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ON THE GENERALIZED QUADRATIC GAUSS SUMS AND ITS UPPER BOUND ESTIMATE

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Abstract. The main purpose of this paper is to study generalized quadratic Gauss sums, then use the analytic methods, the properties of the classical Gauss sums and character sums to give a sharp upper bound estimate for it. In addition, we also give several interesting fourth and sixth power mean formulae for the sums.

1. Introduction

The definition of the classical Gauss sums $\tau(\chi,m;q)$ and the quadratic Gauss sums $G(\chi,m;q)$ are as follows:

$$\tau(\chi,m;q)=\sum_{a=1}^{q}\chi(a)e\left(\frac{ma}{q}\right)$$

and

$$G(\chi,m;q)=\sum_{a=1}^{q}\chi(a)e\left(\frac{ma^2}{q}\right), \tag{1}$$

where q is an integer with $q>1$, χ denotes the Dirichlet character $\bmod\ q$, m is any integer and $e(y)=e^{2\pi iy}$. If χ is a primitive character $\bmod\ q$ or $(m,q)=1$, then we have $\tau(\chi,m;q)=\overline{\chi}(m)\tau(\chi,1;q)=\overline{\chi}(m)\cdot\tau(\chi)$.

These sums and related sums have many important applications in the analytic number theory. Therefore, many scholars are devoted to studying the properties of them. For example, W. P. Zhang [1] studied the properties of the quadratic Gauss sums and proved the identities

$$\sum_{\chi \bmod p}|G(\chi,n;p)|^4=\begin{cases}(p-1)\left(3p^2-6p-1\right) & \text{if } p\equiv 3\ (\bmod\ 4),\\ (p-1)\left(3p^2-6p-1+4\left(\frac{n}{p}\right)\sqrt{p}\right) & \text{if } p\equiv 1\ (\bmod\ 4)\end{cases} \tag{2}$$

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and

$$\frac{1}{p-1}\sum_{\chi \bmod p}|G(\chi,n;p)|^6 = 10p^3-25p^2-4p-1, \text{ if } p\equiv 3 \pmod 4, \tag{3}$$

where p is an odd prime, $\sum_{\chi \bmod p}$ denotes the summation over all character mod p, and $\left(\frac{*}{p}\right)$ denotes the Legendre's symbol mod p.

Furthermore, many interesting identities related to the classical Gauss sums are also obtained in references [2]–[13]. For instance, L. Chen [2] considered the sixth-order character mod p, and proved that for any prime p with $p\equiv 1 \pmod 6$ and any sixth-order character λ mod p, one has the identity

$$\tau^3(\lambda)+\tau^3\left(\overline{\lambda}\right) = \begin{cases} p^{\frac{1}{2}}\left(d^2-2p\right) & \text{if } p=12h+1, \\ -i\cdot p^{\frac{1}{2}}\left(d^2-2p\right) & \text{if } p=12h+7, \end{cases}$$

where $i^2=-1$, d is uniquely determined by $4p=d^2+27b^2$ and $d\equiv 1 \pmod 3$.

Recently, S. M. Shen and W. P. Zhang [3] introduced a generalized quadratic Gauss sums as follows:

$$\begin{aligned} &G(\chi_1,\chi_2,\cdots,\chi_k,m;q) \\ &= \sum_{a_1=1}^{q}\sum_{a_2=1}^{q}\cdots\sum_{a_k=1}^{q}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)e\left(\frac{m(a_1+a_2+\cdots+a_k)^2}{q}\right), \end{aligned} \tag{4}$$

where k and m are integers with $k\geqslant 1$ and $(m,q)=1$, and χ_i mod q, $1\leqslant i\leqslant k$.

In fact, if one take $k=1$, then (4) becomes (1). So (4) is a generalized quadratic Gauss sums, and (1) is a special case of (4). Therefore, $G(\chi_1,\chi_2,\cdots,\chi_k,m;q)$ is a further promotion and extension of $G(\chi,m;q)$.

