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ON THE BOUNDS OF SCALING FACTORS OF AFFINE FRACTAL INTERPOLATION FUNCTIONS

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Abstract. In this paper we obtain an upper bound and a lower bound for each vertical scaling factor s_k of an iterated function system so that the obtained affine fractal interpolation function f_Δ has the property that $R(x)-d \leqslant f_\Delta(x) \leqslant R(x)+D$ for all $x \in I$, where D and d are given positive constants and $R(x) = mx+c$ is a given linear function on I. As an example, we consider the case that the graph of R is the regression line that fits the given data points by least square method.

1. Introduction

Interpolation methods are important techniques to reconstruct a continuous function from a given data set. Fractal interpolation is a modern technique and has been applied to construct irregular and non-smooth approximants. The concept of fractal interpolation functions was first introduced by Barnsley ([1], [2]) and has been developed by many researchers. Interested readers are referred to the survey article [3].

Parameter identification is one of the problems in the theory of fractal interpolations. In [4] Dalla and Drakopoulos gave ranges of vertical scaling factors to ensure that the graphs of affine fractal interpolation functions are contained in a given rectangle. In [5] and [6] the authors investigated methods for determining the vertical scaling factors such that the resulting fractal function provides a good fit to the given data set. In [7], [8], and [9] the authors discussed sufficient conditions on the scaling factors and shape parameters for preserving positivity, monotonicity, and convexity (concavity) through rational cubic fractal interpolation functions. Similar results with bounds on the scaling factors were obtained in [10] for α-fractal functions.

Here is a brief introduction to the construction of affine fractal interpolation functions, see [2] and [4] for more details. Consider the set of points $\Delta = \{(x_k, y_k) \in \mathbb{R}\times\mathbb{R} : k = 0, 1, \cdots, N\}$, where $N \geqslant 2$ is a positive integer and $x_0 < x_1 < x_2 < \cdots < x_N$. Denote $I = [x_0, x_N]$ and $I_k = [x_{k-1}, x_k]$ for each $k = 1, \cdots, N$. Let $L_k(x) = a_k x + b_k$ such

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that $L_k(x_0) = x_{k-1}$, $L_k(x_N) = x_k$. Define a function $M_k : I \times \mathbb{R} \to \mathbb{R}$ with $M_k(x,y) = c_k x + s_k y + e_k$ such that $M_k(x_0,y_0) = y_{k-1}$, $M_k(x_N,y_N) = y_k$. By [2] we see that

$$L_k(x) = \left(\frac{x_k - x_{k-1}}{x_N - x_0}\right)x + \left(\frac{x_N x_{k-1} - x_0 x_k}{x_N - x_0}\right), \tag{1.1}$$

$$M_k(x,y) = \left(\frac{y_k - y_{k-1}}{x_N - x_0} - \frac{s_k(y_N - y_0)}{x_N - x_0}\right)x + s_k y + \left(\frac{x_N y_{k-1} - x_0 y_k}{x_N - x_0} - s_k \frac{x_N y_0 - x_0 y_N}{x_N - x_0}\right). \tag{1.2}$$

If $-1 < s_k < 1$ for $k = 1,\cdots,N$, the iterated function system $\{I \times \mathbb{R}; W_1,\cdots,W_N\}$ admits a unique attractor $G \subseteq I \times \mathbb{R}$, where $W_k(x,y) = (L_k(x), M_k(x,y))$, and G is the graph of a continuous function $f_\Delta : I \to \mathbb{R}$ which satisfies $f_\Delta(x_k) = y_k$ for $k = 0,\cdots,N$. The function f_Δ is called an affine fractal interpolation function [4] corresponding to the system and the set of points Δ. Note that f_Δ satisfies the functional equation

$$f_\Delta(x) = \left(\frac{y_k - y_{k-1}}{x_N - x_0} - \frac{s_k(y_N - y_0)}{x_N - x_0}\right)L_k^{-1}(x) + s_k f_\Delta(L_k^{-1}(x)) + \left(\frac{x_N y_{k-1} - x_0 y_k}{x_N - x_0} - s_k \frac{x_N y_0 - x_0 y_N}{x_N - x_0}\right), \quad x \in I_k. \tag{1.3}$$

