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ABSOLUTELY MONOTONIC FUNCTIONS INVOLVING THE COMPLETE ELLIPTIC INTEGRALS OF THE FIRST KIND WITH APPLICATIONS

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Abstract. Let $\mathcal{K}(r)$ be the complete elliptic integral of the first kind. In this paper, we prove that the function $F_p(x) = (1-x)^p \exp \mathcal{K}(\sqrt{x})$ is absolutely monotonic on $(0,1)$ if and only if $p \leqslant \pi/8$, and $-F_p'(x)$ is absolutely monotonic on $(0,1)$ if and only if $1/2 \leqslant p \leqslant (\pi+4+\sqrt{16-\pi})/8$. This generalizes a known result and gives several new inequalities involving the complete elliptic integral of the first kind.

1. Introduction

For real numbers a, b, and c with $-c \notin \mathbb{N} \cup \{0\}$, the Gaussian hypergeometric function is defined as

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}$$

for $x \in (-1,1)$, where $(a)_n$ denotes Pochhammer symbol defined by

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)},$$

for $n \in \mathbb{N}$ and $(a)_0 = 1$ for $a \neq 0$, here $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ $(x > 0)$ is the gamma function. $F(a,b;a+b;x)$ is called a zero-balanced hypergeometric function.

For later use, we list the behavior of the hypergeometric function near $x = 1$ as follows

$$\begin{cases} F(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \text{if } c > a+b, \\ F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x) & \text{if } c < a+b; \end{cases} \tag{1.1}$$

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in the case of $c = a+b$, $F(a,b;c;x)$ is called zero-balanced function, which satisfies the asymptotic relation

$$F(a,b;a+b;x) = \frac{R(a,b) - \ln(1-x)}{B(a,b)} + O((1-x)\ln(1-x)) \tag{1.2}$$

as $x \to 1$, where

$$B(z,w) = \frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}, \ \operatorname{Re}(z) > 0, \operatorname{Re}(w) > 0$$

is the classical beta function and

$$R(a,b) = -2\gamma - \psi(a) - \psi(b), \tag{1.3}$$

here $\psi(z) = \Gamma'(z)/\Gamma(z)$, $\operatorname{Re}(z) > 0$ is the psi function and γ is the Euler-Mascheroni constant.

The complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ of the first kind and second kind are defined on $(0,1)$ by

$$\mathscr{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1-r^2\sin^2 t}} dt,$$
$$\mathscr{E}(r) = \int_0^{\pi/2} \sqrt{1-r^2\sin^2 t}\, dt,$$

respectively. They can also be expressed by the Gaussian hypergeometric function

$$\mathscr{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2)_n^2}{(n!)^2} r^{2n}, \tag{1.4}$$

$$\mathscr{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(-1/2)_n (1/2)_n}{(n!)^2} r^{2n}. \tag{1.5}$$

By the asymptotic relation (1.2) it holds that

$$\mathscr{K}(r) \sim \ln \frac{4}{r'} \ \text{ as } r \to 1^-, \tag{1.6}$$

where and in what follows $r' = \sqrt{1-r^2}$. Due to the importance of the complete elliptic integrals in different branches of mathematics such as geometric function theory and quasiconformal mappings, a variety of properties of combinations of them and other elementary functions including monotonicity, convexity and inequalities have been widely studied, see for example, [1], [2], [3], [4], [5], [6], [7], [8], and recent papers [9], [10], [11], [12], [13], [14], [15], [16], [17], [18].