For the special cases $q=p$, an odd prime, and $k=2$, $m=1$, S. M. Shen and W. P. Zhang [3] studied the fourth power mean of (4), and obtained an exact calculating formula for it. That is, they proved the following conclusion.

Let p be an odd prime with $p\equiv 3 \pmod 4$. Then for any character ψ mod p, one has the identity

$$\begin{aligned} &\frac{1}{p-1}\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi(a)\psi(b)e\left(\frac{(a+b)^2}{p}\right)\right|^4 \\ &= \begin{cases} p^5-7p^4+17p^3-10p^2-12p-1 & \text{if } \psi=\chi_0, \\ 3p^4-6p^3-p^2 & \text{if } \psi(-1)=-1, \\ 3p^4+E(\psi,p) & \text{if } \psi(-1)=1 \text{ and } \psi\neq\chi_0, \end{cases} \end{aligned}$$

where χ_0 denotes the principal character mod p, and $|E(\psi,p)|\leqslant 23p^3$.

In this paper, we will study the upper bound estimate problem of (4). If $q=p$ is an odd prime and $k=1$, then from A. Weil's classical work [4] or simple elementary method we can deduce the estimate

$$\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right)\right| \leqslant 2\sqrt{p}.$$

Naturally, we ask the following two problems:

1. For $k \geqslant 2$, whether there exists a sharp upper bound estimate for (4)?
2. Whether there exist two similar formulae as in (2) and (3)?

The main purpose of this paper is to study these problems. We will use the analytic method and the properties of the Gauss sums to prove the following results.

THEOREM 1. *Let p be an odd prime, k and m are integers with $k \geqslant 1$, χ_i $(1 \leqslant i \leqslant k)$ denotes any Dirichlet character* mod p. *If one of χ_i is not the principal character* mod p, *then we have the estimate*

$$\left|\sum_{a_1=1}^{p-1} \cdots \sum_{a_k=1}^{p-1} \chi_1(a_1) \cdots \chi_k(a_k)\, e\left(\frac{m(a_1+\cdots+a_k)^2}{p}\right)\right| \leqslant 2 \cdot p^{\frac{k}{2}}.$$

THEOREM 2. *Let p be an odd prime with $p \equiv 3 \pmod 4$, k and m are two integers with $k \geqslant 1$ and $(m,p)=1$, χ_i $(1 \leqslant i \leqslant k)$ denotes any non-principal characters* mod p. *If $\chi_1\chi_2\cdots\chi_k = \chi_0$ is the principal character* mod p, *then we have the identities*

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(\chi,\chi_1,\cdots,\chi_k,m;p)|^4 = p^{2k-2} \cdot \left(3p^4 - 7p^3 - 4p^2 - 3p - 1\right)$$

and

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(\chi,\chi_1,\cdots,\chi_k,m;p)|^6$$
$$= p^{3k-3} \cdot \left(10p^6 - 26p^5 - 8p^4 - 8p^3 - 7p^2 - 4p - 1\right).$$

If $\chi_1\chi_2\cdots\chi_k = \psi \neq \chi_0$, then we have the identities

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(\chi,\chi_1,\chi_2,\cdots,\chi_k,m;p)|^4$$
$$= p^{2k} \cdot \left(3p^2 - 6p - 1\right) - (p+1) \cdot p^{2k-2} \cdot \left|\sum_{a=1}^{p-1} \psi(a) e\left(\frac{ma^2}{p}\right)\right|^4$$

and

$$\frac{1}{p-1} \sum_{\chi \bmod p} |G(\chi,\chi_1,\chi_2,\cdots,\chi_k,m;p)|^6$$
$$= p^{3k} \cdot \left(10p^3 - 25p^2 - 4p - 1\right) - p^{3k-3}\left(p^2+p+1\right)\left|\sum_{a=1}^{p-1} \psi(a) e\left(\frac{ma^2}{p}\right)\right|^6.$$

From Theorem 2 we can deduce the following two corollaries.