Here L_k^{-1}, the inverse function of L_k, is given by

$$L_k^{-1}(x) = \left(\frac{x_N - x_0}{x_k - x_{k-1}}\right)x - \left(\frac{x_N x_{k-1} - x_0 x_k}{x_k - x_{k-1}}\right). \tag{1.4}$$

It is known that the values of vertical scaling factors s_k, $k = 1,\cdots,N$ have main effect on the graph of an affine fractal interpolation function. In [4], Dalla and Drakopoulos obtained a range for each s_k so that the graph of the obtained affine fractal interpolation function remains within a given rectangle $I \times [a,b]$. Similar ideas have been extended and applied to problems of shape preservation through fractal interpolation functions. See [7], [8], [9], and [10].

In this paper we apply the idea given in [4, Theorem 3] and [10, Theorem 3.1] to obtain an upper bound and a lower bound for each s_k so that the obtained affine fractal interpolation function f_Δ satisfies the condition $R(x) - d \leqslant f_\Delta(x) \leqslant R(x) + D$ for all $x \in I$, where D and d are given positive constants and $R(x) = mx + c$ is a linear function on I. We also consider three particular cases as examples: the graph of R is a horizontal line, the straight line that passes through (x_0,y_0) and (x_N,y_N), and the regression line that fits the data points in Δ by least square method.

Throughout this paper, N is an integer greater or equal to 2 and $\Delta = \{(x_i,y_i) \in \mathbb{R} \times \mathbb{R} : i = 0,1,\cdots,N\}$ is a given set of points, where $x_0 < x_1 < x_2 < \cdots < x_N$. We also suppose that all the data points in Δ are non-collinear. Let $I = [x_0,x_N]$ and $I_k = [x_{k-1},x_k]$ for $k = 1,\cdots,N$. Suppose that f_Δ is an affine fractal interpolation function that satisfies (1.3) and $R(x) = mx + c$ is the given linear function on I. Define $\varepsilon_k = y_k - R(x_k)$ for $k = 0,\cdots,N$ and let D and d be positive constants such that $-d \leqslant \varepsilon_k \leqslant D$ for $k = 0,\cdots,N$.

2. Bounds on the scaling factors

In this section we establish an upper bound and a lower bound for each s_k so that $R(x)-d \leqslant f_\Delta(x) \leqslant R(x)+D$ for all $x \in I$.

Let ℓ be the function of the straight line that passes through (x_0,y_0) and (x_N,y_N) and let ℓ_k be the function of the straight line that passes through (x_{k-1},y_{k-1}) and (x_k,y_k). Then

$$\ell(x) = y_0 + \left(\frac{y_N-y_0}{x_N-x_0}\right)(x-x_0), \qquad x \in I,$$
$$\ell_k(x) = y_{k-1} + \left(\frac{y_k-y_{k-1}}{x_k-x_{k-1}}\right)(x-x_{k-1}), \qquad x \in I_k.$$

By (1.3) we rewrite $f_\Delta(x)$ in the form

$$\begin{aligned} f_\Delta(L_k(x)) = &\left(\frac{x-x_0}{x_N-x_0}\right)y_k + \left(\frac{x_N-x}{x_N-x_0}\right)y_{k-1} + s_k f_\Delta(x) \\ &-s_k\left\{\left(\frac{x-x_0}{x_N-x_0}\right)y_N + \left(\frac{x_N-x}{x_N-x_0}\right)y_0\right\}, \quad x \in I. \end{aligned} \tag{2.1}$$

For $x \in I$, equation (2.1) implies

$$f_\Delta(L_k(x)) - \ell_k(L_k(x)) = s_k(f_\Delta(x)-\ell(x)), \tag{2.2}$$

and we have

$$\begin{aligned} &f_\Delta(L_k(x)) - R(L_k(x)) \\ = \ &s_k(f_\Delta(x)-R(x)) + \ell_k(L_k(x)) - R(L_k(x)) + s_k(R(x)-\ell(x)). \end{aligned} \tag{2.3}$$