In 1992, Anderson, Vamanamurthy and Vuorinen [3, Conjecture 3.1 (6)] conjectured that

$$(r')^2 \leqslant \frac{r' \exp \mathscr{K}(r) - 4}{\exp(\pi/2) - 4} \leqslant \frac{2\sqrt{1-r}}{2-r} \tag{1.7}$$

for $r \in (0,1)$. This was solved by Qiu, Vamanamurthy and Vuorinen [6] and was improved as

$$\left(r'\right)^2 < \frac{r' \exp \mathscr{K}(r) - 4}{\exp(\pi/2) - 4} < r' < \frac{2\sqrt{1-r}}{2-r}$$

for $r \in (0,1)$. In 1996, Qiu and Vamanamurthy [6, Theorem 1.2] showed the function $r \mapsto r' \exp \mathscr{K}(r)$ is strictly decreasing and concave from $(0,1)$ onto $\left(4, e^{\pi/2}\right)$.

Recently, Yang, Qian and Chu [10, Theorem 3.1] proved that the function $r \mapsto (r')^p \exp \mathscr{K}(r)$ is strictly increasing on $(0,1)$ if and only if $p \leqslant \pi/4$ and strictly decreasing on $(0,1)$ if and only if $p \geqslant 1$.

Recall that a function f is called absolutely monotonic (AM, for short) on the interval I if it has nonnegative derivatives of all orders in the region, that is,

$$f^{(k)}(x) \geqslant 0 \text{ for } x \in I \text{ and } k = 0,1,2,\ldots$$

(see [19]). Clearly, if $f(x)$ is a power series converging on $(0,c)$ ($c > 0$), then $f(x)$ is AM on $(0,c)$ if and only if all coefficients of $f(x)$ are nonnegative.

The aim of this paper is to study the absolute monotonicity of the functions

$$F_p(x) = (1-x)^p e^{\mathscr{K}(\sqrt{x})} \tag{1.8}$$

and $\ln F_p(x)$ on $(0,1)$. Our results are contained in the following theorems.

THEOREM 1. *Let F_p be defined on $(0,1)$ by (1.8).*
(i) $-(\ln F_p)'$ is AM on $(0,1)$ if and only if $p \geqslant 1/2$.
(ii) F_p is AM on $(0,1)$ if and only if $p \leqslant \pi/8 = 0.392\ldots$.

THEOREM 2. *Let F_p be defined on $(0,1)$ by (1.8).*
(i) $-F_p'$ is AM on $(0,1)$ if and only if $1/2 \leqslant p \leqslant \left(\pi + 4 + \sqrt{16-\pi}\right)/8 = 1.340\ldots$.
(ii) F_p is AM on $(0,1)$ if and only if $p \leqslant \pi/8 = 0.392\ldots$.

REMARK 1. Taking $p = 1/2$ in Theorem 2, we immediately see that the function the function $F_p\left(r^2\right) = r' \exp \mathscr{K}(r)$ is strictly decreasing and concave from $(0,1)$ onto $\left(4, e^{\pi/2}\right)$. So Theorem 2 is a generalization of Qiu and Vamanamurthy's result in [6, Theorem 1.2].

2. Proofs of Theorems 1 and 2

For convenience, we use W_n to denote the Wallis ratio:

$$W_n = \frac{(2n-1)!!}{(2n)!!} = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}.$$

Clearly, W_n has the following properties:

(i) W_n satisfies the recurrence relation

$$W_{n+1} = \frac{n+1/2}{n+1} W_n; \tag{2.1}$$

(ii) W_n satisfies the inequality (see [20])

$$W_n < \frac{1}{\sqrt{\pi(n+1/4)}}. \tag{2.2}$$

Moreover, using the notation W_n, $\mathscr{K}(r)$ and $\mathscr{E}(r)$ can be represented as

$$\mathscr{K}(r) = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; r^2;\right) = \frac{\pi}{2} \sum_{n=0}^{\infty} W_n^2 r^{2n}, \tag{2.3}$$

$$\mathscr{E}(r) = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; r^2\right) = -\frac{\pi}{2} \sum_{n=0}^{\infty} \frac{W_n^2}{2n-1} r^{2n}. \tag{2.4}$$

We now establish the recurrence relations of coefficients of the power series of $F_p(x)$ and $\ln F_p(x)$.