COROLLARY 1. *Let p be an odd prime with $p \equiv 3 \pmod 4$, k and m are two integers with $k \geqslant 1$ and $(m,p)=1$, χ_i $(1 \leqslant i \leqslant k)$ is any non-principal character* mod p *such that $\chi_1\chi_2\cdots\chi_k$ is an odd character* mod p, *then we have*

$$\sum_{\chi \bmod p} |G(\chi,\chi_1,\chi_2,\cdots,\chi_k,m;p)|^4 = (p-1)\cdot p^{2k}\cdot\left(3p^2-6p-1\right)$$

and

$$\sum_{\chi \bmod p} |G(\chi,\chi_1,\chi_2,\cdots,\chi_k,m;p)|^6 = (p-1)\cdot p^{3k}\cdot\left(10p^3-25p^2-4p-1\right).$$

COROLLARY 2. *Let p be an odd prime with $p \equiv 3 \pmod 4$, k and m are two integers with $k \geqslant 1$ and $(m,p)=1$, χ_i $(1 \leqslant i \leqslant k)$ is any non-principal character* mod p *such that $\chi_1\chi_2\cdots\chi_k$ is an even character, then we have the asymptotic formulae*

$$\sum_{\chi \bmod p} |G(\chi,\chi_1,\chi_2,\cdots,\chi_k,m;p)|^4 = 3\cdot p^{2k+3} + E(m,p)$$

and

$$\sum_{\chi \bmod p} |G(\chi,\chi_1,\chi_2,\cdots,\chi_k,m;p)|^6 = 10\cdot p^{3k+4} + H(m,p),$$

where the error terms $E(m,p)$ and $H(m,p)$ satisfies the estimates $|E(m,p)| \leqslant 25\cdot p^{2k+2}$ and $|H(m,p)| \leqslant 99\cdot p^{3k+3}$.

SOME NOTES. If $p \equiv 1 \pmod 4$ in Theorem 2, then we can also deduce some corresponding results. But at this time, the results are not so perfect and beautiful.

For general integer $q > 1$, whether there exist some similar estimates or identities as in Theorem 1 and Theorem 2 are two open problems.

Obviously, whether the constant 2 in Theorem 1 is the best one is also an interesting problem, but we guess it is the best.

2. Several simple lemmas

To prove our theorems, we need two simple lemmas. In the process of proving our lemmas, we need to use some basic properties of the classical Gaussian sums, Jacobi sums and character sums, all of these can be found in references [14] and [15], so there is no need to repeat. First we have the following.

LEMMA 1. *Let p be an odd prime, k and m are integers with $(m,p)=1$ and $k \geqslant 1$. For any character χ_i* mod p *with $1 \leqslant i \leqslant k$, if one of χ_i is not the principal character* mod p *and $\chi_1\chi_2\cdots\chi_k = \chi_0$, then we have the identity*

$$\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)e\left(\frac{m(a_1+a_2+\cdots+a_k)^2}{p}\right)$$
$$=\left(1-\frac{1}{p}\sum_{b=0}^{p-1}e\left(\frac{mb^2}{p}\right)\right)\cdot\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_{k-1})\cdot\tau\left(\overline{\chi_1\chi_2\cdots\chi_{k-1}}\right).$$

If $\chi_1\chi_2\cdots\chi_k \neq \chi_0$, *then we have*

$$\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)e\left(\frac{m\,(a_1+a_2+\cdots+a_k)^2}{p}\right)$$
$$=\frac{\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_k)}{\tau\,(\chi_1\chi_2\cdots\chi_k)}\cdot\left(\sum_{b=1}^{p-1}\chi_1(b)\chi_2(b)\cdots\chi_k(b)e\left(\frac{mb^2}{p}\right)\right).$$