Consider the following condition:

$$\begin{aligned} &x \in I \text{ and } -d \leqslant y-R(x) \leqslant D \\ \Rightarrow &-d \leqslant s_k(y-R(x)) + \ell_k(L_k(x)) - R(L_k(x)) + s_k(R(x)-\ell(x)) \leqslant D. \end{aligned} \tag{2.4}$$

It is easy to verify that if condition (2.4) is satisfied, we have

$$-d \leqslant f_\Delta(x) - R(x) \leqslant D, \qquad x \in I. \tag{2.5}$$

For $k = 1, \cdots, N$, define

$$\alpha_k = \min\{\varepsilon_{k-1} - s_k\varepsilon_0, \varepsilon_k - s_k\varepsilon_N\} \text{ and } \beta_k = \max\{\varepsilon_{k-1} - s_k\varepsilon_0, \varepsilon_k - s_k\varepsilon_N\}.$$

In the following we investigate bounds on each s_k so that condition (2.4) is satisfied. Suppose that $x \in I$ and $-d \leqslant y-R(x) \leqslant D$. Since $\ell_k(L_k(x)) - R(L_k(x)) + s_k(R(x) - \ell(x))$ is a linear function,

$$\alpha_k \leqslant \ell_k(L_k(x)) - R(L_k(x)) + s_k(R(x)-\ell(x)) \leqslant \beta_k. \tag{2.6}$$

We first consider the case $0 \leqslant s_k < 1$. Since $-s_k d \leqslant s_k(y-R(x)) \leqslant s_k D$, by (2.6),

$$\alpha_k - s_k d \leqslant s_k(y-R(x)) + \ell_k(L_k(x)) - R(L_k(x)) + s_k(R(x)-\ell(x)) \leqslant \beta_k + s_k D. \quad (2.7)$$

If $\beta_k + s_k D \leqslant D$ and $-d \leqslant \alpha_k - s_k d$, then condition (2.4) holds. The condition $\beta_k + s_k D \leqslant D$ is equivalent to

$$\varepsilon_{k-1} - s_k\varepsilon_0 + s_k D \leqslant D \text{ and } \varepsilon_k - s_k\varepsilon_N + s_k D \leqslant D \quad (2.8)$$

and (2.8) holds under one of the following cases.

(a) $\varepsilon_0 = \varepsilon_N = D$.

(b) $\varepsilon_0 = D$, $\varepsilon_N < D$, $0 \leqslant s_k \leqslant \frac{D-\varepsilon_k}{D-\varepsilon_N}$,

(c) $\varepsilon_0 < D$, $\varepsilon_N = D$, $0 \leqslant s_k \leqslant \frac{D-\varepsilon_{k-1}}{D-\varepsilon_0}$,

(d) $\varepsilon_0 < D$, $\varepsilon_N < D$, $0 \leqslant s_k \leqslant \min\left\{\frac{D-\varepsilon_{k-1}}{D-\varepsilon_0}, \frac{D-\varepsilon_k}{D-\varepsilon_N}\right\}$.

Similarly, the condition $-d \leqslant \alpha_k - s_k d$ is equivalent to

$$-d \leqslant \varepsilon_{k-1} - s_k\varepsilon_0 - s_k d \text{ and } -d \leqslant \varepsilon_k - s_k\varepsilon_N - s_k d \quad (2.9)$$

and (2.9) holds under one of the following cases.

(e) $\varepsilon_0 = \varepsilon_N = -d$.