LEMMA 1. *Let $x \in (0,1)$ and $p \in \mathbb{R}$. (i) $\ln F_p(x)$ has the power series representation*

$$\ln F_p(x) = \mathscr{K}\left(\sqrt{x}\right) + p\ln(1-x) = \sum_{n=0}^{\infty} b_n x^n, \tag{2.5}$$

with $b_0 = \pi/2$ and for $n \geqslant 1$,

$$b_n = \frac{1}{n}\left(\frac{\pi}{2} n W_n^2 - p\right). \tag{2.6}$$

(ii) $F_p(x)$ has the power series representation

$$F_p(x) = (1-x)^p e^{\mathscr{K}(\sqrt{x})} = \sum_{n=0}^{\infty} a_n x^n, \tag{2.7}$$

where the coefficients $a_n = a_n(p)$ satisfy: $a_0 = e^{\pi/2}$, $a_1 = (\pi/8 - p)e^{\pi/2}$ and for $n \geqslant 1$,

$$a_n = \frac{1}{n} \sum_{k=1}^{n} k b_k a_{n-k}. \tag{2.8}$$

Proof. (i) Using (1.4), (2.3) and

$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n},$$

we immediately get

$$\begin{aligned}\ln F_p(x) = \mathscr{K}\left(\sqrt{x}\right) + p\ln(1-x) &= \frac{\pi}{2}\sum_{n=0}^{\infty} W_n^2 x^n - p\sum_{n=1}^{\infty}\frac{x^n}{n} \\ &= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(\frac{\pi}{2} n W_n^2 - p\right)\frac{x^n}{n},\end{aligned}$$

which implies the first assertion.

(ii) It is clear that

$$F_p(x) = \exp\left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} a_n x^n.$$

Now, differentiation yields

$$\left(\sum_{n=0}^{\infty} n b_n x^{n-1}\right) \exp\left(\sum_{n=0}^{\infty} b_n x^n\right) = \left(\sum_{n=0}^{\infty} n b_n x^{n-1}\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} n a_n x^{n-1},$$

then multiplying x gives

$$\left(\sum_{n=0}^{\infty} n b_n x^n\right)\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=0}^{\infty} n a_n x^n.$$

Using Cauchy product formula and comparing the coefficients of x^n, we have

$$\sum_{k=0}^{n} k b_k a_{n-k} = n a_n,$$

which implies (2.8). This completes the proof. □

LEMMA 2. *Let b_n be defined by (2.6) for $n \in \mathbb{N}$. Then (i) $b_n < 0$ if $p \geqslant 1/2$ and $b_n > 0$ if $p \leqslant \pi/8$; (ii) the sequence $\{(n+1)b_{n+1}/(nb_n)\}_{n\geqslant 1}$ is increasing if $p \geqslant 1/2$.*

Proof. To prove the desired results, we need to prove the sequence $\{nW_n^2\}_{n\geqslant 1}$ is strictly increasing with

$$\frac{1}{4} \leqslant nW_n^2 < \lim_{n\to\infty}\left(nW_n^2\right) = \frac{1}{\pi}.$$

In fact, using the recurrence relation (2.1), we have

$$(n+1)W_{n+1}^2 - nW_n^2 = (n+1)\left(\frac{n+1/2}{n+1}\right)^2 W_n^2 - nW_n^2 = \frac{1}{4}\frac{W_n^2}{n+1} > 0,$$

which implies that the sequence $\{nW_n^2\}_{n\geqslant 1}$ is strictly increasing. It follows that

$$\frac{1}{4} = 1 \times W_1^2 \leqslant nW_n^2 < \lim_{n\to\infty}\left(nW_n^2\right) = \frac{1}{\pi},$$

where the limit relation holds due to

$$nW_n^2 = n\left[\frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}\right]^2 \sim \frac{1}{\pi} n \times \left(n^{1/2-1}\right)^2 = \frac{1}{\pi},$$

as $n \to \infty$.