Proof. Let $\chi=\chi_1\chi_2\cdots\chi_k$, then we have

$$G(\chi_1,\chi_2,\cdots,\chi_k,m;p)=\sum_{n\in\mathbb{F}_p}e\left(\frac{mn^2}{p}\right)\sum_{\substack{a_1,a_2,\cdots,a_k\in\mathbb{F}_p^*\\ a_1+a_2+\cdots+a_k=n}}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)$$
$$=\sum_{n\in\mathbb{F}_p^*}e\left(\frac{mn^2}{p}\right)\sum_{\substack{a_1,a_2,\cdots,a_k\in\mathbb{F}_p^*\\ a_1+a_2+\cdots+a_k=1}}\chi_1(na_1)\chi_2(na_2)\cdots\chi_k(na_k)$$
$$+\sum_{\substack{a_1,a_2,\cdots,a_k\in\mathbb{F}_p^*\\ a_1+a_2+\cdots+a_k=0}}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)$$
$$=\sum_{n\in\mathbb{F}_p^*}\chi(n)e\left(\frac{mn^2}{p}\right)\sum_{\substack{a_1,a_2,\cdots,a_k\in\mathbb{F}_p^*\\ a_1+a_2+\cdots+a_k=1}}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)$$
$$+\sum_{\substack{a_1,a_2,\cdots,a_k\in\mathbb{F}_p^*\\ a_1+a_2+\cdots+a_k=0}}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)$$
$$\equiv\sum_{n\in\mathbb{F}_p^*}\chi(n)e\left(\frac{mn^2}{p}\right)J\,(\chi_1,\chi_2,\cdots,\chi_k)+J_0\,(\chi_1,\chi_2,\cdots,\chi_k),\tag{5}$$

where $J\,(\chi_1,\chi_2,\cdots,\chi_k)$ and $J_0\,(\chi_1,\chi_2,\cdots,\chi_k)$ are the Jacobi sums (see [15]).

From the definition and properties of the classical Gauss sums we can easily deduce (see [15] for details)

$$J\,(\chi_1,\chi_2,\cdots,\chi_k)=\frac{\chi(-1)}{p}\cdot\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_k)\cdot\tau\,(\overline{\chi})\tag{6}$$

and

$$J_0\,(\chi_1,\chi_2,\cdots,\chi_k)=\frac{1}{p}\cdot\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_k)\cdot\sum_{b=1}^{p-1}\overline{\chi}(b).\tag{7}$$

It is clear that if $\chi(-1)=-1$, then we have the identity

$$\sum_{n\in\mathbb{F}_p}\chi(n)e\left(\frac{mn^2}{p}\right)=0.\tag{8}$$

Therefore, from (5), (7) and the properties of character sums we have

$$G(\chi_1,\chi_2,\cdots,\chi_k,m;p)=0. \tag{9}$$

Now if $\chi=\chi_1\chi_2\cdots\chi_k=\chi_0$, then note that $\chi_1\chi_2\cdots\chi_{k-1}=\overline{\chi}_k$ and

$$\sum_{b=1}^{p-1}\overline{\chi}(b)=\sum_{b=1}^{p-1}\chi_0(b)=p-1,$$

from (5), (6) and (7) we have

$$\begin{aligned}&G(\chi_1,\chi_2,\cdots,\chi_k,m;p)\\ =&\left(1-\frac{1}{p}\sum_{b=0}^{p-1}e\left(\frac{mb^2}{p}\right)\right)\cdot\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_{k-1})\cdot\tau\left(\overline{\chi_1\chi_2\cdots\chi_{k-1}}\right).\end{aligned} \tag{10}$$

If $\chi_1\chi_2\cdots\chi_k\neq\chi_0$ and $\chi(-1)=1$, then note that $\tau(\chi)\cdot\tau(\overline{\chi})=\overline{\chi}(-1)\cdot p=p$, from (5), (6) and (7) we have

$$\begin{aligned}&G(\chi_1,\chi_2,\cdots,\chi_k,m;p)\\ =&\frac{\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_k)}{\tau(\chi_1\chi_2\cdots\chi_k)}\cdot\left(\sum_{b=1}^{p-1}\chi_1(b)\chi_2(b)\cdots\chi_k(b)e\left(\frac{mb^2}{p}\right)\right).\end{aligned} \tag{11}$$