(f) $\varepsilon_0 = -d$, $\varepsilon_N > -d$, $0 \leqslant s_k \leqslant \frac{d+\varepsilon_k}{d+\varepsilon_N}$,

(g) $\varepsilon_0 > -d$, $\varepsilon_N = -d$, $0 \leqslant s_k \leqslant \frac{d+\varepsilon_{k-1}}{d+\varepsilon_0}$,

(h) $\varepsilon_0 > -d$, $\varepsilon_N > -d$, $0 \leqslant s_k \leqslant \min\left\{\frac{d+\varepsilon_{k-1}}{d+\varepsilon_0}, \frac{d+\varepsilon_k}{d+\varepsilon_N}\right\}$.

Therefore if

$$0 \leqslant s_k \leqslant \min\left\{\frac{D-\varepsilon_{k-1}}{D-\varepsilon_0}, \frac{D-\varepsilon_k}{D-\varepsilon_N}, \frac{d+\varepsilon_{k-1}}{d+\varepsilon_0}, \frac{d+\varepsilon_k}{d+\varepsilon_N}\right\}, \quad (2.10)$$

the condition (2.4) holds. Here we take $a/0 = \infty$ for any $a \geqslant 0$. Now consider the case $-1 < s_k \leqslant 0$. Since $s_k D \leqslant s_k(y-R(x)) \leqslant -s_k d$, we have

$$\alpha_k + s_k D \leqslant s_k(y-R(x)) + \ell_k(L_k(x)) - R(L_k(x)) + s_k(R(x)-\ell(x)) \leqslant \beta_k - s_k d. \quad (2.11)$$

Therefore if $\beta_k - s_k d \leqslant D$ and $-d \leqslant \alpha_k + s_k D$, then condition (2.4) holds. The condition $\beta_k - s_k d \leqslant D$ is equivalent to

$$\varepsilon_{k-1} - s_k\varepsilon_0 - s_k d \leqslant D \text{ and } \varepsilon_k - s_k\varepsilon_N - s_k d \leqslant D \quad (2.12)$$

and (2.12) holds under one of the following cases.

(i) $\varepsilon_0 = \varepsilon_N = -d$.

(j) $\varepsilon_0 = -d$, $\varepsilon_N > -d$, $\frac{-(D-\varepsilon_k)}{d+\varepsilon_N} \leqslant s_k \leqslant 0$,

(k) $\varepsilon_0 > -d$, $\varepsilon_N = -d$, $\frac{-(D-\varepsilon_{k-1})}{d+\varepsilon_0} \leqslant s_k \leqslant 0$,

(l) $\varepsilon_0 > -d$, $\varepsilon_N > -d$, $\max\left\{\dfrac{-(D-\varepsilon_k)}{d+\varepsilon_N}, \dfrac{-(D-\varepsilon_{k-1})}{d+\varepsilon_0}\right\} \leqslant s_k \leqslant 0$.

Similarly, the condition $-d \leqslant \alpha_k + s_k D$ is equivalent to

$$-d \leqslant \varepsilon_{k-1} - s_k\varepsilon_0 + s_k D \text{ and } -d \leqslant \varepsilon_k - s_k\varepsilon_N + s_k D \tag{2.13}$$

and (2.13) holds under one of the following cases.

(m) $\varepsilon_0 = \varepsilon_N = D$.

(n) $\varepsilon_0 = D$, $\varepsilon_N < D$, $\frac{-(d+\varepsilon_k)}{D-\varepsilon_N} \leqslant s_k \leqslant 0$,

(o) $\varepsilon_0 < D$, $\varepsilon_N = D$, $\frac{-(d+\varepsilon_{k-1})}{D-\varepsilon_0} \leqslant s_k \leqslant 0$,

(p) $\varepsilon_0 < D$, $\varepsilon_N < D$, $\max\left\{\dfrac{-(d+\varepsilon_k)}{D-\varepsilon_N}, \dfrac{-(d+\varepsilon_{k-1})}{D-\varepsilon_0}\right\} \leqslant s_k \leqslant 0$.