Now, with the aid of the above fact, we prove the desired results.

(i) If $p \geqslant 1/2$ then

$$nb_n = \frac{\pi}{2} nW_n^2 - p \leqslant \frac{\pi}{2}\frac{1}{\pi} - \frac{1}{2} = 0.$$

If $p \leqslant \pi/8$, then

$$nb_n = \frac{\pi}{2} nW_n^2 - p \geqslant \frac{\pi}{2}\frac{1}{4} - \frac{\pi}{8} = 0.$$

(ii) To show the sequence $\{(n+1)b_{n+1}/(nb_n)\}_{n\geqslant 1}$ is increasing if $p \geqslant 1/2$, it suffices to check that $(n+1)b_{n+1}(n-1)b_{n-1} - (nb_n)^2 > 0$ for $n \geqslant 2$. Using the recurrence relation (2.1) and simplifying yield

$$\begin{aligned}
&(n+1)b_{n+1}(n-1)b_{n-1} - (nb_n)^2 \\
&= \left[\frac{\pi}{2}(n+1)\left(\frac{n+1/2}{n+1}\right)^2 W_n^2 - p\right]\left[\frac{\pi}{2}(n-1)\left(\frac{n}{n-1/2}\right)^2 W_n^2 - p\right] - \left(\frac{\pi}{2}nW_n^2 - p\right)^2 \\
&= \frac{\pi}{8}\frac{W_n^2}{(n+1)(2n-1)^2}\vartheta_n,
\end{aligned}$$

where

$$\vartheta_n = (8n-1)p - 4\pi n^2 W_n^2.$$

Utilizing the known Wallis inequality (2.2) and $2p \geqslant 1$, we get

$$\vartheta_n > (8n-1)\frac{1}{2} - 4\pi n^2 \frac{1}{\pi(n+1/4)} = \frac{1}{2}\frac{4n-1}{4n+1} > 0,$$

which proves the monotonicity of the sequence $\{(n+1)b_{n+1}/(nb_n)\}_{n\geqslant 1}$. □

We are now in a position to prove Theorems 1 and 2.

Proof of Theorem 1. (i) If $-(\ln F_p)'$ is AM on $(0,1)$, then $b_n \leqslant 0$ for all $n \geqslant 1$. This yields

$$\lim_{n\to\infty}(nb_n) = \lim_{n\to\infty}\left(\frac{\pi}{2}nW_n^2 - p\right) = \frac{1}{2} - p \leqslant 0,$$

that is, $p \geqslant 1/2$. Conversely, when $p \geqslant 1/2$, by (i) of Lemma 2 we see that $nb_n < 0$ for all $n \geqslant 1$, which implies so is b_n.

(ii) If $\ln F_p$ is AM on $(0,1)$, then $b_1 = \pi/8 - p \geqslant 0$, that is, $p \leqslant \pi/8$. Conversely, if $p \leqslant \pi/8$ then $nb_n \geqslant 0$ for all $n \geqslant 1$ due to (i) of Lemma 2, thereby completing the proof. □

Proof of Theorem 2. (i) The recurrence relation (2.8) can be written as

$$a_n = b_n a_0 + \frac{1}{n}\sum_{k=1}^{n-1} k b_k a_{n-k}. \tag{2.9}$$

Then

$$a_{n+1} = b_{n+1} a_0 + \frac{1}{n+1}\sum_{k=0}^{n-1}(k+1) b_{k+1} a_{n-k}. \tag{2.10}$$