Combining (9), (10) and (11) we may immediately deduce Lemma 1. □

LEMMA 2. *Let p be an odd prime, m be any integer with $(m,p)=1$. Then for any Dirichlet character χ* mod p*, we have the estimate*

$$\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|\leqslant 2\sqrt{p}.$$

Proof. It is clear that if $\chi(-1)=-1$, then we have

$$\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)=0. \tag{12}$$

If $\chi(-1)=1$, then there exists a character λ mod p such that $\lambda^2=\chi$. In this case, let χ_2 denote the Legendre's symbol mod p. Then from the properties of the classical Gauss sums we have

$$\begin{aligned}&\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|=\left|\sum_{a=1}^{p-1}\lambda^2(a)e\left(\frac{ma^2}{p}\right)\right|=\left|\sum_{a=1}^{p-1}\lambda(a)\left(1+\chi_2(a)\right)e\left(\frac{ma}{p}\right)\right|\\ =&\left|\sum_{a=1}^{p-1}\lambda(a)e\left(\frac{ma}{p}\right)+\sum_{a=1}^{p-1}\lambda(a)\chi_2(a)e\left(\frac{ma}{p}\right)\right|\\ =&\left|\tau(\lambda)+\chi_2(m)\tau(\lambda\chi_2)\right|\leqslant|\tau(\lambda)|+|\chi_2(m)\tau(\lambda\chi_2)|\leqslant 2\sqrt{p}.\end{aligned} \tag{13}$$

From (12) and (13) we can deduce Lemma 2. □

3. Proofs of the theorems

In this section, we shall complete the proofs of our theorems. First we prove Theorem 1. Note that the estimate for classical Gauss sums is $|\tau(\chi)| \leqslant \sqrt{p}$, and

$$\left|\sum_{a=0}^{p-1} e\left(\frac{ma^2}{p}\right)\right| = \left|\sum_{a=1}^{p-1}\left(\frac{a}{p}\right) e\left(\frac{ma}{p}\right)\right| = \sqrt{p},$$

from Lemma 1 and Lemma 2 we know that if $\chi_1\chi_2\cdots\chi_k = \chi_0$, then we have

$$\begin{aligned}
&\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)e\left(\frac{m(a_1+a_2+\cdots+a_k)^2}{p}\right)\right| \\
&= \left|\left(1-\frac{1}{p}\sum_{b=0}^{p-1}e\left(\frac{mb^2}{p}\right)\right)\cdot\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_{k-1})\cdot\tau\left(\overline{\chi_1\chi_2\cdots\chi_{k-1}}\right)\right| \\
&\leqslant \left(1+\frac{\sqrt{p}}{p}\right)\cdot p^{\frac{k}{2}} \leqslant 2p^{\frac{k}{2}}. \qquad (14)
\end{aligned}$$

If $\chi_1\chi_2\cdots\chi_k \neq \chi_0$, then from Lemma 1 and Lemma 2 we have the estimate

$$\begin{aligned}
&\left|\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi_1(a_1)\chi_2(a_2)\cdots\chi_k(a_k)e\left(\frac{m(a_1+a_2+\cdots+a_k)^2}{p}\right)\right| \\
&= \left|\frac{\tau(\chi_1)\cdot\tau(\chi_2)\cdots\tau(\chi_k)}{\tau(\chi_1\chi_2\cdots\chi_k)}\cdot\left(\sum_{b=1}^{p-1}\chi_1(b)\chi_2(b)\cdots\chi_k(b)e\left(\frac{mb^2}{p}\right)\right)\right| \\
&\leqslant p^{\frac{k-1}{2}}\cdot 2\sqrt{p} = 2p^{\frac{k}{2}}. \qquad (15)
\end{aligned}$$

It is clear that Theorem 1 follows from the estimations (14) and (15).