Therefore if

$$\max\left\{\frac{-(D-\varepsilon_{k-1})}{d+\varepsilon_0}, \frac{-(D-\varepsilon_k)}{d+\varepsilon_N}, \frac{-(d+\varepsilon_{k-1})}{D-\varepsilon_0}, \frac{-(d+\varepsilon_k)}{D-\varepsilon_N}\right\} \leqslant s_k \leqslant 0, \tag{2.14}$$

the condition (2.4) holds. Here we take $-a/0 = -\infty$ for any $a \geqslant 0$. Let

$$s_k^{\text{upp}} = \min\left\{\frac{D-\varepsilon_{k-1}}{D-\varepsilon_0}, \frac{D-\varepsilon_k}{D-\varepsilon_N}, \frac{d+\varepsilon_{k-1}}{d+\varepsilon_0}, \frac{d+\varepsilon_k}{d+\varepsilon_N}\right\}, \tag{2.15}$$

$$s_k^{\text{low}} = \max\left\{\frac{-(D-\varepsilon_{k-1})}{d+\varepsilon_0}, \frac{-(D-\varepsilon_k)}{d+\varepsilon_N}, \frac{-(d+\varepsilon_{k-1})}{D-\varepsilon_0}, \frac{-(d+\varepsilon_k)}{D-\varepsilon_N}\right\}. \tag{2.16}$$

Combine (2.10) and (2.14), we have the following theorem.

THEOREM 2.1. *If $-1 < s_k < 1$ and $s_k^{low} \leqslant s_k \leqslant s_k^{upp}$ for each $k = 1, \cdots, N$, where s_k^{upp} and s_k^{low} are given by (2.15) and (2.16), respectively, then*

$$R(x) - d \leqslant f_\Delta(x) \leqslant R(x) + D, \quad x \in I. \tag{2.17}$$

3. Some examples

COROLLARY 3.1. *Let c be a constant such that $c-d \leqslant y_k \leqslant c+D$ for $k = 0,\cdots,N$. If $s_k^{low} \leqslant s_k \leqslant s_k^{upp}$ and $-1 < s_k < 1$ for each $k = 1,\cdots,N$, where*

$$s_k^{upp} = \min\left\{\frac{D-y_{k-1}+c}{D-y_0+c}, \frac{D-y_k+c}{D-y_N+c}, \frac{d+y_{k-1}-c}{d+y_0-c}, \frac{d+y_k-c}{d+y_N-c}\right\},$$
$$s_k^{low} = \max\left\{\frac{-(D-y_{k-1}+c)}{d+y_0-c}, \frac{-(D-y_k+c)}{d+y_N-c}, \frac{-(d+y_{k-1}-c)}{D-y_0+c}, \frac{-(d+y_k-c)}{D-y_N+c}\right\},$$

then the graph of f_Δ lies within the rectangle $I \times [c-d, c+D]$.

Corollary 3.1 can be obtained by Theorem 2.1 if $R(x) = c$. By setting $a = c-d$ and $b = c+D$, Corollary 3.1 can be reduced to [4, Theorem 3].

EXAMPLE 3.1. Let $\Delta = \{(0,1.4),(1,3.5),(2,2.1),(3,3.1),(4,3.8),(5,3.4)\}$. Then $I = [0,5]$ and $x_k = k$ for $k = 0,\cdots,5$. We choose $R(x) = 3$. Then $\varepsilon_0 = -1.6$, $\varepsilon_1 = 0.5$, $\varepsilon_2 = -0.9$, $\varepsilon_3 = 0.1$, $\varepsilon_4 = 0.8$, $\varepsilon_5 = 0.4$. Let $d = D = 2$. By Corollary 3.1 we see that if

$$\begin{aligned} &-0.11111 \leqslant s_1 \leqslant 0.93750, \quad -0.68750 \leqslant s_2 \leqslant 0.41667,\\ &-0.30556 \leqslant s_3 \leqslant 0.80556, \quad -0.50000 \leqslant s_4 \leqslant 0.52778,\\ &-0.66667 \leqslant s_5 \leqslant 0.33333, \end{aligned}$$

then the graph of f_Δ lies within the rectangle $[0,5] \times [1,5]$. The graph of f_Δ with $s_1 = 0.937$, $s_2 = 0.416$, $s_3 = 0.805$, $s_4 = 0.527$, $s_5 = 0.333$ is plotted in Fig 3.1, and the graph in Fig 3.2 is based on the factors $s_1 = 0.937$, $s_2 = -0.687$, $s_3 = -0.305$, $s_4 = -0.5$, $s_5 = 0.333$.