Eliminating a_0 from the above two relations, we get

$$a_{n+1}-\frac{b_{n+1}}{b_n}a_n=\frac{1}{n+1}\sum_{k=0}^{n-1}(k+1)b_{k+1}a_{n-k}-\frac{1}{n}\frac{b_{n+1}}{b_n}\sum_{k=1}^{n-1}kb_ka_{n-k}$$
$$=\frac{1}{n+1}b_1a_n+\sum_{k=1}^{n-1}\left(\frac{(k+1)b_{k+1}}{n+1}-\frac{b_{n+1}}{nb_n}kb_k\right)a_{n-k},$$

which, by an arrangement, gives

$$a_{n+1}=\left(\frac{b_{n+1}}{b_n}+\frac{b_1}{n+1}\right)a_n+\frac{1}{n+1}\sum_{k=1}^{n-1}\left(\frac{(k+1)b_{k+1}}{kb_k}-\frac{(n+1)b_{n+1}}{nb_n}\right)kb_ka_{n-k}. \tag{2.11}$$

Now, if $-F_p'$ is AM on $(0,1)$, then

$$F_p'(x)=\left[(1-x)\frac{\pi}{8}F\left(\frac{3}{2},\frac{3}{2};2;x\right)-p\right](1-x)^{p-1}e^{K(\sqrt{x})}\leqslant 0$$

for all $x\in(0,1)$. This yields

$$p\geqslant\frac{\pi}{8}(1-x)F\left(\frac{3}{2},\frac{3}{2};2;x\right)$$

for all $x\in(0,1)$. By the formulas (1.1) we have

$$p\geqslant\frac{\pi}{8}\sup_{x\in(0,1)}\left[F\left(\frac{1}{2},\frac{1}{2};2;x\right)\right]=\frac{1}{2}.$$

which gives the first necessary condition. The second follows from

$$a_1=\left(\frac{\pi}{8}-p\right)e^{\pi/2}<0 \text{ and } a_2=\frac{64p^2-16(\pi+4)p+\pi^2+9\pi}{128}e^{\pi/2}\leqslant 0,$$

which implies $p\leqslant\left(\pi+4+\sqrt{16-\pi}\right)/8=1.340\ldots$

Assume that $1/2\leqslant p\leqslant\left(\pi+4+\sqrt{16-\pi}\right)/8$. We have known that $a_1,a_2<0$. Suppose that $a_n\leqslant 0$ for $1\leqslant n\leqslant m$. We prove $a_{m+1}\leqslant 0$ by induction. Let us return to the recurrence relation (2.11).

By Lemma 2, we see that $((k+1)b_{k+1}/(kb_k)-(n+1)b_{n+1}/(nb_n))<0$ and $kb_k<0$ for $1\leqslant k\leqslant n-1$. If we check $(b_{n+1}/b_n+b_1/(n+1))>0$ for $2\leqslant n\leqslant m$, then $a_{m+1}<0$. In fact, we have

$$\frac{b_{n+1}}{b_n}+\frac{b_1}{n+1}=\frac{n}{n+1}\frac{(n+1)b_{n+1}}{nb_n}+\frac{b_1}{n+1}>\frac{n}{n+1}\frac{3b_3}{2b_2}+\frac{b_1}{n+1}$$
$$=\frac{1}{n+1}\left(n\frac{3b_3}{2b_2}+b_1\right)>\frac{1}{n+1}\left[2\frac{75\pi/512-p}{9\pi/64-p}+\left(\frac{1}{8}\pi-p\right)\right]$$
$$=-\frac{512p^2-8p(17\pi+128)+3\pi(3\pi+50)}{8(64p-9\pi)(n+1)}>0$$

due to $1/2 \leqslant p \leqslant \left(\pi+4+\sqrt{16-\pi}\right)/8$. The sufficiency thus follows.

(ii) The necessary condition for F_p to be AM on $(0,1)$ follows from $a_1 = (\pi/8 - p)e^{\pi/2} \geqslant 0$. Suppose that $p \leqslant \pi/8$. A direct verification gives $a_0 = e^{\pi/2} > 0$,

$$a_2 = \frac{64p^2 - 16(\pi+4)p + \pi^2 + 9\pi}{128} e^{\pi/2} > 0.$$

Assume that $a_n \geqslant 0$ for $0 \leqslant n \leqslant m$. By Lemma 2, we see that $b_k \geqslant 0$ for $p \leqslant \pi/8$ and $k \geqslant 1$, by the recurrence formula (2.8) we immediately get $a_{m+1} > 0$. Using the induction we arrive at $a_n \geqslant 0$ for all $n \geqslant 0$.

