the Mittag-Leffler-Hyers-Ulam stability of the non-homogeneous linear differential equation (2) using Mahgoub transform method.

THEOREM 9. *Assume that $r : [0,\infty) \to \mathbb{K}$ is a continuous function of exponential order and that λ and ν are constants satisfying $\Re(\lambda) > 0$ and $\nu > 0$. Then the non-homogeneous linear differential equation (2) has the Mittag-Leffler-Hyers-Ulam stability for the class $\mathscr{F}$.*

Proof. Suppose $x \in \mathscr{F}$ and $x(t)$ satisfies (8) for each $t \geqslant 0$. Consider the function $p : [0,\infty) \to \mathbb{K}$ defined by $p(t) := x'(t)+\lambda x(t)-r(t)$ for all $t \geqslant 0$. Then it follows from (8) that $|p(t)| \leqslant \varepsilon E_\nu(t)$ for all $t \geqslant 0$.

Mahgoub transform of $p(t)$ yields the following result:

$$P(u) = \mathscr{M}\{p(t)\} = \mathscr{M}\{x'(t)+\lambda x(t)-r(t)\} = uX(u)-ux(0)+\lambda X(u)-R(u),$$

which further implies that

$$X(u) = \mathscr{M}\{x(t)\} = \frac{ux(0)+P(u)+R(u)}{\lambda+u}. \tag{19}$$

If we set $y(t) = e^{-\lambda t}x(0)+\left(r(t)*e^{-\lambda t}\right)$, then $y \in \mathscr{F}$. We apply the Mahgoub transform on both sides of the last equality to get

$$Y(u) = \mathscr{M}\{y(t)\} = \frac{ux(0)+R(u)}{\lambda+u}. \tag{20}$$

On the other hand,

$$\mathscr{M}\{y'(t)+\lambda y(t)\} = uY(u)-ux(0)+\lambda Y(u) = (\lambda+u)Y(u)-ux(0).$$

Then, by using (20), we have

$$\mathscr{M}\{y'(t)+\lambda y(t)\} = R(u) = \mathscr{M}\{r(t)\}$$

and thus, $y'(t)+\lambda y(t) = r(t)$ for all $t \geqslant 0$. Hence, $y(t)$ is a solution of the differential equation (2).

In addition, by applying (19) and (20), we get

$$\mathscr{M}\{x(t)-y(t)\} = \frac{P(u)}{\lambda+u} = \frac{1}{u}\mathscr{M}\{p(t)\}\mathscr{M}\{q(t)\},$$

where $\mathscr{M}\{q(t)\} = \frac{u}{\lambda+u}$ which gives $q(t) = e^{-\lambda t}$. Therefore, $\mathscr{M}\{x(t)-y(t)\} = \mathscr{M}\{p(t) * q(t)\}$ which yields $x(t)-y(t) = p(t)*q(t)$ for each $t \geqslant 0$. Furthermore,

$$\begin{aligned}|x(t)-y(t)| &= \left|p(t)*q(t)\right| = \left|\int_0^t p(s)q(t-s)ds\right| \leqslant \int_0^t |p(s)||q(t-s)|ds \\ &\leqslant \varepsilon E_\nu(t)e^{-\Re(\lambda)t}\int_0^t e^{\Re(\lambda)s}ds = \varepsilon E_\nu(t)\frac{1}{\Re(\lambda)}\left(1-e^{-\Re(\lambda)t}\right) \\ &\leqslant K\varepsilon E_\nu(t)\end{aligned}$$

for all $t \geqslant 0$, where we set $K = \frac{1}{\Re(\lambda)}$. This completes the proof. □

Similar to the case of Theorem 9, the Mittag-Leffler-Hyers-Ulam ϕ-stability of the linear differential equation (2) is proved. For the sake of this paper's completeness, we present the whole proof.