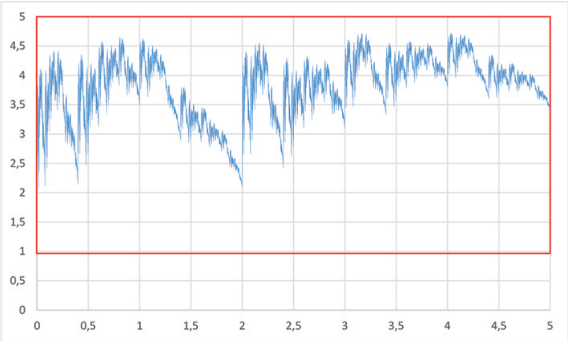

Figure 3.1: $R(x) = 3,\ s_1 = 0.937,\ s_2 = 0.416,\ s_3 = 0.805,\ s_4 = 0.527,\ s_5 = 0.333$

Let R be the function of the straight line passing through (x_0, y_0) and (x_N, y_N). Then

$$R(x) = y_0 + \left(\frac{y_N - y_0}{x_N - x_0}\right)(x - x_0), \qquad x \in I. \tag{3.1}$$

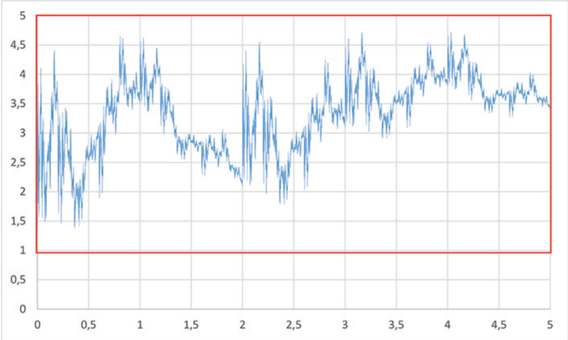

Figure 3.2: $R(x)=3,\ s_1=0.937,\ s_2=-0.687,\ s_3=-0.305,\ s_4=-0.5,\ s_5=0.333$

In this case, $\varepsilon_0=\varepsilon_N=0$. For $k=2,\cdots,N-1$,

$$s_k^{\text{upp}}=\min\left\{\frac{D-\varepsilon_{k-1}}{D},\frac{D-\varepsilon_k}{D},\frac{d+\varepsilon_{k-1}}{d},\frac{d+\varepsilon_k}{d}\right\},$$
$$s_k^{\text{low}}=\max\left\{\frac{-(D-\varepsilon_{k-1})}{d},\frac{-(D-\varepsilon_k)}{d},\frac{-(d+\varepsilon_{k-1})}{D},\frac{-(d+\varepsilon_k)}{D}\right\}.$$

For $k=1$,

$$s_1^{\text{upp}}=\min\left\{1,\frac{D-\varepsilon_1}{D},\frac{d+\varepsilon_1}{d}\right\},$$
$$s_1^{\text{low}}=\max\left\{\frac{-D}{d},\frac{-(D-\varepsilon_1)}{d},\frac{-d}{D},\frac{-(d+\varepsilon_1)}{D}\right\},$$

and for $k=N$,

$$s_N^{\text{upp}}=\min\left\{1,\frac{D-\varepsilon_{N-1}}{D},\frac{d+\varepsilon_{N-1}}{d}\right\},$$
$$s_N^{\text{low}}=\max\left\{\frac{-(D-\varepsilon_{N-1})}{d},\frac{-D}{d},\frac{-(d+\varepsilon_{N-1})}{D},\frac{-d}{D}\right\}.$$