This completes the proof. □

3. Several functional inequalities

As applications, we give several functional inequalities involving the complete elliptic integral of the first kind $\mathscr{K}(r)$.

PROPOSITION 1. *Let $a_k = a_k(p)$ for $k \geqslant 0$ be defined by (2.8) with $a_0 = e^{\pi/2}$. The double inequality*

$$\ln \frac{\sum_{k=0}^{n} a_k(q) r^{2k}}{(r')^{2q}} < \mathscr{K}(r) < \ln \frac{\sum_{k=0}^{n} a_k(p) r^{2k}}{(r')^{2p}} \tag{3.1}$$

holds for $r \in (0,1)$ if $1/2 \leqslant p \leqslant \left(\pi+4+\sqrt{16-\pi}\right)/8$ and $q \leqslant \pi/8$.

Proof. By Theorem 2 the inequality

$$G_{p,n}(x) := (1-x)^p e^{\mathscr{K}(\sqrt{x})} - \sum_{k=0}^{n} a_k(p) x^k = \sum_{k=n+1}^{\infty} a_k(p) x^k < 0 \tag{3.2}$$

holds for $x \in (0,1)$ if $1/2 \leqslant p \leqslant \left(\pi+4+\sqrt{16-\pi}\right)/8$. Set $x = r^2$. Then this inequality implies the second one of (3.1). Similarly, the inequality $G_{q,n}(x) > 0$ for $x \in (0,1)$ if $q \leqslant \pi/8$ implies the first one of (3.1). This completes the proof. □

REMARK 2. Taking $n = 0$ in the double inequality (3.1) yields

$$\frac{\pi}{2} + 2q \ln \frac{1}{r'} < \mathscr{K}(r) < \frac{\pi}{2} + 2p \ln \frac{1}{r'} \tag{3.3}$$

for $r \in (0,1)$ with the best constants $p = 1/2$ and $q = \pi/8$. This result was proven in [10, Corollary 3.6]. Evidently, Proposition (1) is a generalization of [10, Corollary 3.6].

Note that the function $x \mapsto x^{-n-1} G_{1/2,n}(x)$ is strictly decreasing with

$$\lim_{x\to 0} \frac{G_{1/2,n}(x)}{x^{n+1}} = a_{n+1}\left(\frac{1}{2}\right) \text{ and } \lim_{x\to 1^-} \frac{G_{1/2,n}(x)}{x^{n+1}} = 4 - \sum_{k=0}^{n} a_k\left(\frac{1}{2}\right).$$

We have

$$\left[4-\sum_{k=0}^{n} a_k\left(\frac{1}{2}\right)\right] x^{n+1} < \sqrt{1-x}e^{\mathscr{K}(\sqrt{x})} - \sum_{k=0}^{n} a_k\left(\frac{1}{2}\right) x^k < a_{n+1}\left(\frac{1}{2}\right) x^{n+1}$$

for $x \in (0,1)$. We obtain therefore the following proposition.

PROPOSITION 2. *Let $a_k = a_k(p)$ for $k \geqslant 0$ be defined by (2.8) with $a_0 = e^{\pi/2}$. The double inequality*

$$\ln \frac{\sum_{k=0}^{n} a_k(1/2) r^{2k} + [4-\sum_{k=0}^{n} a_k(1/2)] r^{2n+2}}{r'} < \mathscr{K}(r) < \ln \frac{\sum_{k=0}^{n+1} a_k(1/2) r^{2k}}{r'}$$

holds for $r \in (0,1)$. The lower and upper bounds are sharp.