Now we prove Theorem 2. If $p \equiv 3 \pmod 4$, let $\left(\frac{*}{p}\right) = \lambda$ denote the Legendre's symbol $\bmod\ p$, then $\tau(\lambda) = i\cdot\sqrt{p}$ (where $i^2 = -1$). So if $\chi_1\chi_2\cdots\chi_k = \chi_0$, then from Lemma 1 and its proving method we have

$$\begin{aligned}
&\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi_0(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m(a+a_1+\cdots+a_k)^2}{p}\right)\right|^4 \\
&= \frac{(p+1)^2}{p^2}\cdot p^{2k} = (p+1)^2\cdot p^{2(k-1)}. \qquad (16)
\end{aligned}$$

In this time, from (2), (16), Lemma 1 and Lemma 2 we have

$$\begin{aligned}
&\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m(a+a_1+\cdots+a_k)^2}{p}\right)\right|^4 \\
&= \sum_{\substack{\chi \bmod p \\ \chi\neq\chi_0}}\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m(a+a_1+\cdots+a_k)^2}{p}\right)\right|^4
\end{aligned}$$

$$+\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi_0(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m\,(a+a_1+\cdots+a_k)^2}{p}\right)\right|^4$$

$$= p^{2k}\cdot\sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left|\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\cdots\chi_k(a)e\left(\frac{ma^2}{p}\right)\right|^4 + (p+1)^2\cdot p^{2(k-1)}$$

$$= p^{2k}\cdot\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\cdots\chi_k(a)e\left(\frac{ma^2}{p}\right)\right|^4$$

$$+(p+1)^2\cdot p^{2(k-1)} - p^{2k}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^2}{p}\right)\right|^4$$

$$= p^{2k}\cdot\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^4 + (p+1)^2\cdot p^{2(k-1)} - p^{2k}(p+1)^2$$

$$= (p-1)\cdot p^{2k}\cdot\left(3p^2-6p-1\right)-(p+1)^3\cdot p^{2k-2}\cdot(p-1)$$

$$= (p-1)\cdot p^{2k-2}\cdot\left(3p^4-7p^3-4p^2-3p-1\right). \tag{17}$$

Similarly, from (3) and the method of proving (17) we can also deduce the identity

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m\,(a+a_1+\cdots+a_k)^2}{p}\right)\right|^6$$

$$= p^{3k}\cdot\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\cdots\chi_k(a)e\left(\frac{ma^2}{p}\right)\right|^6$$

$$+(p+1)^3\cdot p^{3(k-1)} - p^{3k}\left|\sum_{a=1}^{p-1}e\left(\frac{ma^2}{p}\right)\right|^6$$

$$= p^{3k}\cdot\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^6 + (p+1)^3\cdot p^{3(k-1)} - p^{3k}(p+1)^3$$

$$= (p-1)\cdot p^{3k}\cdot\left(10p^3-25p^2-4p-1\right)-(p+1)^3\cdot p^{3k-3}\cdot(p^3-1)$$

$$= (p-1)\cdot p^{3k-3}\cdot\left(10p^6-26p^5-8p^4-8p^3-7p^2-4p-1\right). \tag{18}$$

If $\chi_1\chi_2\cdots\chi_k=\psi\neq\chi_0$, then from Lemma 1, Lemma 2 and the method of proving (17) we have

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m\,(a+a_1+\cdots+a_k)^2}{p}\right)\right|^4$$

$$= \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m\,(a+a_1+\cdots+a_k)^2}{p}\right)\right|^4$$

$$+\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi_0(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m(a+a_1+\cdots+a_k)^2}{p}\right)\right|^4$$

$$= p^{2k}\cdot \sum_{\substack{\chi \bmod p\\ \chi\neq\chi_0}}\left|\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\cdots\chi_k(a)e\left(\frac{ma^2}{p}\right)\right|^4$$

$$+p^{2(k-1)}\left|\sum_{a=1}^{p-1}\chi_1(a)\cdots\chi_k(a)e\left(\frac{ma^2}{p}\right)\right|^4$$

$$= p^{2k}\cdot \sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)\chi_1(a)\cdots\chi_k(a)e\left(\frac{ma^2}{p}\right)\right|^4$$

$$-\left(p^{2k}-p^{2k-2}\right)\left|\sum_{a=1}^{p-1}\chi_1(a)\cdots\chi_k(a)e\left(\frac{ma^2}{p}\right)\right|^4$$

$$= p^{2k}\cdot \sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^4-\left(p^{2k}-p^{2k-2}\right)\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^4$$

$$= (p-1)\cdot p^{2k}\cdot\left(3p^2-6p-1\right)-\left(p^{2k}-p^{2k-2}\right)\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^4. \qquad (19)$$