COROLLARY 3.2. *Let R be given by (3.1). If $s_k^{low}\leqslant s_k\leqslant s_k^{upp}$ and $-1<s_k<1$ for each $k=1,\cdots,N$, then the graph of f_Δ lies within the parallelogram with vertices (x_0,y_0-d), (x_0,y_0+D), (x_N,y_N-d), and (x_N,y_N+D).*

EXAMPLE 3.2. Let Δ be given in Example 3.1. Then $R(x)=0.4x+1.4$. We have $\varepsilon_0=0$, $\varepsilon_1=1.7$, $\varepsilon_2=-0.1$, $\varepsilon_3=0.5$, $\varepsilon_4=0.8$, $\varepsilon_5=0$. Let $d=1$ and $D=2$. By Corollary 3.2 we see that if

$$\begin{aligned}
&-0.30000\leqslant s_1\leqslant 0.15000, && -0.30000\leqslant s_2\leqslant 0.15000,\\
&-0.45000\leqslant s_3\leqslant 0.75000, && -0.75000\leqslant s_4\leqslant 0.60000,\\
&-0.50000\leqslant s_5\leqslant 0.60000,
\end{aligned}$$

then the graph of f_Δ lies within the parallelogram with vertices $(0,0.4)$, $(0,3.4)$, $(5,2.4)$, $(5,5.4)$. The graph of f_Δ with $s_1=0.15$, $s_2=0.15$, $s_3=0.75$, $s_4=0.60$, $s_5=0.60$ is plotted in Fig 3.3, and the graph in Fig 3.4 is based on the factors $s_1=-0.30$, $s_2=0.15$, $s_3=0.75$, $s_4=0.60$, $s_5=-0.50$.

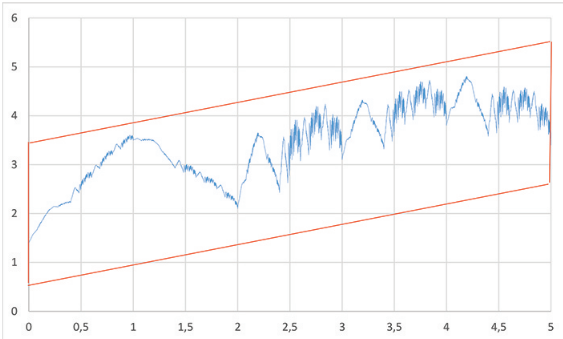

Figure 3.3: $R(x)=0.4x+1.4$, $s_1=0.15$, $s_2=0.15$, $s_3=0.75$, $s_4=0.60$, $s_5=0.60$

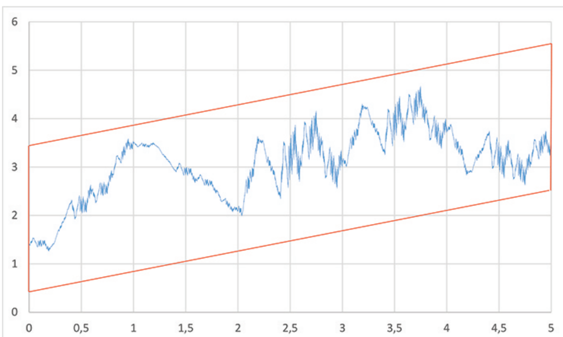

Figure 3.4: $R(x)=0.4x+1.4$, $s_1=-0.30$, $s_2=0.15$, $s_3=0.75$, $s_4=0.60$, $s_5=-0.50$

In the following case, we consider that R is the regression line that fits the data points in Δ by the method of least squares. We choose the coefficients m and c of R to minimize the sum of squared errors $\mathrm{SSE}=\sum_{i=0}^{N}\varepsilon_i^2$. This problem can be solved by taking $\partial\mathrm{SSE}/\partial m=0$ and $\partial\mathrm{SSE}/\partial c=0$. We have

$$m=\frac{\sum_{i=0}^{N}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=0}^{N}(x_i-\bar{x})^2},\quad c=\bar{y}-m\bar{x}, \tag{3.2}$$

where $\bar{x}=\frac{1}{N+1}\sum_{i=0}^{N}x_i$ and $\bar{y}=\frac{1}{N+1}\sum_{i=0}^{N}y_i$. By Theorem 2.1, if $-1<s_k<1$ and $s_k^{\mathrm{low}}\leqslant s_k\leqslant s_k^{\mathrm{upp}}$ for each $k=1,\cdots,N$, where s_k^{upp} and s_k^{low} are given by (2.15) and (2.16), respectively, the graph of f_Δ lies within the parallelogram with vertices $(x_0,mx_0+$