THEOREM 10. *Assume that $r:[0,\infty)\to\mathbb{K}$ is a continuous function of exponential order, $\phi:[0,\infty)\to(0,\infty)$ is an increasing function, and that λ and ν are constants which satisfy $\Re(\lambda)>0$ and $\nu>0$. Then the non-homogeneous differential equation (2) has the Mittag-Leffler-Hyers-Ulam ϕ-stability for the class $\mathscr{F}$.*

Proof. Assume that $x\in\mathscr{F}$ and it satisfies the inequality (10) for any $t\geqslant 0$. It is to be proved that there exist a constant $K>0$ (independent of ε) and a solution $y:[0,\infty)\to\mathbb{K}$ of the differential equation (2) such that $y\in\mathscr{F}$ and

$$|x(t)-y(t)| \leqslant K\phi(t)\varepsilon E_\nu(t)$$

for all $t\geqslant 0$.

If we define a function $p:[0,\infty)\to\mathbb{K}$ by $p(t) := x'(t)+\lambda x(t)-r(t)$ for each $t\geqslant 0$, then we have $|p(t)|\leqslant\phi(t)\varepsilon E_\nu(t)$ for every $t\geqslant 0$. By applying similar methods as Theorem 9, we can easily prove that there exists a solution $y:[0,\infty)\to\mathbb{K}$ of (2) satisfying $y\in\mathscr{F}$ and

$$\begin{aligned}|x(t)-y(t)| &= \left|p(t)*e^{-\lambda t}\right| = \left|\int_0^t p(s)e^{-\lambda(t-s)}ds\right| \leqslant \int_0^t |p(s)|\left|e^{-\lambda(t-s)}\right|ds \\ &\leqslant \phi(t)\varepsilon E_\nu(t)\frac{1}{\Re(\lambda)}\left(1-e^{-\Re(\lambda)t}\right) \\ &\leqslant K\phi(t)\varepsilon E_\nu(t)\end{aligned}$$

for all $t\geqslant 0$, where we set $K = \frac{1}{\Re(\lambda)}$. □

5. Examples and remarks

In this section, we will introduce some examples to make it easier to understand the main results of this paper.

EXAMPLE 1. We consider the following non-homogeneous linear differential equation

$$x'(t) + x(t) = 2\cos t. \tag{21}$$

We know that $r(t) = 2\cos t$ is a function of exponential order and $\lambda = 1$.

If a continuously differentiable function $z : [0,\infty) \to \mathbb{K}$ of exponential order satisfies

$$|z'(t) + z(t) - 2\cos t| \leqslant \varepsilon$$

for all $t \geqslant 0$ and for some $\varepsilon > 0$, then Theorem 7 implies that there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (21) such that $y(t)$ is of exponential order and

$$|z(t) - y(t)| \leqslant K\varepsilon$$

for all $t \geqslant 0$, where $K = \frac{1}{\Re(\lambda)} = 1$. In particular, $y(t) = ce^{-t} + \sin t + \cos t$ for some constant $c \in \mathbb{K}$.

EXAMPLE 2. We consider the following non-homogeneous linear differential equation

$$x'(t) + 3x(t) = t, \tag{22}$$

where $r(t) = t$ is a function of exponential order and $\lambda = 3$.

If a continuously differentiable function $z : [0,\infty) \to \mathbb{K}$ of exponential order satisfies

$$|z'(t) + 3z(t) - t| \leqslant \varepsilon$$

for all $t \geqslant 0$ and for some $\varepsilon > 0$, then Theorem 7 implies that there exists a solution $y : [0,\infty) \to \mathbb{K}$ of differential equation (22) such that $y(t)$ is of exponential order and

$$|z(t) - y(t)| \leqslant K\varepsilon$$

for all $t \geqslant 0$, where we set $K = \frac{1}{\Re(\lambda)} = \frac{1}{3}$. In particular, $y(t) = ce^{-3t} + \frac{1}{3}t - \frac{1}{9}$ for some constant $c \in \mathbb{K}$.

REMARK 1. The above examples are also true when we replace ε and $K\varepsilon$ with $\phi(t)\varepsilon$ and $K\phi(t)\varepsilon$, respectively, where $\phi(t)$ is an increasing function. In this case, we see that the corresponding differential equations have the Hyers-Ulam ϕ-stability for the class $\mathscr{F}$.