Similarly, we also have the identity

$$\sum_{\chi \bmod p}\left|\sum_{a=1}^{p-1}\sum_{a_1=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}\chi(a)\chi_1(a_1)\cdots\chi_k(a_k)\, e\left(\frac{m(a+a_1+\cdots+a_k)^2}{p}\right)\right|^6$$

$$= (p-1)\cdot p^{3k}\cdot\left(10p^3-25p^2-4p-1\right)-\left(p^{3k}-p^{3k-3}\right)\left|\sum_{a=1}^{p-1}\psi(a)e\left(\frac{ma^2}{p}\right)\right|^6, \quad (20)$$

where $\psi=\chi_1\chi_2\cdots\chi_k$.

Now Theorem 2 follows from (17), (18), (19) and (20).

4. Conclusion

The main results of this paper are Theorem 1 and Theorem 2. Theorem 1 obtained a sharper upper bound estimate for (4) with $q=p$, an odd prime. Theorem 2 proved several identities for the fourth power mean and the sixth power mean of (4) with $q=p$. Especially Corollary 1, the result is very simple and beautiful. These works have a good reference for further research on generalized multivariate quadratic Gauss sums. In addition, these theorems also profoundly reveal the law of the value distribution of this kind new Gauss sums.

For the general integer $q>1$ (or $q=p$ and $k\geqslant 4$), whether there exits a mean value formula or asymptotic formula for (4) is an open problem. These will contribute to the further study of these contents.

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REFERENCES

[1] W. P. ZHANG, *Moments of Generalized Quadratic Gauss Sums Weighted by L-functions*, Journal of Number Theory, 92 (2002), 304–314.

[2] L. CHEN, *On the classical Gauss sums and their some properties*, Symmetry 2018, 10, 625.

[3] S. M. SHEN AND W. P. ZHANG, *The generalized quadratic Gauss sums and its fourth power mean*, Mathematics 2019, 7, 258.

[4] A. WEIL, *On some exponential sums*, Proc. Nat. Acad. Sci. U.S.A., 34 (1948), 203–210.

[5] Z. Y. CHEN AND W. P. ZHANG, *On the fourth-order linear recurrence formula related to classical Gauss sums*, Open Mathematics, 2017, 15, 1251–1255.

[6] B. C. BERNDT AND R. J. EVANS, *The determination of Gauss sums*, Bulletin of the American Mathematical Society, 1981, 5, 107–128.

[7] H. BAI AND J. Y. HU, *On the classical Gauss sum and the recursive properties*, Advances in Difference Equations, 2018, 2018: 387.

[8] W. P. ZHANG AND J. Y. HU, *The number of solutions of the diagonal cubic congruence equation* mod p, Mathematical Reports, 2018, 20, 73–80.

[9] S. M. SHEN AND W. P. ZHANG, *On the quartic Gauss sums and their recurrence property*, Advances in Difference Equations, 2017, 2017: 43.

[10] L. CHEN AND J. Y. HU, *A linear Recurrence Formula Involving Cubic Gauss Sums and Kloosterman Sums*, Acta Mathematica Sinica (Chinese Series), 61 (2018), 67–72.

[11] T. T. WANG AND G. H. CHEN, *A note on the classical Gauss sums*, Mathematics, 2018, 6, 313.

[12] S. CHOWLA, J. COWLES AND M. COWLES, *On the number of zeros of diagonal cubic forms*, Journal of Number Theory, 9 (1977), 502–506.

[13] B. C. BERNDT AND R. J. EVANS, *Sums of Gauss, Jacobi, and Jacobsthal*, Journal of Number Theory, 1979, 11, 349–389.

[14] T. M. APOSTOL, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

[15] K. IRELAND AND M. ROSEN, *A classical introduction to modern number theory*, Springer-Verlag, New York, 1982.

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