$c-d)$, (x_0, mx_0+c+D), (x_N, mx_N+c-d), (x_N, mx_N+c+D), where m and c are given by (3.2).

EXAMPLE 3.3. Let Δ be given in Example 3.1. Then $m \approx 0.34$ and $c \approx 2$. We choose $R(x) = 0.34x+2$. Then $\varepsilon_0 = -0.6$, $\varepsilon_1 = 1.16$, $\varepsilon_2 = -0.58$, $\varepsilon_3 = 0.08$, $\varepsilon_4 = 0.44$, $\varepsilon_5 = -0.3$. Let $d = 1$ and $D = 2$. If

$$\begin{aligned} &-0.15385 \leqslant s_1 \leqslant 0.36522, \quad -0.18261 \leqslant s_2 \leqslant 0.32308, \\ &-0.16154 \leqslant s_3 \leqslant 0.83478, \quad -0.41538 \leqslant s_4 \leqslant 0.67826, \\ &-0.30435 \leqslant s_5 \leqslant 0.60000, \end{aligned}$$

then the graph of f_Δ lies within the parallelogram with vertices $(0,1)$, $(0,4)$, $(5,2.7)$, $(5,5.7)$. The graph of f_Δ with $s_1 = 0.365$, $s_2 = 0.323$, $s_3 = 0.834$, $s_4 = 0.678$, $s_5 = 0.60$ is plotted in Fig 3.5, and the graph in Fig 3.6 is based on the factors $s_1 = 0.365$, $s_2 = 0.323$, $s_3 = -0.161$, $s_4 = 0.678$, $s_5 = 0.60$.

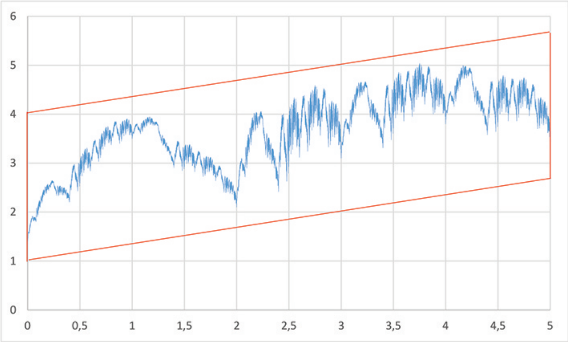

Figure 3.5: $R(x) = 0.34x+2$, $s_1 = 0.365$, $s_2 = 0.323$, $s_3 = 0.834$, $s_4 = 0.678$, $s_5 = 0.60$

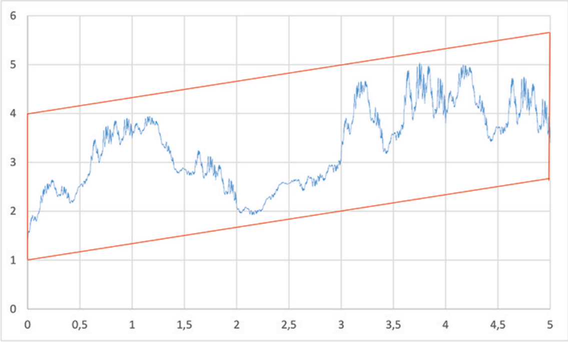

Figure 3.6: $R(x) = 0.34x+2$, $s_1 = 0.365$, $s_2 = 0.323$, $s_3 = -0.161$, $s_4 = 0.678$, $s_5 = 0.60$

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