REMARK 2. The differential equations (21) and (22) have the Mittag-Leffler-Hyers-Ulam stability for the class $\mathscr{F}$ if $\nu > 0$. In particular, they also have the Mittag-Leffler-Hyers-Ulam ϕ-stability for the class $\mathscr{F}$ when $\phi(t)$ is an increasing function and $\nu > 0$.

6. Discussion

What results can be expected for the Hyers-Ulam stability of the differential equation (2) when the relevant domain is the set of all non-positive real numbers?

For any given constant $\varepsilon > 0$, we consider the following inequality

$$\left|x'(t) + \lambda x(t) - r(t)\right| \leqslant \varepsilon \quad (\text{for } t \leqslant 0), \tag{23}$$

where $x : (-\infty, 0] \to \mathbb{K}$ is a continuously differentiable function of exponential order and $r : (-\infty, 0] \to \mathbb{K}$ is a continuous function of exponential order.

If we set $x_1(t) = x(-t)$ and $r_1(t) = r(-t)$ for all $t \geqslant 0$, then it follows from (23) that

$$\left|x_1'(t) - \lambda x_1(t) + r_1(t)\right| \leqslant \varepsilon \quad (\text{for } t \geqslant 0). \tag{24}$$

Since $x_1(t)$ is a continuously differentiable function of exponential order and $r_1(t)$ is a continuous function of exponential order, if we additionally assume that $\Re(\lambda) < 0$, then Theorem 7 and its proof imply that there exists a continuously differentiable function $y : [0, \infty) \to \mathbb{K}$ of exponential order which satisfies

$$y'(t) - \lambda y(t) + r_1(t) = 0 \tag{25}$$

and

$$|x_1(t) - y(t)| \leqslant \frac{1}{|\Re(\lambda)|}\varepsilon \tag{26}$$

for all $t \geqslant 0$.

If we define a function $z : (-\infty, 0] \to \mathbb{K}$ by $z(t) = y(-t)$ for each $t \leqslant 0$, then $z'(t) = -y'(-t)$ and so, by (25), we see that $z'(t) + \lambda z(t) = -y'(-t) + \lambda y(-t) = r(t)$ for any $t \leqslant 0$, *i.e.*, $z(t)$ satisfies the differential equation (2) for all $t \leqslant 0$. Moreover, it follows from (26) that

$$|x(t) - z(t)| \leqslant \frac{1}{|\Re(\lambda)|}\varepsilon$$

for all $t \leqslant 0$.

Putting all of the above facts together, we get the following theorem.

THEOREM 11. *Assume that $r : (-\infty, 0] \to \mathbb{K}$ is a continuous function of exponential order and λ is a constant with $\Re(\lambda) < 0$. The differential equation (2) has the Hyers-Ulam stability for the class of all continuously differentiable functions $x : (-\infty, 0] \to \mathbb{K}$ of exponential order.*

Assuming $r(t) \equiv 0$ in Theorem 11, we obtain the Hyers-Ulam stability of the differential equation (1) when the relevant domain is the set of all non-positive real numbers. Other types of Hyers-Ulam stability of the differential equation (2) can be similarly established when the relevant domain is the set of all non-positive real numbers.

7. Conclusions

In this paper, we proved the Hyers-Ulam stability, Hyers-Ulam ϕ-stability, Mittag-Leffler-Hyers-Ulam stability, and Mittag-Leffler-Hyers-Ulam ϕ-stability of the linear differential equations of first order with constant coefficients using Mahgoub transform method. In other words, we established sufficient criteria for the Hyers-Ulam stability of first-order linear differential equations with constant coefficients using the Mahgoub transform method.

Moreover, this paper provides a new method to investigate the Hyers-Ulam stability of differential equations. This is the first attempt to use the Mahgoub transformation to prove the Hyers-Ulam stability for linear differential equations of the first order. Furthermore, this paper shows that the Mahgoub transform method is more convenient for investigating the stability problems for linear differential equations with constant coefficients.

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