REMARK 3. Taking $n = 0$ in Proposition 2 yields

$$\ln \frac{e^{\pi/2} + \left(4-e^{\pi/2}\right) r^2}{r'} < \mathscr{K}(r) < \ln \frac{e^{\pi/2} + (\pi/8-1/2) e^{\pi/2} r^2}{r'} \tag{3.4}$$

for $r \in (0,1)$. The left hand side inequality of (3.4) was proven in [6, Eq. (1.4)], while the right hand side one seems to be a newcomer. Furthermore, we have the following corollary.

COROLLARY 1. *Let $\alpha, \beta \geqslant 0$. The double inequality*

$$\ln\left(\frac{\alpha}{r'} + \left(e^{\pi/2}-\alpha\right) r'\right) < \mathscr{K}(r) < \ln\left(\frac{\beta}{r'} + \left(e^{\pi/2}-\beta\right) r'\right) \tag{3.5}$$

holds for $r \in (0,1)$ if and only if $\alpha \leqslant 4$ and $\beta \geqslant \beta_0 = (\pi+4) e^{\pi/2}/8 = 4.294\ldots$.

Proof. Necessity. The necessary conditions for the double inequality (3.5) to hold for $r \in (0,1)$ can be deduced from the following limit relations

$$\lim_{r\to 1^-} \left[\ln\left(\frac{\alpha}{r'} + \left(e^{\pi/2}-\alpha\right) r'\right) - \mathscr{K}(r)\right] \leqslant 0, \tag{3.6}$$

$$\lim_{r\to 0} \frac{\mathscr{K}(r) - \ln\left(\beta/r' + \left(e^{\pi/2}-\beta\right) r'\right)}{r^2} \leqslant 0. \tag{3.7}$$

Using the asymptotic formula (1.6) we have, as $r \to 1^-$

$$\ln\left(\frac{\alpha}{r'} + \left(e^{\pi/2}-\alpha\right) r'\right) - \mathscr{K}(r) \sim \ln\left(\frac{\alpha}{r'} + \left(e^{\pi/2}-\alpha\right) r'\right) - \ln\frac{4}{r'}$$

$$= \ln\left(\frac{\alpha}{4} + \frac{e^{\pi/2}-\alpha}{4}\left(r'\right)^2\right) \to \ln\frac{\alpha}{4}.$$

This together with (3.6) implies $\alpha \leqslant 4$. Expanding in power series leads to

$$\mathscr{K}(r) - \ln\left(\beta/r' + \left(e^{\pi/2} - \beta\right) r'\right) = \frac{\pi}{2}\left(1 + \frac{1}{4}r^2\right) - \frac{\pi}{2} - \left(\beta e^{-\pi/2} - \frac{1}{2}\right) r^2 + O\left(r^4\right),$$

as $r \to 0$, which gives

$$\lim_{r\to 0} \frac{\mathscr{K}(r) - \ln\left(\beta/r' + \left(e^{\pi/2} - \beta\right) r'\right)}{r^2} = -\left(\beta - \frac{\pi+4}{8} e^{\pi/2}\right) e^{-\pi/2}.$$

This in combination with (3.7) indicates $\beta \geqslant (\pi+4) e^{\pi/2}/8 = \beta_0$.

Sufficiency. Since the function $\alpha \mapsto \ln\left(\alpha/r' + \left(e^{\pi/2} - \alpha\right) r'\right)$ is increasing on $[0,\infty)$, it suffices to prove the double inequality holds for $r \in (0,1)$ when $\alpha = 4$ and $\beta = \beta_0$. Since the double inequality (3.4) can be written as

$$\ln\left(\frac{4}{r'} + \left(e^{\pi/2} - 4\right) r'\right) < \mathscr{K}(r) < \ln\left(\frac{\beta_0}{r'} + \left(e^{\pi/2} - \beta_0\right) r'\right),$$

the sufficiency follows. This completes the proof. □

A function $f : (a,\infty) \to \mathbb{R}$ is said to be superadditive if

$$f(x) + f(y) \leqslant f(x+y) \text{ for } x, y \in (a,\infty). \tag{3.8}$$

If $-f$ is superadditive, then f is called subadditive on (a,∞) (see [21]). Petrović [22] showed that every convex function $f : [0,\infty) \to \mathbb{R}$ satisfies a functional inequality

$$f(x) + f(y) \leqslant f(0) + f(x+y) \text{ for } x, y \in [0,\infty) \tag{3.9}$$

(see [22]). In fact, this conclusion is also true for any interval $[0,a)$ $(a > 0)$. Let $G_{p,n}(x)$ be as in (3.2). It is clear that

$$\lim_{x\to 0} G_{p,n}(x) = 0.$$

By Theorem 2 we immediately get the double inequality

$$2G_{p,n}\left(\frac{x+y}{2}\right) < G_{p,n}(x) + G_{p,n}(y) < G_{p,n}(x+y)$$

for $x, y, x+y \in (0,1)$ if $p \leqslant \pi/8$. The double inequality is reversed if $1/2 \leqslant p \leqslant \left(\pi + 4 + \sqrt{16-\pi}\right)/8$.

PROPOSITION 3. *If $p \leqslant \pi/8$, then the double inequality*

$$\begin{aligned} &2\left(1 - \frac{x+y}{2}\right)^p e^{\mathscr{K}\left(\sqrt{(x+y)/2}\right)} + 2\textstyle\sum_{k=0}^n a_k(p)\left[\dfrac{x^k+y^k}{2} - \left(\dfrac{x+y}{2}\right)^k\right] \\ &< (1-x)^p e^{\mathscr{K}(\sqrt{x})} + (1-y)^p e^{\mathscr{K}(\sqrt{y})} \\ &< (1-x-y)^p e^{\mathscr{K}(\sqrt{x+y})} + \textstyle\sum_{k=0}^n a_k(p)\left[x^k + y^k - (x+y)^k\right] \end{aligned} \tag{3.10}$$

holds for $x, y, x+y \in (0,1)$. The inequalities (3.10) are reversed if

$$1/2 \leqslant p \leqslant \left(\pi + 4 + \sqrt{16-\pi}\right)/8.$$

Taking $n=0$ in Proposition 3 and noting $a_0(p)=e^{\pi/2}$, we have

COROLLARY 2. *If $p \leqslant \pi/8$, then the double inequality*

$$2\left(1-\frac{x+y}{2}\right)^p e^{\mathscr{K}\left(\sqrt{(x+y)/2}\right)} < (1-x)^p e^{\mathscr{K}(\sqrt{x})} + (1-y)^p e^{\mathscr{K}\left(\sqrt{y}\right)}$$
$$< (1-x-y)^p e^{\mathscr{K}(\sqrt{x+y})} + e^{\pi/2}$$

holds for $x,y,x+y \in (0,1)$. It is reversed if $1/2 \leqslant p \leqslant \left(\pi+4+\sqrt{16-\pi}\right)/8$.

REMARK 4. Putting $p=1/2$ and letting $y=1-r^2$, $x \to r^2$ in the above corollary we obtain

$$4+e^{\pi/2} < r' e^{\mathscr{K}(r)} + r e^{\mathscr{K}\left(r'\right)} < \sqrt{2} e^{\mathscr{K}\left(1/\sqrt{2}\right)} = \sqrt{2}\exp\left(\frac{\Gamma(1/4)^2}{4\sqrt{\pi}}\right) \tag{3.11}$$

for $r \in (0,1)$, here we have used the asymptotic formula (1.6) and the identity

$$\mathscr{K}\left(\frac{1}{\sqrt{2}}\right) = \sqrt{2}\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}.$$

Clearly, the lower and upper bounds are sharp.

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